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# RESEARCH PROBLEMS IN NUMBER THEORY

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**Abstract.** We formulate some open problems, conjectures in the field of arithmetic functions.

## 1. Notation

Let $\mathcal{P}$ = set of primes; $\mathbb{N}$ = set of positive integers; $\mathbb{Z}$ = set of integers; $\mathbb{Q}$ = set of rational numbers; $\mathbb{R}$ = set of real numbers; $\mathbb{C}$ = set of complex numbers. Let $G$ be an Abelian group, $\mathcal{A}_G$ = set of additive arithmetical functions mapping into $G$. $f:\mathbb{N}\to G$ belongs to $\mathcal{A}_G$, if $f(mn)=f(m)+f(n)$ whenever $m,n$ are coprimes. Let $\mathcal{A}_G^*$ = set of completely additive arithmetical functions, $f:\mathbb{N}\to G$. $f\in\mathcal{A}_G^*\quad(\subseteq\mathcal{A}_G)$ if $f(mn)=f(m)+f(n)$ holds for every $m$ and $n$. We shall write simply $\mathcal{A},\mathcal{A}^*$ if $G=\mathbb{R}$.

Let $\mathcal{M}$ be the set of multiplicative functions. We say that $g\in\mathcal{M}$ if $g:\mathbb{N}\to\mathbb{C}$ and $g(nm)=g(n)\cdot g(m)$ for every $n,m$ coprime pairs of integers. Let

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$\mathcal{M}^*$ be the set of completely multiplicative functions. We say that $g : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{M}^*$ if $g(nm) = g(n) \cdot g(m)$ holds for every $n, m \in \mathbb{N}$.

$p$ and $q$, with and without suffixes always denote primes. For some $x \in \mathbb{R}$ let $\{x\}$ be the fractional part of $x$, and $\| x \| = \min(\{x\}, 1 - \{x\})$.

## 2. On the iteration of multiplicative functions

Let $\vartheta$ be a completely multiplicative function taking positive integer values. We shall define a directed graph $G_\vartheta$ on the set of primes according to the following rule: if $q$ is a prime divisor of $\vartheta(p)$ then we lead an edge from $p$ to $q$. Let $E_p$ denote the set of those primes $q$ which can be reached from $p$ walking on $G_\vartheta$. Let furthermore $K$ be the set of that primes which are located on some circles.

The properties of $K$ and $E_p$ were investigated in [17], [35], [18] in the case $\vartheta(p) = p + a, \quad a \in \mathbb{N}$. It was proved that $K$ is a finite set, and that for every prime $p$ there is a $k$ such that all the prime factors of $\vartheta^{(k)}(p)$ belong to $K$. Here $\vartheta^{(k)}(n)$ is the $k$-fold iterate of $\vartheta(n)$, i.e. $\vartheta^{(0)}(n) = n, \quad \vartheta^{(k+1)}(n) = \vartheta(\vartheta^{(k)}(n))$.

**Conjecture 2.1.** *Let $\vartheta$ be a completely multiplicative function defined at prime places $p$ by $\vartheta(p) = ap + b$, where $a \geq 2, \quad 2a + b \geq 1, \quad a, b$ be integers, $(a, b) = 1$. Then*

$$(1) \quad E_p \quad \text{is a finite set for every} \quad p \in \mathcal{P},$$

$$(2) \quad K \quad \text{is a finite set.}$$

**Conjecture 2.2.** *Let $\vartheta$ be completely multiplicative, $\vartheta(p) = p^2 + 1$. Then $K$ is infinite, and there is a $q \in \mathcal{P}$ for which $E_q$ is an infinite set.*

## 3. $\log n$ as an additive function

It is clear that $c \log n$ is an additive function. Erdős proved in [12] that if an additive function $f(n)$ satisfies $f(n + 1) - f(n) \to 0 \quad (n \to \infty)$, or $f(n + 1) \geq f(n) \quad (n \in \mathbb{N})$, then $f(n) = c \log n$. In the same paper Erdős formulated the conjecture that

$$\frac{1}{x} \sum_{n \leq x} |f(n + 1) - f(n)| \to 0 \qquad (x \to \infty)$$

implies that $f(n) = c\log n$. This was proved by I. Kátai [19], and in a more general form by E. Wirsing ([38], [39]).

