THE FUNCTIONAL EQUATION $f(\mathcal{A} + \mathcal{B}) = g(\mathcal{A}) + h(\mathcal{B})$

Imre Kátai and Bui Minh Phong (Budapest, Hungary)

Dedicated to Dr. László Szili on his 60th anniversary

Communicated by László Germán

(Received May 30, 2014; accepted July 15, 2014)

Abstract. We shall investigate the functional equation f(a + b) = g(a) + h(b) for all $a \in A, b \in B$, where A and B are subsets of natural numbers satisfying some condition.

1. Introduction

Let \mathcal{P}, \mathbb{N} and \mathbb{C} be the set of primes, positive integers and complex numbers, respectively. For the sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$ we define $\mathcal{A} + \mathcal{B}, \mathcal{A} - \mathcal{B}$ as follows:

 $\mathcal{A} + \mathcal{B} := \{ a + b \mid a \in \mathcal{A}, b \in \mathcal{B} \} \text{ and } \mathcal{A} - \mathcal{B} := \{ a - b \mid a \in \mathcal{A}, b \in \mathcal{B}, a > b \}.$

In this paper we shall investigate those subsets $\mathcal{A}, \ \mathcal{B} \subseteq \mathbb{N}$ for which if the functions

 $f: \mathcal{A} + \mathcal{B} \to \mathbb{C}, \quad g: \mathcal{A} \to \mathbb{C} \quad \text{and} \quad h: \mathcal{B} \to \mathbb{C}$

https://doi.org/10.71352/ac.43.287

Key words and phrases: Multiplicative functions, additive functions, continuous homomorphism.

²⁰¹⁰ Mathematics Subject Classification: 11A07, 11A25, 11N25, 11N64.

This work was completed with the support of the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645).

satisfy the condition

$$f(a+b) = g(a) + h(b)$$
 for all $a \in \mathcal{A}, b \in \mathcal{B}$,

then there is a complex number A such that

$$f(a+b) - A(a+b) = O(1), g(a) - Aa = O(1)$$
 and $h(b) - Ab = O(1)$

for all $a \in \mathcal{A}, b \in \mathcal{B}$.

Many results concerning this topics are known for multiplicative functions, for example C. Spiro proved that if f is a multiplicative, $f(p_0) \neq 0$ for some prime p_0 and f(p+q) = f(p) + f(q) for all primes p,q, then f(n) = n for all $n \in \mathbb{N}$. For some similar results, we refer the works of P.V. Chung [2], K.K. Chen and Y.G. Chen [3], P.V. Chung and B.M. Phong [4], J-H. Fang [5], K-H. Indlekofer and B.M. Phong [6], J.M. De Koninck, I. Kátai and B.M. Phong [8], B.M. Phong [13]–[18], C. Spiro [19].

Bojan Basic [1] investigated a function $f : \mathbb{N} \to \mathbb{C}$ such that $f(n^2 + m^2) = f(n)^2 + f(m)^2$ is satisfied for all n, m. Let

$$\mathfrak{M} := \{ p_1 + p_2 + p_3 \mid p_1, p_2, p_3 \in \mathcal{P} \}.$$

Recently, by using result of H. A. Helfgott concerning the ternary Goldbach conjecture, I. Kátai and B. M. Phong [7] proved that if the function $f : \mathfrak{M} \to \mathbb{C}$, $g : \mathcal{P} \to \mathbb{C}$ satisfy the condition

$$f(p_1 + p_2 + p_3) = g(p_1) + g(p_2) + g(p_3)$$

for every $p_1, p_2, p_3 \in \mathcal{P}$, then there exist suitable constants $A, B \in \mathbb{C}$ such that

$$f(n) = An + 3B$$
 and $g(p) = Ap + B$ for all $n \in \mathfrak{M}, p \in \mathcal{P}$.

Our purpose of this paper is to prove the following result:

Theorem 1. Assume that the sets

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A} + \mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ and $h : \mathcal{B} \to \mathbb{C}$ satisfy the equation

$$f(a+n^2) = g(a) + h(n^2)$$
 for all $a \in \mathcal{A}, n \in \mathbb{N}$.

