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# SOME RELATIONS AMONG ARITHMETICAL FUNCTIONS

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**Abstract.** We consider some possible relations among $q$-additive and completely multiplicative functions. We proved that if $f$ is completely multiplicative and $q$-additive function, then either $f(n)=n$ for every $n\in\mathbb{N}$ or $f(n)$ is the Dirichlet character $\pmod{q_0}$, where $q_0|q$.

## 1. Notation and some notions

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ be the set of primes, positive integers, integers, rational numbers and complex numbers, respectively. Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ be the set of non-negative integers.

Let $\mathcal{A}$ (resp. $\mathcal{A}^*$) be the class of additive (completely additive) functions, $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the class of multiplicative (completely multiplicative) functions.

For some integer $q\geq 2$ let $\mathcal{A}_q$ be the set of $q$-additive function. Every $n\in\mathbb{N}_0$ can be uniquely represented in the form

$$n=\sum_{r=0}^{\infty}a_r(n)q^r \quad \text{with} \quad a_r(n)\in\{0,1,...,q-1\}(=A_q)$$

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and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(\epsilon_r(n) q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}.$$

These functions were first introduced and studied by A. O. Gelfond [6].

**Definition 1.** (Set of uniqueness for completely additive functions.) We say that $A \subseteq \mathbb{N}$ is a *set of uniqueness for completely additive functions* if $f \in \mathcal{A}^*$, $f(a) = 0$ for every $a \in A$ implies that $f(n) = 0$ for all $n \in \mathbb{N}$.

**Definition 2.** (Set of uniqueness for completely additive functions $\pmod 1$.) We say that $B \subseteq \mathbb{N}$ is a *set of uniqueness for completely additive functions* $\pmod 1$ if $f \in \mathcal{A}^*$, $f(b) \equiv 0 \pmod 1$ for all $b \in B$ implies that $f(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$.

D. Wolke [16] proved that $A$ is a set of uniqueness if and only if every $n \in \mathbb{N}$ can be written as $n = \prod_{i=1}^{k} a_i^{r_i}$, where $a_i \in A$ and $r_i \in \mathbb{Q}$.

K.-H. Indlekofer [8], [9], P. Hoffman [7], F. Dress and B. Volkmann [2] proved independently that $B$ is a set of uniqueness for completely additive functions $\pmod 1$ if and only if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^{k} b_j^{\ell_j}, \quad \text{where} \quad \ell_j \in \mathbb{Z},\ b_j \in B.$$

I. Kátai [11], [12] formulated the conjecture in 1969 that

$$\mathcal{P}_{+1} = \{p + 1 \mid p \in \mathcal{P}\}$$

is a set of uniqueness for additive functions, and proved that there exists such a finite set $Q$ of primes for which $\mathcal{P}_{+1} \cup Q$ is a set of uniqueness $\pmod 1$. P. D. T. A. Elliott [3] proved that $Q = \{p \mid p \le 10^{387}, p \in \mathcal{P}\}$ is an appropriate choice, that is every $n \in \mathbb{N}$ can be written as

$$n = t \cdot \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}, \quad \epsilon_i \in \{-1, 1\},$$

and $t$ is such a rational number the largest prime factor of which does not exceed $10^{387}$.

Furthermore, in [4] he proved that for every rational number $r$, we can find such primes $p_1, \cdots, p_k$ and $\epsilon_j \in \{-1, 1\}$ $(j = 1, 2, \cdots, k)$, for which

$$r^g = \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}. \tag{1.1}$$

Here $g$ is a constant, $g \in \{1, 2, 3\}$.

A direct consequence of this assertion is

**Theorem 1.** *Let $f \in \mathcal{M}^*$, $f(p+1) = p+1$ $(\forall p \in \mathcal{P})$. Then*

$$f(n) = nH(n), \quad H \in \mathcal{M}^*, \quad H(n)^g = 1 \quad \textit{for every} \quad n \in \mathbb{N}.$$