Iványi and Kátai proved the following result in [16].

*If $f(n)$ is a completely additive function, $N_1 < N_2 < \dots$ an infinite sequence of integers, $\varepsilon > 0$ is an arbitrary positive constant, such that*

$$f(n) \le f(n+1) \qquad \textit{when} \quad n \in [N_j, N_j + (2+\varepsilon)\sqrt{N_j}],$$

$j = 1, 2, \dots,$ *then $f(n)$ is a constant multiple of* $\log n$ *(Theorem 1 in [16]).*

Let $\Psi(N) = \exp\left(\frac{c\log N}{\log\log N}\right)$, $\quad c > 0$ an absolute constant.

They proved: *if $f$ is an additive function,*

$$f(n) \le f(n+1) \qquad \textit{in} \quad [N_j, N_j + \Psi(N_j)\sqrt{N_j}]$$

*for $j = 1, 2, \dots,$ where $N_1 < N_2 < \dots \quad (N_j \to \infty)$, then $f(n)$ is a constant multiple of* $\log n$.

Recently, it is proved in [3] that if $f(n)$ is a completely additive function, $c$, $d$ are positive integers with $c > 2d$, an infinite sequence of integers $1 < N_1 < N_2 < \dots$ and an infinite sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \dots$ satisfies the relation

$$f\big(n\big) \ \le \ f\big(n+d\big) \ \text{ if } \ n \in [N_\nu, N_\nu + c\sqrt{N_\nu}] \ \text{ and } \ n \equiv \ell_\nu \pmod d \ \ (\nu \in \mathbb{N}),$$

then there exists a constant $c$ such that $f(n) = c\log n$ for all $n \in \mathbb{N}$, $(n,d) = 1$.

**Conjecture 3.1.** *If $f(n)$ is an additive function such that*

$$f(n) \le f(n+1) \qquad \textit{for} \quad n \in [N_j, N_j + N_j^\varepsilon],$$

*where $N_j \to \infty$, and $\varepsilon > 0$ is an arbitrary constant, then $f(n) = c\log n$.*

Wirsing proved that $f \in \mathcal{A}^*$, $f(n+1) - f(n) = o(\log n) \quad (n \to \infty)$ implies that $f(n) = c\log n$ ([39]). This theorem is very deep, it is based upon the Bombieri-Vinogradov theorem. Hence we could deduce simply that, if $f, g \in \mathcal{A}^*$, $g(n+1) - f(n) = o(\log n) \ \ (n \to \infty)$, then $f(n) = g(n) = c\log n$.

In some of his papers Kátai asked for a characterization of those $f_i \in \mathcal{A} \quad (i = 1, \dots, k)$ which satisfy

$$l(n) := f_1(n+1) + f_2(n+2) + \dots + f_k(n+k) \to 0 \qquad \text{as} \quad n \to \infty. \tag{3.1}$$

**Conjecture 3.2.** *Assume that $f_1, \dots, f_k \in \mathcal{A}$, and $l(n) \to 0 \quad (n \to \infty)$. Then there exist appropriate constants $c_1, \dots, c_k$, and $v_1, \dots, v_k \in \mathcal{A}$, such that $f_i(n) = c_i \log n + v_i(n)$, where $c_1 + \dots + c_k = 0$,*

$$\sum_{i=1}^{k} v_i(n+i) = 0 \qquad (n = 0, 1, 2, \dots), \tag{3.2}$$

*and* $v_1, \dots, v_k$ *are of finite support.*

**Definition 1.** *An additive function* $f$ *is said to be of finite support, if* $f(p^\alpha) = 0 \quad (\alpha = 1, 2, \dots)$ *holds for all but finitely many primes* $p$.