If

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A},$$

then there is a complex number A such that

$$g(a) = Aa + \widetilde{g}(a), \ h(n^2) = An^2 + \widetilde{h}(n) \quad and \quad f(a+n^2) = A(a+n^2) + \widetilde{g}(a) + \widetilde{h}(n)$$

hold for all $a \in \mathcal{A}, n \in \mathbb{N}$, furthermore

$$\widetilde{g}(a) = \widetilde{g}(b) \quad if \quad a \equiv b \pmod{120}, \quad (a, b \in \mathcal{A})$$

and

$$h(n) = h(m)$$
 if $n \equiv m \pmod{60}$, $(n, m \in \mathbb{N})$.

Maillet [10] formulated in 1905 the following

Conjecture 1. Every even number is the difference of two primes.

J. Pintz [4] proved that a positive proportion of even numbers in an interval of type $[x, x + (\log x)^C]$ can be written as the difference of two primes if $C > C_0$ and $x > x_0$.

Conjecture 1 is clearly weaker than

Conjecture 2. (Kronecker [9], 1901) Every even number can be expressed in infinitely many ways as the difference of two primes,

or

Conjecture 3. (Polignac [12], 1912) Every even number can be written in infinitely many ways as the difference of two consecutive primes.

A weaker form of Conjecture 1 is

Conjecture 4. Every positive number of the form 8n is the difference of two primes.

We obtain from Theorem 1 the following

Theorem 2. Assume that the sets

$$\mathcal{A} = \mathcal{P}, \quad \mathcal{B} := \{ m^2 \mid m \in \mathbb{N} \}$$

and the arithmetical functions $F : \mathcal{P} + \mathcal{B} \to \mathbb{C}$, $G : \mathcal{P} \to \mathbb{C}$ and $H : \mathcal{B} \to \mathbb{C}$ satisfy the equation

$$F(p+n^2) = G(p) + H(n^2)$$
 for all $p \in \mathcal{P}, n \in \mathbb{N}$.

For each odd prime $p \text{ let } \overline{p} \in \{1,3\} \text{ such that } p \equiv \overline{p} \pmod{4}$.

If Conjecture 4 holds, then there are complex numbers A, A_2, D such that

$$G(p) = Ap + G(\bar{p}) - A\bar{p}, G(2) = A + G(1) + A_2,$$
$$H(n^2) = An^2 + A_2\chi_2(n) + D,$$
$$F(p + n^2) = A(p + n^2) + G(\bar{p}) - A\bar{p} + A_2\chi_2(n) + D$$

for all $p \in \mathcal{P} \setminus \{2\}$, $n \in \mathbb{N}$, where $\chi_2(n)$ is the Dirichlet character (mod 2), that is $\chi_2(0) = 0, \chi_2(1) = 1$.

2. Lemmata

In this section we denote by \mathcal{A}, \mathcal{B} the following sets:

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N} \text{ and } \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}.$$

We consider those arithmetical functions

$$U: \mathcal{A} \to \mathbb{C} \quad \text{and} \quad V: \mathcal{B} \to \mathbb{C},$$

for which

(1)
$$U(a) + V(n^2) = U(b) + V(m^2)$$
 if $a + n^2 = b + m^2$, $a, b \in \mathcal{A}, n, m \in \mathbb{N}$.
Let $S_m := V(m^2) \quad (m \in \mathbb{N}).$

•

Lemma 1. Assume that the arithmetical functions U, V satisfy (1). If

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A}$$

is satisfied, then

(3)
$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n$$

holds for all $n \in \mathbb{N}$ and

(4)
$$\begin{cases} S_7 = 2S_5 - S_1 \\ S_8 = 2S_5 + S_4 - 2S_1 \\ S_9 = S_6 + 2S_5 - S_2 - S_1 \\ S_{10} = S_6 + 3S_5 - S_3 - 2S_1 \\ S_{11} = S_6 + 4S_5 - S_3 - S_2 - 2S_1 \\ S_{12} = S_6 + 4S_5 + S_4 - S_2 - 4S_1 \end{cases}$$

Proof. The proof of this result is similar to the proof of Lemma 1 in [18]. First we infer from (1) and (2) that for each $m \in \mathbb{N}, m > 7$ we have

(5)
$$S_{2m+1} - S_{2m-1} = S_{m+2} - S_{m-2}$$
 and $S_{2m+1} - S_{2m-5} = S_{m+5} - S_{m-7}$

