ON THE LAPLACE GENERALIZED CONVOLUTION TRANSFORM

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Abstract. Several classes of integral transforms related to Laplace generalized convolution are studied. Necessary and sufficient conditions to ensure that the transformation is unitary are obtained, and a formula for the inverse transformation is derived in this case. In the application, we obtain solutions of several classes of integro-differential equations in closed form.

1. Introduction

Laplace transform is an integral transform method which is of the form

$$\big(\mathcal{L}f\big)(y) = \int_0^\infty f(x)e^{-yx}dx,\ y > 0.$$

Here, the integral converges for functions f of the exponential order $\alpha > 0$, i.e., there exists a positive M and $x_0 \geqslant 0$ such that $|f(x)| \leqslant Me^{\alpha x}, x \geqslant x_0$. This transform finds wide applications in various terms of electrical engineering, optics, signal processing, partial different equation, integral equation, inverse problems and so on (see [3, 5, 6, 9, 10, 11, 12, 15]).

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Although, so far there have been many articles about convolution transforms (see [1, 2, 4, 7, 12, 13, 14, 17]). But generalized convolution transform related to the Laplace transform has not been studied. In this paper, we introduce several generalized convolutions related to Laplace transform. Then, we study classes of integral transforms related to these generalized convolutions, which are Laplace generalized convolution transforms.

This paper consists of four sections. The first section, reflects the content of the paper. The second section, we recall several fundamental notations used in this paper. The third section, we introduce several new generalized convolutions for the Fourier sine, Fourier cosine and the Laplace transforms; as well as together with the related integral transforms. The Watson's type theorem which gives the necessary and sufficient condition to ensure that the above integral transforms are unitary. Additionally, the Plancherel's type theorem is also obtained. In the final section, as application, we obtain solutions of several classes of integro-differential equations in closed form. The research is interested in $L_2(\mathbb{R}_+)$ space and the space of functions of exponential order $\alpha > 0$.

2. Preliminaries

The Fourier cosine and Fourier sine transforms are defined as follows (see [11, 12])

$$(2.1)\quad \big(F_c f\big)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos xy dx = \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int_0^\infty f(x) \frac{\sin xy}{x} dx,\ y > 0,$$

$$(2.2)\quad \big(F_s f\big)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin xy dx = \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int_0^\infty f(x) \frac{1-\cos xy}{x} dx,\ y > 0,$$

for $f \in L_1(\mathbb{R}_+)$. The second integrals in (2.1) and (2.2) are also well-defined for $f \in L_2(\mathbb{R}_+)$.

The Fourier sine and cosine generalized convolution of f and k is defined as in [11] by

$$(2.3)\quad (f \underset{1}{*} k)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[k(x+y) + \operatorname{sign}(x-y)k(|x-y|)]dy,\ x > 0,$$

which satisfies the following factorization and Parseval's type identities (see [1])

$$F_s\big(f \underset{1}{*} k\big)(y) = \big(F_c f\big)(y)\big(F_s k\big)(y),\ \forall y > 0,\ f, k \in L_2(\mathbb{R}_+), \tag{2.4}$$

$$\big(f \underset{1}{*} k\big)(x) = F_s\big[(F_c f)(F_s k)\big](x),\ \forall x > 0,\ f, k \in L_2(\mathbb{R}_+). \tag{2.5}$$

The Fourier cosine and sine generalized convolution of f and k is defined as in [8] by

$$(f \underset{2}{*} k)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[k(x+y) - \operatorname{sign}(x-y)k(|x-y|)]dy,\ x > 0, \tag{2.6}$$

which satisfies the following factorization and Parseval's type identities (see [2])

$$F_c\big(f \underset{2}{*} k\big)(y) = \big(F_s f\big)(y)\big(F_s k\big)(y),\ \forall y > 0,\ f, k \in L_2(\mathbb{R}_+), \tag{2.7}$$

$$\big(f \underset{2}{*} k\big)(x) = F_c\big[(F_s f)(F_s k)\big](x),\ \forall x > 0,\ f, k \in L_2(\mathbb{R}_+). \tag{2.8}$$