*Especially, if $f(n)$ is a positive real number for every $n \in \mathbb{N}$, then*

$$f(n) = n \quad \textit{for every} \quad n \in \mathbb{N}.$$

**Proof.** Since, for every $q \in \mathcal{P}$ there is at least one $p \in \mathcal{P}$ such that $q|p+1$, therefore $f(q) \neq 0$, and so $f(n) \neq 0$ $(n \in \mathbb{N})$. From (1.1) we obtain that

$$f(r)^g = \prod_{i=1}^{k} f(p_i+1)^{\epsilon_i} = \prod_{i=1}^{k} (p_i+1)^{\epsilon_i} = r^g.$$

Thus

$$1 = \left(\frac{f(r)}{r}\right)^g = H(r)^g \quad \text{for every positive rational number.}$$

The proof of Theorem 1 is complete. ■

The conjecture that $\mathcal{P}_{+1}$ is a set of uniqueness (mod 1) is formulated by several mathematicians. It would follow from the conjecture of A. Schinzel and W. Sierpinnski [14], namely that every positive integer has infinitely many representations of the form $\frac{p+1}{q+1}$ $(p, q \in \mathcal{P})$.

T. Csajbók, A. Járai and J. Kasza [1] proved that every prime $Q \in [2, 10^{14}]$ can be written as $\frac{p+1}{q+1}$ $(p, q \in \mathcal{P})$, and every $n \in [2, 10^{11}]$ can be written also as $\frac{p+1}{q+1}$ $(p, q \in \mathcal{P})$.

## 2. On *q*-additive functions

**Definition 3.** (Set of uniqueness for $q$-additive functions.) We say that $D \subseteq \mathbb{N}_0$ is a *set of uniqueness for the set of $q$-additive functions*, if $f \in \mathcal{A}_q$, $f(d) = 0$ for all $d \in D$ implies that $f(n) = 0$ for all $n \in \mathbb{N}_0$.

The functions $f(n) = cn$ belong to $\mathcal{A}_q$ for every $q \geq 2$. J. C. Puchta and J. Spilker [13] gave all the functions belonging to $\mathcal{A}_{q_1} \cap \mathcal{A}_{q_2}$.

The following question seems to be interesting. Let $(q_1, q_2) = 1$, $q_1$, $q_2 \geq 2$. Let $\mathcal{K}$ be such a subset of $\mathbb{N}_0$ for which the following assertion holds:

*If*

$$f_1 \in \mathcal{A}_{q_1}, f_2 \in \mathcal{A}_{q_2} \quad \textit{and} \quad f_1(k) = f_2(k) \quad \textit{for every} \quad k \in \mathcal{K},$$

*then*

$$f_1(n) = f_2(n) = cn \quad \textit{for every} \quad n \in \mathbb{N}_0,$$

*where* $c \in \mathbb{C}$ *is a suitable number.*

Assume that $q_2 > q_1 \geq 2$,

$$E_1 = \{aq_1^n | a = 1, \cdots, q_1 - 1, n = 0, 1, \cdots\}$$

and

$$E_2 = \{bq_2^m | b = 1, \cdots, q_2 - 1, m = 0, 1, \cdots\}.$$

Let $bq_2^m \in E_2$ with $m \geq 1$. Let $L(bq_2^m) = aq_1^n$ be the largest element of $E_1$, for which

$$aq_1^n < bq_2^m.$$

Observe that $aq_1^n = bq_2^m$ or $(a+1)q_1^n = bq_2^m$ cannot occur. Let

$$J_{bq_2^m} = \Big(bq_2^m, (a+1)q_1^n\Big), \quad \text{where} \quad L(bq_2^m) = aq_1^n.$$

It is obvious that $a + 1 = q_1$ can be occur.

We know that the interval $J_{bq_2^m}$ is quite a large interval. This follows from an important theorem of R. Tijdeman [15], which is stated now as

**Theorem A.** *Let* $p$ *be a prime,* $p \geq 3$ *and let* $1 = n_1 < n_2 < \cdots$ *be the sequence of all positive integers composed of primes* $\leq p$*. Then there exists an effectively computable constant* $C = C(p)$ *such that*

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^C} \quad \textit{for every} \quad n_i \geq 3.$$

**Corollary.** *Let $p$ be the largest prime factor of $q_2!$, $C = C(p)$ (defined in Theorem A). Let $\mathcal{K}$ be such a set for which*

$$\mathcal{K} \cap J_{bq_2^m} \neq \emptyset \quad \textit{if} \quad bq_2^m > K.$$