The conjecture is true in the special case, when $f_i(n) = \lambda_i f_1(n)$, where $(1 =)\lambda_1, \lambda_2, \dots, \lambda_k$ are constants. This assertion has been proved by Elliott [10], and by Kátai [20]. Let $E$ be the shift operator $E$ defined over $\{a_n\}$, by $a'_n := Ea_n = a_{n+1} \quad (n = 1, 2, \dots)$. If $P(x) = \lambda_0 + \lambda_1 x + \dots + \lambda_k x^k \in \mathbb{R}[x]$, then let $(a'_n :) = P(E)a_n := \sum_{j=0}^{k} \lambda_j a_{n+j}$. The developed method was suitable to prove the following assertions:

I. *If* $P(x) \in \mathbb{R}[x], \quad f \in \mathcal{A}$, *and*

$$\frac{1}{x} \sum_{n \le x} |P(E)f(n)| \to 0,$$

*then* $f(n) = c \log n + u(n)$, *where* $P(E)u(n) = 0 \quad (n = 1, 2, \dots)$. *If* $P(1) \neq 0$, *then* $c = 0$. *Furthermore,* $u$ *is of finite support.*

II. *If* $f \in \mathcal{A}^*, P(x) \in \mathbb{R}[x]$, *and* $P(E)f(n) = o(\log n) \quad$ *as* $\quad n \to \infty$, *then* $f(n) = c \log n$.

For further generalization of these questions see the excellent book of Elliott [11].

## 4. Characterization of $n^s$ as a multiplicative function $\mathbb{N} \to \mathbb{C}$

In a series of papers [21-26] there were considered functions $f \in \mathcal{M}$ under the conditions that $\Delta f(n) = f(n+1) - f(n)$ tends to zero in some sense. There were determined all the functions $f, g \in \mathcal{M}$ for which the relation

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+k) - f(n)| < \infty \tag{4.1}$$

with some fixed $k \in \mathbb{N}$ holds. In the special case $k = 1, \quad f, g \in \mathcal{M}^*$ the relation (4.1) implies that either

$$\sum \frac{|f(n)|}{n} < \infty, \quad \sum \frac{|g(n)|}{n} < \infty,$$

or

$$f(n) = g(n) = n^{\sigma - i\tau}, \quad \sigma, \tau \in \mathbb{R}, \quad 0 \le \sigma < 1.$$

Hence it follows especially that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda(n+1) - \lambda(n)| = \infty,$$

where $\lambda(n)$ is the Liouville function, i.e. $\lambda \in \mathcal{M}^*$, $\lambda(p) = -1$ for every $p \in \mathcal{P}$.

In [13-15] the following assertion has been proved: *if* $f, g \in \mathcal{M}^*$ *and*

$$\sum_{n \le x} |g(n+1) - f(n)| = O(x),$$

*then either* $\sum_{n \le x} |f(n)| = O(x)$, $\quad \sum_{n \le x} |g(n)| = O(x)$, *or*

$$f(n) = g(n) = n^s, \quad 0 \le Re\ s < 1.$$

**Conjecture 4.1.** *Let* $f, g \in \mathcal{M}$, $\quad k \in \mathbb{N}$ *such that* $\liminf \frac{1}{x} \sum_{n \le x} |f(n)| > 0$, *and*

$$\frac{1}{x} \sum_{n \le x} |g(n+k) - f(n)| \to 0 \qquad (x \to \infty). \tag{4.2}$$

*Then there exist* $U, V \in \mathcal{M}$ *and* $s \in \mathbb{C}$ *with* $0 \le Re\ s < 1$ *such that* $f(n) = U(n) \cdot n^s$, $\quad g(n) = V(n) \cdot n^s$, *and*

$$V(n+k) = U(n) \qquad (n \in \mathbb{N}) \tag{4.3}$$

*holds.*

Even a complete solution of (4.3) is not trivial. (4.3) was treated and all solutions found in the papers [28], [29], [30].

Celebrating P. Erdős on his 70th anniversary in a conference in Ootacamund (India) Kátai gave a talk, proving that $f \in \mathcal{M}$, $|\Delta f(n)|(\log n)^2 = O(1)$ implies that either $f(n) = n^s$, $\quad 0 \le Re\ s < 1$, or $f(n) \to 0$, and formulated the conjecture:

**Conjecture 4.2.** *If* $f \in \mathcal{M}$, $\quad \Delta f(n) \to 0 \quad (n \to \infty)$ , *then either* $f(n) = n^s$, $\quad 0 \le Re\ s < 1$, *or* $f(n) \to 0 \quad (n \to \infty)$.

This conjecture was proved by E. Wirsing. He sent the proof to Kátai [40]. Tang Yuansheng and Shao Pintsung, being unaware of the existing proof of his conjecture, gave an independent proof. They have written a paper together with Wirsing [42]. In a paper of B. M. Phong and I. Kátai [30] they characterized all those $f, g \in \mathcal{M}$ for which $g(n+k) - f(n) \to 0 \quad (n \to \infty)$.