The Fourier cosine convolution of two functions f and k is defined as in [11] by

$$(f \underset{F_c}{*} k)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[k(x+y) + k(|x-y|)]dy, \quad x > 0,$$

which satisfies the following factorization identity

$$F_c\big(f \underset{F_c}{*} k\big)(y) = \big(F_c f\big)(y)\big(F_c k\big)(y),\ \forall y > 0, \quad f, k \in L_2(\mathbb{R}_+). \tag{2.9}$$

The convolution of two functions f and k for the Laplace transform (see [5, 10])

$$(f \underset{\mathcal{L}}{*} k)(x) = \int_0^x f(x-y)k(y)dy,\ x > 0,$$

this convolution satisfies the factorization identity

$$\mathcal{L}\big(f \underset{\mathcal{L}}{*} k\big)(y) = (\mathcal{L}f)(y)(\mathcal{L}k)(y), y > 0. \tag{2.10}$$

This factorization identity holds for all functions f and k of exponential order $\alpha > 0$. Moreover, $(f \underset{\mathcal{L}}{*} k)(x)$ of exponential order $\alpha > 0$ (see Theorem 2.39, p.92 in [10]).

3. The Laplace generalized convolution transform

In this section, we introduce several new generalized convolutions related to the Laplace transform and study classes related to integral transforms.

Definition 3.1. *The generalized convolutions with a weight function* $\gamma(y) = \sin y$ *of two functions* f, k *for the Fourier sine, Fourier cosine and Laplace transforms are defined by*

$$(f \overset{\gamma}{*} k)_{\left\{\substack{1\\2}\right\}}(x) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty \Big\{\Big[\frac{v}{v^2+(x-1-u)^2} \pm \frac{v}{(v^2+(x-1+u)^2}\Big]$$

$$- \Big[\frac{v}{v^2+(x+1-u)^2} \pm \frac{v}{v^2+(x+1+u)^2}\Big]\Big\} f(u)k(v)dudv, \tag{3.1}$$

where $x > 0$.

Theorem 3.1. *Suppose that* $f(x) \in L_2(\mathbb{R}_+)$ *and* $k(x)$ *is a function of exponential order* $\alpha > 0$. *Then, the generalized convolutions* $(f \overset{\gamma}{*} k)_{\left\{\substack{1\\2}\right\}} \in L_2(\mathbb{R}_+)$ *satisfy the Parseval's type identities*

$$(f \overset{\gamma}{*} k)_{\left\{\substack{1\\2}\right\}}(x) = F_{\left\{\substack{s\\c}\right\}}\big[\pm \sin y (F_{\left\{\substack{c\\s}\right\}} f)(\mathcal{L}k)\big](x), \ \ \forall x > 0, \tag{3.2}$$

and the following factorization identities hold

$$F_{\left\{\substack{s\\c}\right\}}(f \overset{\gamma}{*} k)_{\left\{\substack{1\\2}\right\}}(y) = \pm \sin y (F_{\left\{\substack{c\\s}\right\}} f)(y)(\mathcal{L}k)(y), \ \ \forall y > 0. \tag{3.3}$$

Proof. From (3.1) and by using the formula (2.13.5) in [5]

$$\int_0^\infty e^{-\alpha x}\cos xy dx = \frac{\alpha}{\alpha^2+y^2}, \ \ \alpha > 0,$$