*Assume that $g_1 \in \mathcal{A}_{q_1}$,$g_2 \in \mathcal{A}_{q_2}$,*

$$g_1(n) = g_2(n) \quad \textit{if} \quad n \in \mathcal{K}$$

*and*

$$g_1(n) = g_2(n) = cn \quad \textit{for every} \quad n \leq K \quad (K > q_2).$$

*Then*

$$g_1(n) = g_2(n) = cn \quad \textit{for every} \quad n \in \mathbb{N}_0.$$

**Proof.** We can see it by using induction. ■

Assume that $g_1(k) = g_2(k) = ck$ holds for every $k < bq_2^m$. Let $k < (a+1)q_1^n$. Since

$$g_1(uq_1^\nu) = \begin{cases} cuq_1^\nu & \text{for} \quad \nu < n \quad \text{and} \quad u \in \{1, \cdots, q_1 - 1\} \\ cuq_1^n & \text{for every} \quad u \leq a, \end{cases}$$

therefore

$$g_1(k) = ck \quad \text{if} \quad k < (a+1)q_1^n.$$

Let $\kappa \in \mathcal{K} \cap J_{bq_2^m}$. Thus

$$g_1(\kappa) = g_2(\kappa) = c\kappa, \kappa = bq_2^m + h, h < q_2^m,$$

and so

$$g_2(\kappa) = g_2(bq_2^m) + g_2(h) = g_2(bq_2^m) + ch.$$

Consequently

$$g_2(bq_2^m) = cbq_2^m.$$

**Remark.** Let $\mathcal{E} = \{a_1 < a_2 < \cdots\}$ be such a sequence of integers for which $a_{n+1} - a_n < a_n^{1-\epsilon}$ for some $\epsilon > 0$. Then $\mathcal{E}$ is a $\mathcal{K}$-sequence, that is, if $K$ is a suitable constant, $f_1 \in \mathcal{A}_{q_1}$, $f_2 \in \mathcal{A}_{q_2}$, and

$$f_1(n) = f_2(n) \ \ (\forall n < K) \quad \text{and} \quad f_1(a_j) = f_2(a_j) \ \ (j = 1, 2, \cdots),$$

then

$$f_1(n) = f_2(n) = c \cdot n \quad \text{for every} \quad n \in \mathbb{N}_0.$$

Specially, if $Q(x) \in \mathbb{Z}[x]$,

$$\mathcal{E}_1 = \{Q(n) | n = 1, 2, \cdots\} \quad \text{and} \quad \mathcal{E}_2 = \{Q(p) | p \in \mathcal{P}\},$$

then both of $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{K}$-sequences.

## 3. What are the $q$-additive multiplicative functions?

In this section let $I(n)$ be the identity function and $\chi_q(n)$ be a Dirichlet character function. It is clear that

$$\{I, \chi_q\} \subseteq \mathcal{M}^* \cap \mathcal{A}_q.$$

We shall prove the following

**Theorem 2.** *Let $\mathcal{M}^*$ the set of all completely multiplicative functions $f$ with $f(1) = 1$. Then there exists a $q_0|q$ such that*

$$\mathcal{M}^* \cap \mathcal{A}_q \subseteq \{I, \chi_{q_0}\}.$$

**Proof.** Let $f \in \mathcal{M}^* \cap \mathcal{A}_q$. First we consider the case $f(q) = 0$. Let $q = q_0 q_1$, where $f(q_1) \neq 0$ and $f(p) = 0$ for all primes $p, p|q_0$. It is clear that

$$f(q_1)f(q_0 m + 1) = f(qm + q_1) = f(q_1) + f(qm) = f(q_1) + f(q)f(m) = f(q_1)$$

for every $m \in \mathbb{N}$. This implies that $f(q_0 m + 1) = 1$, which with Lemma 19.3 of [5] proves that $f = \chi_{q_0}$.