Indeed, for $m \in \mathbb{N}, m > 7$ we have

$$\begin{cases} 8m &= (2m+1)^2 - (2m-1)^2 = (m+2)^2 - (m-2)^2 \\ 24m &= (2m+1)^2 - (2m-5)^2 = (m+5)^2 - (m-7)^2, \end{cases}$$

and so (2) implies that there are $x,x',y,y'\in \mathcal{A}, x>x',y>y'$ such that

$$x - x' = (2m + 1)^2 - (2m - 1)^2 = (m + 2)^2 - (m - 2)^2,$$

and

$$y - y' = (2m + 1)^2 - (2m - 5)^2 = (m + 5)^2 - (m - 7)^2.$$

These with (1) imply that

$$U(x) - U(x') = S_{2m+1} - S_{2m-1} = S_{m+2} - S_{m-2}$$

and

$$U(y) - U(y') = S_{2m+1} - S_{2m-5} = S_{m+5} - S_{m-7}.$$

Hence (5) is proved.

Finally, we infer from (5) that

$$S_{m+5} - S_{m-7} = S_{2m+1} - S_{2m-5} =$$

= $\left(S_{2m+1} - S_{2m-1}\right) + \left(S_{2m-1} - S_{2m-3}\right) + \left(S_{2m-3} - S_{2m-5}\right) =$
= $\left(S_{m+2} - S_{m-2}\right) + \left(S_{m+1} - S_{m-3}\right) + \left(S_m - S_{m-4}\right).$

This with m := n + 7 proves that (3) holds for all $n \in \mathbb{N}$.

Now we prove (4). Indeed, by using (5), we have

$$S_7 = S_{2.3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2.4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2.5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Since the following three numbers

$$8^2 - 4^2 = 7^2 - 1^2$$
, $10^2 - 2^2 = 11^2 - 5^2$, $12^2 - 8^2 = 9^2 - 1^2$

are in $\mathcal{A} - \mathcal{A}$, therefore from (1) and (2) we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1.$$

which completes the proof (4).

Lemma 1 is proved.

Lemma 2. Assume that the arithmetical functions U, V satisfy (1). Let

(6)
$$\begin{cases} A := \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1), \\ A_2 := \frac{-1}{8}(S_6 - 4S_5 + 4S_4 - S_3 - 3S_1), \\ A_3 := \frac{-1}{3}(S_6 - 2S_5 + 2S_3 - S_2), \\ A_4 := \frac{1}{4}(S_6 - 2S_4 - S_3 + S_2 + S_1), \\ A_5 := \frac{1}{5}(S_6 - S_5 - S_3 - S_2 + 2S_1), \\ D := \frac{1}{4}(S_6 - 4S_5 + 2S_4 + 3S_3 + S_2 + S_1), \\ B_k := A_2\chi_2(k) + A_3\chi_3(k) + A_4\chi_4(k-1) + A_5\chi_5(k) + D, \end{cases}$$

where $\chi_2(k) \pmod{2}$, $\chi_3(k) \pmod{3}$ are the principal Dirichlet characters and $\chi_4(k) \pmod{4}$, $\chi_5(k) \pmod{5}$ are the real, non-principal Dirichlet characters, *i.e.*

$$\chi_2(0) = 0, \chi_2(1) = 1, \chi_3(0) = 0, \chi_3(1) = \chi_3(2) = 1,$$

$$\chi_4(0) = \chi_4(2) = 0, \chi_4(1) = 1, \chi_4(3) = -1,$$

$$\chi_5(0) = 0, \chi_5(2) = \chi_5(3) = -1, \chi_5(1) = \chi_5(4) = 1.$$

If (2) holds, then

(7)
$$S_k = Ak^2 + B_k \quad for \ all \quad k \in \mathbb{N}.$$

Proof. By using (3) and (4), the proof Lemma 2 is similar to the proof of Lemma 2 in [18]. For the sake of completeness, we give the proof here.

From the definition of B_k , using Maple program, we may compute the values of B_k for $k = 1, 2, \dots, 12$. We have