we get

$$\begin{aligned}\big(f \overset{\gamma}{*} k\big)_{\left\{\substack{1\\2}\right\}}(x) =& \frac{1}{2\pi}\int_0^\infty\int_0^\infty\int_0^\infty f(u)k(v)e^{-vy}\Big\{\big[\cos(x-1-u)y \pm \cos(x-1+u)y\big]\\ &- \big[\cos(x+1-u)y \pm \cos(x+1+u)y\big]\Big\}dudvdy\\ =& \pm\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty f(u)k(v)e^{-vy}\begin{Bmatrix}\sin xy.\sin y.\cos uy\\ \cos xy.\sin y.\sin uy\end{Bmatrix}dudvdy\\ =& \pm\frac{2}{\pi}\int_0^\infty\Big[\int_0^\infty f(u)\begin{Bmatrix}\cos uy\\ \sin uy\end{Bmatrix}du.\int_0^\infty k(v)e^{-vy}dv\Big]\sin y\begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix}dy\\ =& \pm\sqrt{\frac{2}{\pi}}\int_0^\infty (F_{\left\{\substack{c\\s}\right\}}f)(y)(\mathcal{L}k)(y)\sin y\begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix}dy.\end{aligned}$$

Therefore Parseval's type identities (3.2) hold. On the other hand, $(\mathcal{L}k)(y)$ vanishes at infinity and $f \in L_2(\mathbb{R}_+)$ therefore $\sin y(F_{\left\{\substack{c\\s}\right\}}f)(y)(\mathcal{L}k)(y) \in L_2(\mathbb{R}_+)$. Combining with (3.2) we have $\big(f \overset{\gamma}{*} k\big)_{\left\{\substack{1\\2}\right\}} \in L_2(\mathbb{R}_+)$ and factorization identities (3.3) hold.

In the next part of this section, we reflect several properties of the integral transforms related to convolutions (2.3), (2.6) and generalized convolutions (3.1), namely, transformations of the form

(3.4)
$$f(x) \mapsto g(x) = \big(T_{k_1,k_2}f\big)(x) = \Big(1-\frac{d^2}{dx^2}\Big)\Big\{(f \overset{\gamma}{*} k_1)_{\left\{\substack{1\\2}\right\}}(x) + (f \underset{\left\{\substack{1\\2}\right\}}{*} k_2)(x)\Big\},$$

$$x > 0.$$

Theorem 3.2 (Watson's type theorem)**.** *Suppose that $k_1(x)$ is a function of exponential order $\alpha > 0$ and $k_2(x) \in L_2(\mathbb{R}_+)$, then necessary and sufficient conditions to ensure that the transforms* (3.4) *are unitary on $L_2(\mathbb{R}_+)$ are that*

$$\big|\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big| = \frac{1}{1+y^2}. \tag{3.5}$$

Moreover, the inverse transforms have the form

$$f(x) = \Big(1-\frac{d^2}{dx^2}\Big)\Big\{(g \overset{\gamma}{*} \overline{k_1})_{\left\{\substack{1\\2}\right\}}(x) + (g \underset{\left\{\substack{1\\2}\right\}}{*} \overline{k_2})(x)\Big\}. \tag{3.6}$$

Proof. *Necessity.* Assume that k_1 and k_2 satisfy conditions (3.5). We known that $h(y), yh(y), y^2h(y) \in L_2(\mathbb{R})$ if and only if $(Fh)(x)$, $\frac{d}{dx}(Fh)(x)$, $\frac{d^2}{dx^2}(Fh)(x) \in L_2(\mathbb{R})$ (Theorem 68, pp.92, [12]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = \frac{1}{\sqrt{2\pi}}\frac{d^2}{dx^2}\int\limits_{-\infty}^{+\infty} h(y)e^{-ixy}dy = F\big[(-iy)^2h(y)\big](x).$$

Specially, if h is an even or odd function such that $h(y), y^2h(y) \in L_2(\mathbb{R}_+)$, then the following equalities hold

$$\left(1-\frac{d^2}{dx^2}\right)\left(F_{\left\{\substack{c\\s}\right\}}h\right)(x) = F_{\left\{\substack{c\\s}\right\}}\big[(1+y^2)h(y)\big](x). \tag{3.7}$$