Now we assume that $\xi := f(q) \neq 0$. For each $a \in \{0, 1, \cdots q-2\}$, we have

$$f(q+1) = f(1) + f(q) = 1 + \xi, \quad f(q+a) = f(a) + f(q) = f(a) + \xi$$

and

$$(q+1)(q+a) = q^2 + (a+1)q + a.$$

Thus, we infer that

$$f\Big((q+1)(q+a)\Big) = f\Big(q^2 + (a+1)q + a\Big) =$$

$$= f(q^2) + f\Big((a+1)q\Big) + f(a) = \xi^2 + f(a+1)\xi + f(a),$$

consequently

$$(1+\xi)(f(a)+\xi) = \xi^2 + f(a+1)\xi + f(a).$$

This shows that

$$f(a+1) = f(a) + 1 \quad \text{for} \quad a \in \{0, 1, \cdots q-2\},$$

and so

$$f(m) = m \quad \text{for} \quad m \in \{0, 1, \cdots q-1\}.$$

Since $f \in \mathcal{A}_{q^2}$, therefore the above result shows that

$$f(M) = M \quad \text{for every} \quad M < q^2,$$

and so $\xi = f(q) = q$, because $q < q^2$.

Theorem 2 is proved. ■

The following problem seems to be interesting. Characterize those subsets $\mathcal{D}$ of $\mathbb{N}_0$ for which if $f \in \mathcal{M}^*$, $g \in \mathcal{A}_q$, and

$$f(d) = g(d) \quad (\forall d \in \mathcal{D}),$$

then

$$f(n) = g(n) \quad \text{for every} \quad n \in \mathbb{N}.$$

**Theorem 3.** *Let $c, N_0$ be given numbers, $J_N = (2^N, 2^{N+1})$. Let $\mathcal{D} \subseteq \mathbb{N}_0$ be such a set of integers for which if $N > N_0$, then there exist $m_1, m_2 \in J_N \cap \mathcal{D}$, $m_1 \neq m_2$ such that*

$$\frac{m_1}{m_2} = \frac{A}{B}, \quad A, B < c, A, B \in \mathbb{N}.$$

*Assume that*

$$f \in \mathcal{M}^*, \quad g \in \mathcal{A}_2, \quad f(n) = g(n) = n \quad \text{if} \quad n \leq \max(c, 2^{N_0}).$$

*Then*

$$f(d) = g(d) \quad \text{for every} \quad d \in \mathcal{D},$$

*implies that*

$$f(n) = n \quad \text{for every} \quad n \in \mathcal{D}.$$

**Proof.** We shall use induction. Assume that

$$g(2^n) = 2^n \quad \text{for every} \quad n = 0, 1, \cdots, N-1.$$

Then clearly

$$f(u) = g(u) = u \quad \text{if} \quad u < 2^N.$$

Let $m_1 = 2^N + l, m_2 = 2^N + r, 0 < l, r < 2^N$, $l \neq r$. We have

$$g(m_1) = g(2^N) + g(l) = g(2^N) + l,$$

similarly

$$g(m_2) = g(2^N) + g(r) = g(2^N) + r.$$

We can choose $m_1, m_2$ such that $\frac{m_1}{m_2} = \frac{A}{B}$, $A, B < c$. Thus $m_1 B = m_2 A$, $Bf(m_1) = Af(m_2)$, $f(m_j) = g(m_j)$, and so

$$B\Big(g(2^N) + l\Big) = A\Big(g(2^N) + r\Big).$$

Since $B(2^N + l) = A(2^N + r)$, the above relation implies that $g(2^N) = 2^N$. The proof of Theorem 3 is complete. ■

We guess that the following conjecture is true.

**Conjecture 1.** *There exist such constants $c > 0$ and $N_0$ such that if $N \geq N_0$, $N \in \mathbb{N}$, then in the interval $[2^N, 2^{N+1})$ there exist $p-1, q-1$ $(p, q \in \mathcal{P})$ for which*

$$\frac{p-1}{q-1} = \frac{A}{B}, \quad A, B \in \mathbb{N}, A, B < c.$$

**Corollary.** *Assume that Conjecture 1 is true. Let*

$$f \in \mathcal{M}^*, \quad g \in \mathcal{A}_2, \quad f(n) = g(n) = n \quad \text{if} \quad n \leq \max(c, 2^{N_0}).$$

*If*

$$f(p-1) = g(p-1) \quad \text{for every} \quad p \in \mathcal{P},$$

*then*

$$f(p-1) = p-1 \quad \text{for every} \quad p \in \mathcal{P}.$$

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