$$B_1 = -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1,$$

$$\begin{split} B_2 &= -\frac{1}{30}S_6 - \frac{2}{15}S_5 + \frac{1}{30}S_3 + \frac{31}{30}S_2 + \frac{1}{10}S_1, \\ B_3 &= -\frac{3}{30}S_6 - \frac{3}{10}S_5 + \frac{43}{40}S_3 + \frac{3}{40}S_2 + \frac{9}{40}S_1, \\ B_4 &= -\frac{2}{15}S_6 - \frac{8}{15}S_5 + S_4 + \frac{2}{15}S_3 + \frac{2}{15}S_2 + \frac{2}{5}S_1, \\ B_5 &= -\frac{5}{24}S_6 + \frac{1}{6}S_5 + \frac{5}{24}S_3 + \frac{5}{24}S_2 + \frac{5}{8}S_1, \\ B_6 &= \frac{7}{10}S_6 - \frac{6}{5}S_5 + \frac{3}{10}S_3 + \frac{3}{10}S_2 + \frac{9}{10}S_1, \\ B_7 &= -\frac{49}{120}S_6 + \frac{11}{30}S_5 + \frac{49}{120}S_3 + \frac{49}{120}S_2 + \frac{9}{40}S_1, \\ B_8 &= -\frac{8}{15}S_6 - \frac{2}{15}S_5 + S_4 + \frac{8}{15}S_3 + \frac{8}{15}S_2 - \frac{2}{5}S_1, \\ B_9 &= \frac{13}{40}S_6 - \frac{7}{10}S_5 + \frac{27}{40}S_3 - \frac{13}{40}S_2 + \frac{41}{40}S_1, \\ B_{10} &= \frac{1}{6}S_6 - \frac{1}{3}S_5 - \frac{1}{6}S_3 + \frac{5}{6}S_2 + \frac{1}{2}S_1, \\ B_{11} &= -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1, \\ B_{12} &= -\frac{1}{5}S_6 - \frac{4}{5}S_5 + S_4 + \frac{6}{5}S_3 + \frac{1}{5}S_2 - \frac{2}{5}S_1. \\ Consequently, we obtain from (4) and $A = \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1)$ that $A \cdot k^2 + B_k = S_k$ for all $1 \le k \le 6, \\ A \cdot 7^2 + B_7 = 2S_5 - S_1 = S_7, \\ A \cdot 8^2 + B_8 = 2S_5 + S_4 - 2S_1 = S_8, \\ A \cdot 9^2 + B_9 = S_6 + 2S_5 - S_2 - S_1 = S_9, \\ A \cdot 10^2 + B_{10} = S_6 + 3S_5 - S_3 - 2S_1 = S_{10}, \\ A \cdot 11^2 + B_{11} = S_6 + 4S_5 - S_3 - S_2 - 2S_1 = S_{11}, \\ \end{split}$$$

 $A \cdot 12^2 + B_{12} = S_6 + 4S_5 + S_4 - S_2 - 4S_1 = S_{12}.$

Therefore, we proved that (7) holds for $1 \le k \le 12$.

Assume that $Ak^2 + B_k = S_k$ holds for $n \le k \le n + 11$, where $n \ge 1$. Then we deduce from (4) that

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n =$$

= $A \Big[(n+9)^2 + (n+8)^2 + (n+7)^2 - (n+5)^2 - (n+4)^2 - (n+3)^2 + n^2 \Big] +$
+ $\Big[B_{n+9} + B_{n+8} + B_{n+7} - B_{n+5} - B_{n+4} - B_{n+3} + B_n \Big] =$
= $A (n+12)^2 + B_{n+12},$

which proves that (7) holds for n + 12 and so it is true for all n. In the last relation, by using (6), we have

$$\begin{split} B_{n+9} + B_{n+8} + B_{n+7} - B_{n+5} - B_{n+4} - B_{n+3} + B_n &= \\ &= A_2 \Big[\sum_{k=n+6}^{n+9} \chi_2(k) - \sum_{k=n+3}^{n+6} \chi_2(k) + \chi_2(n) \Big] + \\ &+ A_3 \Big[\sum_{k=n+7}^{n+9} \chi_3(k) - \sum_{k=n+3}^{n+5} \chi_3(k) + \chi_3(n) \Big] + \\ &+ A_4 \Big[\sum_{k=n+6}^{n+9} \chi_4(k-1) - \sum_{k=n+3}^{n+6} \chi_4(k-1) + \chi_4(n-1) \Big] + \\ &+ A_5 \Big[\sum_{k=n+6}^{n+10} \chi_5(k) - \sum_{k=n+2}^{n+6} \chi_5(k) - \chi_5(n+10) + \chi_5(n+2) + \chi_5(n) \Big] + D = \\ &= A_2 \chi_2(n) + A_3 \chi_3(n) + A_4 \chi_4(n-1) + A_5 \chi_5(n+2) + D = \\ &= A_2 \chi_2(n+12) + A_3 \chi_3(n+12) + A_4 \chi_4(n+11) + A_5 \chi_5(n+12) + D = B_{n+12}. \\ \text{Lemma 2 is proved.} \end{split}$$