From conditions (3.5), therefore $\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)$ are bounded, and hence $(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big)(F_{\left\{\substack{c\\s}\right\}}f)(y) \in L_2(\mathbb{R}_+)$. Since (3.4), by using Parsevals' type properties (2.5), (2.8), (3.2), and formula (3.7), we have

$$\begin{aligned}g(x) = &\left(1-\frac{d^2}{dx^2}\right)F_{\left\{\substack{s\\c}\right\}}\Big[\pm\sin y(F_{\left\{\substack{c\\s}\right\}}f)(y)(\mathcal{L}k_1)(y) + (F_{\left\{\substack{c\\s}\right\}}f)(y)(F_sk_2)(y)\Big](x)\\ =&F_{\left\{\substack{s\\c}\right\}}\Big[(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big)(F_{\left\{\substack{c\\s}\right\}}f)(y)\Big](x).\end{aligned}$$

Therefore, the Parseval identities $||f||_{L_2(\mathbb{R}_+)} = ||F_{\left\{\substack{c\\s}\right\}}f||_{L_2(\mathbb{R}_+)}$ and conditions (3.5) give

$$\begin{aligned}||g||_{L_2(\mathbb{R}_+)} =&||(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big)(F_{\left\{\substack{c\\s}\right\}}f)(y)||_{L_2(\mathbb{R}_+)}\\ =&||(F_{\left\{\substack{c\\s}\right\}}f)(y)||_{L_2(\mathbb{R}_+)} = ||f||_{L_2(\mathbb{R}_+)}.\end{aligned}$$

It shows that the transforms (3.4) are isometric.

On the other hand, since

$$(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big)(F_{\left\{\substack{c\\s}\right\}}f)(y) \in L_2(\mathbb{R}_+),$$

we have

$$(F_{\left\{\substack{s\\c}\right\}}g)(y) = (1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y) + (F_sk_2)(y)\big)(F_{\left\{\substack{c\\s}\right\}}f)(y).$$

Using conditions (3.5), we have

$$(F_{\left\{\substack{s\\c}\right\}}f)(y) = (1+y^2)\big(\pm\sin y(\mathcal{L}\overline{k_1})(y) + (F_s\overline{k_2})(y)\big)(F_{\left\{\substack{c\\s}\right\}}g)(y).$$

Again, conditions (3.5) show that

$$(1+y^2)\big(\pm\sin y(\mathcal{L}\overline{k_1})(y)+(F_s\overline{k_2})(y)\big)(F_{\left\{ {c \atop s} \right\}}g)(y)\in L_2(\mathbb{R}_+).$$

By using (2.5), (2.8), (3.2), and formulas (3.7), we have

$$\begin{aligned} f(x) =& F_{\left\{ {s \atop c} \right\}}\Big[(1+y^2)\big(\pm\sin y(\mathcal{L}\overline{k_1})(y)+(F_s\overline{k_2})(y)\big)(F_{\left\{ {c \atop s} \right\}}g)(y)\Big] \\ =& \left(1-\frac{d^2}{dx^2}\right)F_{\left\{ {s \atop c} \right\}}\Big[\pm\sin y(F_{\left\{ {c \atop s} \right\}}g)(y)(\mathcal{L}\overline{k_1})(y)+(F_{\left\{ {c \atop s} \right\}}g)(y)(F_s\overline{k_2})(y)\Big] \\ =& \left(1-\frac{d^2}{dx^2}\right)\Big\{(g\overset{\gamma}{*}\overline{k_1})_{\left\{ {1 \atop 2} \right\}}(x)+(g\underset{\left\{ {1 \atop 2} \right\}}{*}\overline{k_2})(x)\Big\}. \end{aligned}$$

Thus, the transforms (3.4) are unitary on $L_2(\mathbb{R}_+)$ and the inverse transforms have the form (3.6).
Sufficiency. Assume that, the transforms (3.4) are unitary on $L_2(\mathbb{R}_+)$. Then the Parseval identities for Fourier cosine and sine transforms yield

$$\begin{aligned} ||g||_{L_2(\mathbb{R}_+)} =& ||(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y)+(F_sk_2)(y)\big)(F_{\left\{ {c \atop s} \right\}}f)(y)||_{L_2(\mathbb{R}_+)} \\ =& ||(F_{\left\{ {c \atop s} \right\}}f)(y)||_{L_2(\mathbb{R}_+)}=||f||_{L_2(\mathbb{R}_+)}. \end{aligned}$$