As an immediate consequence of the theorem of E. Wirsing the following assertion is true:

*If* $F(n) \in \mathcal{A}$, *and* $\| \Delta F(n) \| \to 0$, *then either* $\| F(n) \| \to 0$ *or* $F(n) - \tau \log n \equiv O \pmod 1$ *for every* $n$, *with suitable* $\tau \in \mathbb{R}$.

**Conjecture 4.3.** *Let* $f$ *be a completely multiplicative function,* $|f(n)| = 1 \quad (n \in \mathbb{N}), \quad \delta_f(n) = f(n+1)\overline{f}(n)$.

*Let* $\mathcal{A}_k = \{\alpha_1, \ldots, \alpha_k\}$ *be the set of limit points of* $\{\delta_f(n) | n \in \mathbb{N}\}$. *Then* $\mathcal{A}_k = S_k$, *where* $S_k$ *is the set of k'th complex units, i.e.* $S_k = \{w | w^k = 1\}$, *furthermore* $f(n) = n^{i\tau}F(n)$ *with a suitable* $\tau \in \mathbb{R}$, *and* $F(N) = S_k$, *and for every* $w \in S_k$ *there exists a sequence* $n_\nu \nearrow \infty$ *such that* $F(n_\nu + 1)\overline{F}(n_\nu) = w \quad (\nu = 1, 2, \ldots)$.

The motivation of this problem, and partial results can be read in [31], [32]. E. Wirsing [41] proved a very important result by proving that if the conditions of Conjecture 4.3 are satisfied, then $f(n) = n^{i\tau}F(n)$, and $F^l(n) = 1 \quad (n \in \mathbb{N})$ holds with a fixed $l$. He was not able to prove that $l = k$. Even he proved this theorem in the more general setting of additive functions mapping into a locally compact Abelian group.

## 5. Additive functions (mod 1)

Let $T = \mathbb{R}/\mathbb{Z}$. We say that $F \in \mathcal{A}_T$ (= set of additive functions mapping into $T$) is of finite support if $F(p^\alpha) = 0$ holds for every large prime $p$.

Let $F_0, F_1, \ldots, F_{k-1} \in \mathcal{A}_T$, and

$$L_n(F_0, \ldots, F_{k-1}) := F_0(n) + F_1(n+1) + \ldots + F_{k-1}(n+k-1). \tag{5.1}$$

**Conjecture 5.1.** *Let* $\mathcal{L}_0$ *be the space of those* $k-$*tuples* $(F_0, \ldots, F_{k-1}), \quad F_\nu \in \mathcal{A}_T \quad (\nu = 0, \ldots, k-1)$ *for which*

$$L_n(F_0, \ldots, F_{k-1}) = 0 \qquad (n \in \mathbb{N})$$

*holds. Then every* $F_\nu$ *is of finite support, and* $\mathcal{L}_0$ *is a finite dimensional* $\mathbb{Z}$ *module.*

The function $F(n) := \tau \log n \pmod 1$ can be extended to $\mathbb{R}_x$(= multiplicative group of positive reals) continuously, namely by defining $F(x) := \tau \log x \pmod 1$. We say that $\tau \log n = F(n)$ is the restriction of a continuous homomorphism from $\mathbb{R}_x$ to $\mathbb{N}$.

It is clear that if $\tau_0, \ldots, \tau_{k-1}$ are such that $\tau_0 + \ldots + \tau_{k-1} = 0$, then

$$L_n(\tau_0 \log ., \tau_1 \log, \ldots, \tau_{k-1} \log .) \to 0 \qquad (n \to \infty).$$