Lemma 3. Let

$$T(k) := B_{k+1} - B_{k-1},$$

where

$$B_m = A_2\chi_2(m) + A_3\chi_3(m) + A_4\chi_4(m-1) + A_5\chi_5(m) + D$$

and A_2, A_3, A_4, A_5, D are defined in (6) of Lemma 2. Then

(8)
$$T(k) = A_3 \chi_3^*(k) + 2A_4 \chi_4(k) + A_5 E_5(k),$$

where $\chi_3^*(k)$ and $\chi_4(k)$ are the real, non-principal Dirichlet characters, i.e.

$$\chi_3^*(0) = 0, \quad \chi_3^*(1) = 1, \quad \chi_3^*(2) = -1,$$

$$\chi_4(0) = \chi_4(2) = 0, \quad \chi_4(1) = 1, \quad \chi_4(3) = -1$$

and

$$E_5(k) := r \in \{0, \pm 1, \pm 2\} \quad if \quad k + r \equiv 0 \pmod{5}.$$

Proof. It is easy to check that

$$\chi_2(k+1) - \chi_2(k-1) = 0, \quad \chi_3(k+1) - \chi_3(k-1) = \chi_3^*(k)$$

and

$$\chi_4(k) - \chi_4(k-2) = 2\chi_4(k)$$

hold for all $k \in \mathbb{N}$. Finally, we infer from the definition of χ_5 that

$$\chi_5(k+1) - \chi_5(k-1) = \begin{cases} 1-1=0 & \text{if } k \equiv 0 \pmod{5} \\ -1-0=-1 & \text{if } k \equiv 1 \pmod{5} \\ -1-1=-2 & \text{if } k \equiv 2 \pmod{5} \\ 1-(-1)=2 & \text{if } k \equiv 3 \pmod{5} \\ 0-(-1)=1 & \text{if } k \equiv 4 \pmod{5}, \end{cases}$$

consequently $\chi_5(k+1) - \chi_5(k-1) = E_5(k)$. Thus, we have

$$T(k) = B_{k+1} - B_{k-1} =$$

$$= A_2 \Big(\chi_2(k+1) - \chi_2(k-1) \Big) + A_3 \Big(\chi_3(k+1) - \chi_3(k-1) \Big) +$$

$$+ A_4 \Big(\chi_4(k) - \chi_4(k-2) \Big) + A_5 \Big(\chi_5(k+1) - \chi_5(k-1) \Big) =$$

$$= A_3 \chi_3^*(k) + 2A_4 \chi_4(k) + A_5 E_5(k).$$

Therefore, (8) and Lemma 3 is proved.

In the following for each $a \in \mathcal{A}$ we denote by \overline{a} the smallest element of \mathcal{A} for which $a \equiv \overline{a} \pmod{4}$.

Lemma 4. Assume that the arithmetical functions U, V satisfy (1). If (2) holds, then

(9)
$$U(a) - Aa = U(\overline{a}) - A\overline{a} + T\left(\frac{a - \overline{a}}{4}\right) \text{ for all } a \in \mathcal{A},$$

where T(k) is defined in Lemma 3. Consequently

(10)
$$U(a) - Aa = U(r) - Ar \quad if \quad a, r \in \mathcal{A}, \quad a \equiv r \pmod{120}.$$

Proof. For each $a \in \mathcal{A}$, we have

$$a + \left(\frac{a - \overline{a}}{4} - 1\right)^2 = \overline{a} + \left(\frac{a - \overline{a}}{4} + 1\right)^2,$$

which with (1) implies that

$$U(a) = U(\overline{a}) + S_{\frac{a-\overline{a}}{4}+1} - S_{\frac{a-\overline{a}}{4}-1}$$

From (7) we have

$$S_{\frac{a-\overline{a}}{4}+1} - S_{\frac{a-\overline{a}}{4}-1} = A\left[\left(\frac{a-\overline{a}}{4}+1\right)^2 - \left(\frac{a-\overline{a}}{4}-1\right)^2\right] + B_{\frac{a-\overline{a}}{4}+1} - B_{\frac{a-\overline{a}}{4}-1} = Aa - A\overline{a} + T\left(\frac{a-\overline{a}}{4}\right),$$

which proves (9).