Therefore, the operators $M_\theta[f](y)=\theta(y)f(y)$, here

$$\theta(y)=(1+y^2)\big(\pm\sin y(\mathcal{L}k_1)(y)+(F_sk_2)(y)\big)$$

are unitary on $L_2(\mathbb{R}_+)$, or equivalent, the conditions (3.5) hold.

Theorem 3.3 (Plancherel's type theorem)**.** *We set*

$$\begin{aligned} \theta_{\left\{ {1 \atop 2} \right\}}(x,u,v) =& \Big[\frac{v}{v^2+(x-1-u)^2}\pm\frac{v}{(v^2+(x-1+u)^2}\Big] \\ & -\Big[\frac{v}{v^2+(x+1-u)^2}\pm\frac{v}{v^2+(x+1+u)^2}\Big], \end{aligned}$$

and suppose that $k_1(x)$ is a function of exponential order $\alpha>0$ and two times continuously differentiable, $k_2(x)\in L_2(\mathbb{R}_+)$ and two times continuously differentiable, satisfying conditions (3.5) *and*

$$\Theta_{\left\{ {1 \atop 2} \right\}}(x,u,v)=\left(1-\frac{d^2}{dx^2}\right)\theta_{\left\{ {1 \atop 2} \right\}}(x,u,v),\ K_2(x)=\left(1-\frac{d^2}{dx^2}\right)k_2(x)$$

are bounded functions. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer N, set

$$
\begin{aligned}
g_N(x) =& \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^N \Theta_{\left\{\frac{1}{2}\right\}}(x,u,v)f(u)k_1(v)dudv \\
&+ \frac{1}{\sqrt{2\pi}}\int\limits_0^N f(u)\big[K_2(x+u) \pm \mathrm{sign}(x-u)K_2(|x-u|)\big]du.
\end{aligned}
$$

Then:
1) We have $g_N \in L_2(\mathbb{R}_+)$, and if $N \to \infty$ then g_N converges in $L_2(\mathbb{R}_+)$ norm to a function $g \in L_2(\mathbb{R}_+)$ with $||g||_{L_2(\mathbb{R}_+)} = ||f||_{L_2(\mathbb{R}_+)}$.
2) Set $g^N = g.\chi(0,N)$, then

$$
\begin{aligned}
f_N(x) =& \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty \Theta_{\left\{\frac{1}{2}\right\}}(x,u,v)g^N(u)\overline{k_1}(v)dudv \\
&+ \frac{1}{\sqrt{2\pi}}\int\limits_0^\infty g^N(u)\big[\overline{K_2}(x+u) \pm \mathrm{sign}(x-y)\overline{K_2}(|x-u|)\big]du,
\end{aligned}
$$

also belong to $L_2(\mathbb{R}_+)$, and if $N \to \infty$ then f_N converges in norm to f.

Proof. Because of the definitions of f_N and g_N, these integrals are over finite intervals and therefore converge.
Set $f^N = f.\chi(0,N)$, we have

$$
\begin{aligned}
g_N(x) =& \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^N \Theta_{\left\{\frac{1}{2}\right\}}(x,u,v)f(u)k_1(v)dudv \\
&+ \frac{1}{\sqrt{2\pi}}\int\limits_0^N f(u)\big[K_2(x+u) \pm \mathrm{sign}(x-u)K_2(|x-u|)\big]du \\
=& \left(1 - \frac{d^2}{dx^2}\right)\left\{ \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty \theta_{\left\{\frac{1}{2}\right\}}(x,u,v)f^N(u)k_1(v)dudv \right. \\
&\left. + \frac{1}{\sqrt{2\pi}}\int\limits_0^\infty f^N(u)\big[k_2(x+u) \pm \mathrm{sign}(x-u)k_2(|x-u|)\big]du \right\}.
\end{aligned}
$$