The proof of (10) is clear. Indeed, if $a \equiv r \pmod{120}$, then $\frac{a-r}{4} \equiv 0 \pmod{30}$, and so

$$T\left(\frac{a-r}{4}\right) = A_3\chi_3^*\left(\frac{a-r}{4}\right) + 2A_4\chi_4\left(\frac{a-r}{4}\right) + A_5E_5\left(\frac{a-r}{4}\right) = 0.$$

Thus, (10) and Lemma 4 is proved.

3. The proof of Theorem 1.

Assume that the sets

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A} + \mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ and $h : \mathcal{B} \to \mathbb{C}$ satisfy the equation

$$f(a+n^2) = g(a) + h(n^2)$$
 for all $a \in \mathcal{A}, n \in \mathbb{N}$,

furthermore we assume that

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A}.$$

It is obvious that (1) is true for U = g and V = h. In the notations of Lemma 2 for U = g, V = h and $S_n = h(n^2)$ $(n \in \mathbb{N})$, we put

$$\widetilde{g}(a) := g(a) - Aa$$
 and $\widetilde{h}(n) := h(n^2) - An^2$ for all $a \in \mathcal{A}, n \in \mathbb{N}$.

Then it follows from Lemma 3 and Lemma 4 that

$$\widetilde{h}(n) = A_2\chi_2(n) + A_3\chi_3(n) + A_4\chi_4(n-1) + A_5\chi_5(n) + D,$$
$$\widetilde{g}(a) = \widetilde{g}(\overline{a}) + T\left(\frac{a-\overline{a}}{4}\right)$$

and

$$f(a+n^2) = A(a+n^2) + \widetilde{g}(a) + \widetilde{h}(n)$$

hold for all $a \in \mathcal{A}, n \in \mathbb{N}$, furthermore

$$\widetilde{g}(a) = \widetilde{g}(r)$$
 if $a, r \in \mathcal{A}$ and $a \equiv r \pmod{120}$.

Thus the proof of Theorem 1 is completes.

4. The proof of Theorem 2.

Assume that the sets

$$\mathcal{A} = \mathcal{P}, \quad \mathcal{B} := \{ m^2 \mid m \in \mathbb{N} \}$$

and the arithmetical functions $F : \mathcal{P} + \mathcal{B} \to \mathbb{C}$, $G : \mathcal{P} \to \mathbb{C}$ and $H : \mathcal{B} \to \mathbb{C}$ satisfy the equation

(11)
$$F(p+n^2) = G(p) + H(n^2) \text{ for all } p \in \mathcal{P}, n \in \mathbb{N}.$$

Let

$$S_m := H(m^2)$$
 and $S := G(3) - G(2).$

First we prove that

(12)
$$\begin{cases} S_3 = 2S_2 - S_1 + 2S, \\ S_4 = 4S_2 - 3S_1 + 3S, \\ S_5 = 6S_2 - 5S_1 + 6S, \\ S_6 = 9S_2 - 8S_1 + 8S, \end{cases}$$

It follows from (11) that

(13)
$$G(p) - G(q) = S_m - S_n$$
 if $p - q = m^2 - n^2$

where $p, q \in \mathcal{P}$ and $n, m \in \mathbb{N}$.

From (13) we have $G(7) = S_3 - S_2 + G(2)$, $G(11) = S_3 - S_1 + G(3)$ and $G(19) = S_3 - S_1 + G(11) = 2S_3 - 2S_1 + G(3)$. Therefore, the relation $G(7) + S_4 = G(19) + S_2$ implies that

(14)
$$S_4 - S_3 - 2S_2 + 2S_1 - G(3) + G(2) = 0.$$

On the other hand, we also get from (13) that

(15)
$$S_5 = S_4 + G(11) - G(2) = S_4 + S_3 - S_1 + G(3) - G(2)$$

and so

(16)
$$S_5 + G(3) - (G(19) + S_3) = S_4 - 2S_3 + S_1 + G(3) - G(2) = 0.$$

Then (14) and (16) imply that

$$S_3 = 2S_2 - S_1 + 2G(3) - 2G(2) = 2S_2 - S_1 + 2S_2$$

and

$$S_4 = 4S_2 - 3S_1 + 3G(3) - 3G(2) = 4S_2 - 3S_1 + 3S,$$

which prove (12) for m = 3, 4. Finally, we infer from (15) that

$$S_5 = S_4 + S_3 - S_1 + G(3) - G(2) = 6S_2 - 5S_1 + 6G(3) - 6G(2) = 6S_2 - 5S_1 + 6S_2 - 5S_1 + 6S_2 - 5S_1 + 6S_2 - 5S_1 - 6G(2) = 6S_2 - 5S_1 - 6G(2) = 6S_2 - 5S_1 - 6S_1 - 5S_1 - 6S_1 - 5S_1 -$$

which proves (12) for m = 5.

Since $G(13) = S_3 - S_1 + G(5) = 3S_2 - 3S_1 + 2G(3) - G(2)$, we have

$$S_6 = G(13) + S_5 - G(2) = 9S_2 - 8S_1 + 8S_5,$$

which proves (12) for m = 6.

Assume that Conjecture 4 holds, then the condition (2) is true, i.e.

$$8\mathbb{N}\subseteq\mathcal{P}-\mathcal{P}.$$

Thus we infer from (6) and (12) that $A_3 = A_4 = A_5 = 0$ and

$$A = \frac{1}{4} \Big(S_2 - S_1 + S \Big), \quad A_2 = \frac{-1}{4} \Big(S_2 - S_1 - 3S \Big), D = S_1 - S,$$

which with Lemma 2 implies that

(17)
$$H(n^2) = An^2 + A_2\chi_2(n) + D \quad \text{for all} \quad n \in \mathbb{N}.$$

Finally, by using the fact $A_3 = A_4 = A_5 = 0$, we infer from the relations (9) and (10) of Lemma 4 that

(18)
$$G(p) - Ap = G(\overline{p}) - A\overline{p} \text{ for all } p \in \mathcal{P} \setminus \{2\},$$

where $\overline{p} \in \{3, 5\}$ such that $p \equiv \overline{p} \pmod{4}$.

For p = 2, we infer from (13), (17) and (18) that

$$G(2) = G(5) + H(1) - H(2^2) =$$

= $\left[5A + G(1) - A\right] + \left[A(1-4) + A_2\right] = A + G(1) + A_2$

The proof of Theorem 2 is completed.

5. Remarks and corollaries

1. Theorem 1 and Theorem 2 remain valid if f, g, h and F, G, H maps into an arbitrary Abelian group.

2. Let \mathbb{N}_k be the set of those integers n for which the number of prime power divisors of $n \ (= \Omega(n))$ is at most k. By using the method of Chen one can deduce that $\mathbb{N}_k - \mathbb{N}_k$ contains every even numbers if $k \ge 2$. Thus Theorem 1 is true for $\mathcal{A} = \mathbb{N}_k$, if $k \ge 2$.

3. Similar results are true if in Theorem 1 and Theorem 2 the set \mathcal{B} is of the form $\{n^2 + M \mid n \in \mathbb{N}\}$, where $M \ge 0$ is a given integer.

4. Now we infer from Theorem 1 the following result of Bojan Basic [1]

Corollary. Assume that $f : \mathbb{N} \to \mathbb{C}$ satisfies the relation

(19)
$$f(n^2 + m^2) = f(n)^2 + f(m)^2 \text{ for all } n, m \in \mathbb{N}.$$

Then one of the following cases holds:

- (a) f(n) = 0 for all $n \in \mathbb{N}$,
- (b) $f(n)^2 = \frac{1}{4}$ for all $n \in \mathbb{N}$,
- (c) $f(n)^2 = n^2$ for all $n \in \mathbb{N}$.

Proof. Let $\mathcal{A} = \{n^2 \mid n \in \mathbb{N}\}$. It is obvious that the condition (2) is satisfied. Let $S_k := f(k)^2$. From $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, $8 = 2^2 + 2^2$, $10 = 1^2 + 3^2$ and $13 = 2^2 + 3^2$ we get from (19) that

$$S_2 = 4S_1^2$$
, $S_5 = (S_1 + S_2)^2$, $S_8 = 4S_2^2$, $S_{10} = (S_1 + S_3)^2$ and $S_{13} = (S_2 + S_3)^2$

Using Lemma 2, we obtain from $S_{13} = (S_2 + S_3)^2$ the equation

$$S_1(4S_1 - 1)(4S_1^2 + 4S_1 + 1 - S_3) = 0.$$

First assume that $S_1 = 0$. Then it follows from the fact $S_{20} = (S_2 + S_4)^2$ that $S_3 = 0$, consequently f(n) = 0 for all $n \in \mathbb{N}$.