In view of Watson's type theorem, we conclude that $g_N \in L_2(\mathbb{R}_+)$. Let g be the transform of f under transformations (3.4), we have $||g||_{L_2(\mathbb{R}_+)} = ||f||_{L_2(\mathbb{R}_+)}$, and the reciprocal formulas (3.6) hold. We have

$$(g-g_N)(x)=\left(1-\frac{d^2}{dx^2}\right)\left\{\frac{1}{2\pi}\int_0^\infty\int_0^\infty\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)(f-f^N)(u)k_1(v)dudv\right.$$
$$\left.+\frac{1}{\sqrt{2\pi}}\int_0^\infty(f-f^N)(u)\big[k_2(x+u)\pm\operatorname{sign}(x-u)k_2(|x-u|)\big]du\right\}.$$

By using Watson's type theorem, we get $(g-g_N)(x)\in L_2(\mathbb{R}_+)$ and

$$||g-g_N||_{L_2(\mathbb{R}_+)}=||f-f^N||_{L_2(\mathbb{R}_+)}.$$

Since $||g-g_N||_{L_2(\mathbb{R}_+)}\to 0$ as $N\to\infty$ then g_N converges in $L_2(\mathbb{R}_+)$ norm to $g\in L_2(\mathbb{R}_+)$.

Similarly, one can obtain the second part of the theorem.

The following example shows the existence k_1 and k_2 which satisfy conditions (3.5).

Example 1.

We choose $k_1(x)=i\sin x$ is a function of exponential order $\alpha>0$, by using (3.2.9) in [5], we have

$$(\mathcal{L}k_1)(y)=\frac{i}{1+y^2}. \tag{3.8}$$

We choose the function $k_2(x)\in L_2(\mathbb{R}_+)$ such that

$$(F_sk_2)(y)=\frac{\cos y}{1+y^2}. \tag{3.9}$$

By using (2.2.14) in [5], we have

$$\begin{aligned}k_2(x)=&F_s\Big[\frac{\cos y}{1+y^2}\Big]=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{\sin y(x+1)+\sin y(x-1)}{1+y^2}dy\\=&\frac{1}{2\sqrt{2\pi}}\Big[e^{-(x+1)}\overline{Ei}(x+1)-e^{(x+1)}Ei(-x-1)\\&+e^{-(x-1)}\overline{Ei}(x-1)-e^{(x-1)}Ei(-x+1)\Big]\in L_2(\mathbb{R}_+),\end{aligned}$$

here, for real nonzero values of x, the exponential integral $Ei(x)$ is defined as (see [5])

$$Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt.$$

When, since (3.8), (3.9), therefore conditions (3.5) are satisfied, i.e.,

$$\big| \pm \sin y(\mathcal{L}k_1)(y) + (F_s k_2)(y)\big| = \frac{1}{1+y^2}.$$

4. A class of integro-differential equations

Not many integro-differential equations can be solved in closed form. In this section, we consider the following integro-differential equations related to the transforms (3.4)

$$f(x) + \frac{d}{dx}\big(T_{\varphi,\psi} f\big)(x) = g(x), \ x > 0. \tag{4.1}$$

Here, $\varphi(x) = \big(\varphi_1 \underset{\mathcal{L}}{*} \varphi_2\big)(x)$, φ_1 is given function of exponential order $\alpha > 0$, $\varphi_2(x) = \big(\sin t \underset{\mathcal{L}}{*} \sin t\big)(x)$ and $\psi(x) = \big(\operatorname{sech} t \underset{1}{*} \psi_1\big)(x)$, $\psi_1(x) \in L_2(\mathbb{R}_+)$. $g(x)$ is given function in $L_2(\mathbb{R}_+)$, and $f(x)$ is unknown function.

Theorem 4.1. *Suppose the following conditions hold*

$$1 + (y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big] \neq 0, \ \forall y > 0. \tag{4.2}$$

Then equations (4.1) *have unique solutions in* $L_2(\mathbb{R}_+)$. *Moreover, the solutions can be presented in closed form as follows*

$$f(x) = g(x) - \big(q \underset{\left\{ {F_c \atop 1} \right\}}{*} g\big)(x),$$

where $q(x) \in L_2(\mathbb{R}_+)$ *is defined by*

$$(F_c q)(y) = \frac{(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}{1 + (y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}.$$

Proof. The equations (4.1) can be rewritten in the form

$$f(x) + \left(\frac{d}{dx} - \frac{d^3}{dx^3}\right)\big[(f \overset{\gamma}{*} \varphi)_{\left\{ {1 \atop 2} \right\}}(x) + \big(f \underset{\left\{ {1 \atop 2} \right\}}{*} \psi\big)(x)\big] = g(x). \tag{4.3}$$

By using Parseval's type identities (3.2), (2.5) and (2.8), we have

(4.4)

$$\begin{aligned}\left(\frac{d}{dx} - \frac{d^3}{dx^3}\right)\big(f \overset{\gamma}{*} \varphi\big)_{\left\{ {1 \atop 2} \right\}}(x) &= \left(\frac{d}{dx} - \frac{d^3}{dx^3}\right) F_{\left\{ {s \atop c} \right\}}\big[\big(\pm \sin y F_{\left\{ {c \atop s} \right\}} f\big)\big(\mathcal{L}\varphi\big)\big](x)\\ &= F_{\left\{ {c \atop s} \right\}}\big[(y+y^3)\sin y\big(F_{\left\{ {c \atop s} \right\}} f\big)\big(\mathcal{L}\varphi\big)\big](x),\end{aligned}$$

and

$$(4.5)\quad \left(\frac{d}{dx}-\frac{d^3}{dx^3}\right)\big(f \underset{\left\{\substack{1\\2}\right\}}{*} \psi\big)(x) = \left(\frac{d}{dx}-\frac{d^3}{dx^3}\right) F_{\left\{\substack{s\\c}\right\}}\big[\big(F_{\left\{\substack{c\\s}\right\}}f\big)\big(F_s\psi\big)\big](x)$$
$$= \pm F_{\left\{\substack{c\\s}\right\}}\big[(y+y^3)(F_{\left\{\substack{c\\s}\right\}}f\big)\big(F_s\psi\big)\big](x).$$

From (4.3), (4.4) and (4.5), we get

$$f(x) + F_{\left\{\substack{c\\s}\right\}}\big[(y+y^3)\sin y(F_{\left\{\substack{c\\s}\right\}}f\big)\big(\mathcal{L}\varphi\big)\big](x) \pm F_{\left\{\substack{c\\s}\right\}}\big[(y+y^3)(F_{\left\{\substack{c\\s}\right\}}f\big)\big(F_s\psi\big)\big](x)$$
$$= g(x).$$

Therefore,

$$(F_{\left\{\substack{c\\s}\right\}}f)(y) + (y+y^3)\big[\sin y\big(F_{\left\{\substack{c\\s}\right\}}f\big)(y)(\mathcal{L}\varphi)(y) \pm (F_{\left\{\substack{c\\s}\right\}}f)(y)(F_s\psi)(y)\big]$$
$$= (F_{\left\{\substack{c\\s}\right\}}g)(y),$$

or equivalent,

$$(4.6)\quad (F_{\left\{\substack{c\\s}\right\}}f)(y)\big[1+(y+y^3)\big(\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big)\big] = (F_{\left\{\substack{c\\s}\right\}}g)(y).$$

From conditions (4.2) and (4.6), we have

(4.7)

$$(F_{\left\{\substack{c\\s}\right\}}f)(y) = (F_{\left\{\substack{c\\s}\right\}}g)(y)\left[1 - \frac{(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}{1+(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}\right].$$