If $S_1 = \frac{1}{4}$, then we get from the relations $S_{17} = (S_1 + S_4)^2$ and $S_{18} = 4S_3^2$ the following system of equations:

$$\begin{cases} (4S_3 + 3)(4S_3 - 1) &= 0\\ (4S_3 - 9)(4S_3 - 1) &= 0. \end{cases}$$

This implies that $S_3 = \frac{1}{4}$, which with Lemma 2 shows that A = 0 and $A_2 = A_3 = A_4 = A_5 = 0$ and $D = \frac{1}{4}$. Thus we have $f(n)^2 = \frac{1}{4}$ for all $n \in \mathbb{N}$.

Finally, assume that $S_3 = 4S_1^2 + 4S_1 + 1$, then we get from $S_{17} = (S_1 + S_4)^2$ and $S_{20} = (S_2 + S_4)^2$ that $S_1 = 1$ and $S_3 = 9$. In this case we have A = 1 and $A_2 = A_3 = A_4 = A_5 = D = 0$, and so $f(n)^2 = n^2$ for all $n \in \mathbb{N}$.

References

- Bojan Basic, Characterization of arithmetic functions that preserve the sum-of-squares operation, *Acta Mathematica Sinica*, English Series Apr., 2014, Vol. 30, No. 4, 689–695.
- [2] Chung, P.V., Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, Math. Slovaca, 46 (1996), 165–171.
- [3] Chen, K.K. and Y.G. Chen, On f(p) + f(q) = f(p+q) for all odd primes p and q, Publ. Math. Debrecen, 76 (2010), 425–430.
- [4] Chung, P.V. and B.M. Phong, Additive uniqueness sets for multiplicative functions, *Publ. Math. Debrecen*, 55 (1999), no. 3-4, 237–243.
- [5] Fang, J-H., A characterisation of the identity function with equation f(p+q+r) = f(p) + f(q) + f(r), Combinatorica, **31** (2011), 697–701.
- [6] Indlekofer, K-H. and B.M. Phong, Additive uniqueness set for multiplicative functions, Annales Univ. Sci. Budapest. Sect. Comp., 26 (2006), 65–77.
- [7] Kátai, I. and B.M. Phong, A consequence of the ternary Goldbach theorem, (Submitted to Publ. Math, Debrecen)
- [8] De Koninck, J.-M., I. Kátai and B.M. Phong, A new characteristic of the identity function, *Journal of Number Theory*, 63 (1997), 325–338.
- [9] Kronecker, L., Vorlesungen über Zahlentheorie, I., p. 68, Teubner, Leipzig, 1901.
- [10] **Maillet, E.,** L'intermédiaire des math. 12 (1905), p. 108.

- [11] **Pintz**, **J**., On the difference of primes, arxiv.orgpdf1206.0149v1.pdf
- [12] Polignac, A. de, Recherches nouvelles sur les nombres premiers, Comptes Rendus Acad. Sci. Paris, 29 (1849), 397–401, Rectification: ibid. pp. 738–739.
- [13] Phong, B.M., A characterization of the identity function, Acta Acad. Paed. Agriensis, Sect. Matematicae, Eger (1997), 1–9.
- [14] Phong, B.M., On sets characterizing the identity function, Annales Univ. Sci. Budapest. Sect. Comp., 24 (2004), 295-306.
- [15] Phong, B.M., A characterization of the identity function with the equation of Hosszú type, Publ. Math. Debrecen, 69 (2006), no. 1-2, 219–226.
- [16] Phong, B.M., A characterization of the identity function with functional equations, Annales Univ. Sci. Budapest. Sect. Comp., 32 (2010), 247–252.
- [17] Phong, B.M., A characterization of the identity with functional equations II., (Submitted to J. Number Theory).
- [18] **Phong, B.M.**, Additive uniqueness sets for a pair of multiplicative functions,(Submitted to *Mathematics of computation*).
- [19] Spiro, C., Additive uniqueness sets for arithmetic functions, J. Number Theory, 42 (1992), 232–246.

I. Kátai and B.M. Phong

Department of Computer Algebra Faculty of Informatics Eötvös Loránd University H-1117 Budapest, Pázmány Péter sétány 1/C Hungary katai@compalg.inf.elte.hu bui@compalg.inf.elte.hu