On the other hand, by using (1.7) in [10] and factorization identity (2.10), we have

$$(4.8)\quad \begin{aligned}(\mathcal{L}\varphi)(y) &= (\mathcal{L}\varphi_1)(y)(\mathcal{L}\varphi_2)(y)\\ &= (\mathcal{L}\varphi_1)(y)\mathcal{L}\big(\sin t\big)(y)\mathcal{L}\big(\sin t\big)(y)\\ &= \frac{1}{(1+y^2)^2}(\mathcal{L}\varphi_1)(y).\end{aligned}$$

Moreover, from formula (1.9.1) in [3]

$$F_c(\operatorname{sech} t)(y) = \sqrt{\frac{\pi}{2}}\operatorname{sech}\frac{\pi y}{2},$$

and formula (1.9.4) in [3] for $n=1$

$$\frac{\sqrt{2\pi}}{4}(1+y^2)\operatorname{sech}\frac{\pi y}{2} = F_c(\operatorname{sech}^3 t)(y),$$

combining with factorization identity (2.4), we have

$$(F_s\psi)(y) = F_c(\operatorname{sech} t)(y)(F_s\psi_1)(y) = \frac{2}{1+y^2}F_c(\operatorname{sech}^3 t)(y)(F_s\psi_1)(y). \tag{4.9}$$

From (4.8) and (4.9), we have

$$\begin{aligned}&(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]\\ &= \sin y\frac{y}{1+y^2}(\mathcal{L}\varphi_1)(y) \pm 2yF_c(\operatorname{sech}^3 t)(F_s\psi_1)(y).\end{aligned}$$

Then, since by the formula (2.13.6) in [5]

$$(F_s e^{-t})(y) = \frac{y}{1+y^2},$$

and by using partial integration, we easily prove the following formula

$$yF_c\big(\operatorname{sech}^3 t\big)(y) = -3F_s\big(\sinh t \operatorname{sech}^4 t\big)(y),$$

combining with factorization identities (3.3), (2.7), we have

$$\begin{aligned}&(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]\\ &= \sqrt{\frac{\pi}{2}}\sin y(F_s e^{-t})(y)(\mathcal{L}\varphi_1)(y) \mp 6F_s(\sinh t \operatorname{sech}^4 t)(F_s\psi_1)(y)\\ &= \sqrt{\frac{\pi}{2}}F_c(e^{-t} \overset{\gamma}{*} \varphi_1)_2(y) \mp 6F_c\big((\sinh t \operatorname{sech}^4 t) \underset{2}{*} \psi_1\big)(y)\\ &= F_c\Big[\sqrt{\frac{\pi}{2}}(e^{-t} \overset{\gamma}{*} \varphi_1)_2 \mp 6(\sinh t \operatorname{sech}^4 t) \underset{2}{*} \psi_1\Big](y) \in L_2(\mathbb{R}_+).\end{aligned} \tag{4.10}$$

From (4.10), conditions (4.2) and Wiener-Levy theorem in [16], there exists a function $q(x) \in L_2(\mathbb{R}_+)$ such that

$$(F_c q)(y) = \frac{(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}{1+(y+y^3)\big[\sin y(\mathcal{L}\varphi)(y) \pm (F_s\psi)(y)\big]}. \tag{4.11}$$

From (4.7), (4.11) and ussing factorization identities (2.4), (2.9), we have

$$\begin{aligned}(F_{\{^c_s\}}f)(y) =& (F_{\{^c_s\}}g)(y) - (F_{\{^c_s\}}g)(y)(F_c q)(y)\\ =& (F_{\{^c_s\}}g)(y) - F_{\{^c_s\}}\big(q \underset{\{^{F_c}_1\}}{*} g\big)(y).\end{aligned}$$

Therefore,

$$f(x) = g(x) - \big(q \underset{\{^{F_c}_1\}}{*} g\big)(x),\ f(x) \in L_2(\mathbb{R}_+).$$

The proof is complete.

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