**Conjecture 5.2.** *If $F_\nu \in \mathcal{A}_T \quad (\nu = 0, \ldots, k-1)$,*

$$L_n(F_0, \ldots, F_{k-1}) \to 0 \quad (n \to \infty),$$

*then there exist suitable real numbers $\tau_0, \ldots, \tau_{k-1}$ such that $\tau_0 + \ldots + \tau_{k-1} = 0$, and if $H_j(n) := F_j(n) - \tau_j \log n$, then*

$$L_n(H_0, \ldots, H_{k-1}) = 0 \quad (n = 1, 2, \ldots).$$

**Remarks.**

1. *Conjecture 5.2 for $k = 1$ can be deduced from Wirsing's theorem.*
2. *Conjecture 5.1 for $k = 3$ was proved assuming that $F_\nu$ are completely additive ([27]).*
3. *Conjecture 5.1 for $k = 4, 5$ was proved assuming that $F_\nu$ are completely additive functions which are defined in the set of non-zero integers by $F_\nu(-n) := F_\nu(n) \quad (n \in \mathbb{N})$ and $F_\nu(0) := 0$ ([1], [34]).*
4. *Conjecture 5.1 for $k = 2$ has been proved by R. Styer [36].*
5. *Marijke van Rossum treated similar problems for functions defined on the set of Gaussian integers. See [37], [33].*
6. *It is proved in [2] that if an additive commutative semigroup $\mathbb{G}$ (with identity element 0) and $\mathbb{G}$-valued completely additive functions $f_0, f_1, f_2$ satisfy the relation $f_0(n) + f_1(2n+1) + f_2(n+2) = 0$ for all $n \in \mathbb{N}$, then $f_0(n) = f_1(2n+1) = f_2(n) = 0$ for all $n \in \mathbb{N}$. The same result is proved when the relation $f_0(n) + f_1(2n-1) + f_2(n+2) = 0$ holds for all $n \in \mathbb{N}$.*

Let $K$ be the closure of the set $\{L_n(F_0, \ldots, F_{k-1}) | n \in \mathbb{N}\}$.

**Conjecture 5.3.** *If $F_0, \ldots, F_{k-1} \in \mathcal{A}_T^*$ and $K$ contains an element of infinite order, then $K = T$.*

## 6. Characterizations of continuous homomorphism as elements of $\mathcal{A}_G$, where $G$ is a compact Abelian group

This topic is investigated in a series of papers written by Z. Daróczy and I. Kátai [4]-[9].

Assume in §6 that $G$ is a metrically compact Abelian group supplied with some translation invariant metric $\varrho$. An infinite sequence $\{x_n\}_{n=1}^{\infty}$ in $G$ is said to belong to $\mathcal{E}_D$, if for every convergent subsequence $x_{n_1}, x_{n_2}, \dots$ the "shifted subsequence" $x_{n_1+1}, x_{n_2+1}, \dots$ is convergent, too. Let $\mathcal{E}_\Delta$ be the set of those sequences $\{x_n\}_{n=1}^{\infty}$ for which $\Delta x_n = x_{n+1} - x_n \to 0 \quad (n \to \infty)$ holds. Then $\mathcal{E}_\Delta \subseteq \mathcal{E}_D$. We say that $f \in \mathcal{A}_G^*$ belongs to $\mathcal{A}_G^*(\Delta)$ (resp. $\mathcal{A}_G^*(\mathcal{D})$) if the sequence $\{f(n)\}_{n=1}^{\infty}$ belongs to $\mathcal{E}_\Delta$ (resp. $\mathcal{E}_D$).

The following results are proved:

*1.* $\mathcal{A}_G^*(\Delta) = \mathcal{A}_G^*(\mathcal{D})$.

*2. If* $f \in \mathcal{A}_G^*(\mathcal{D})$, *then there exists a continuous homomorphism* $\Phi : \mathbb{R}_x \to G$ *such that* $f(n) = \Phi(n)$ *for every* $n \in \mathbb{N}$.

The proof is based upon the theorem of Wirsing [42].

The set of all limit points of $\{f(n)\}_{n=1}^{\infty}$ form a compact subgroup in $G$ which is denoted by $S_f$.

*3.* $f \in \mathcal{A}_G^*(\mathcal{D})$ *if and only if there exists a continuous functions* $H : S_f \to S_f$ *such that* $f(n+1) - H(f(n)) \to 0$ *as* $n \to \infty$.

Now we formulate the main unsolved problems.

Let $f_j \in \mathcal{A}_{G_j} \quad (j = 0, 1, \dots, k-1)$, and consider the sequence $e_n := \{f_0(n), f_1(n+1), \dots, f_{k-1}(n+k-1)\}$. Then $e_n \in S_{f_0} \times S_{f_1} \times \dots \times S_{f_{k-1}} = U$. What can we say about the functions $f_j$, if the set of limit points of $e_n$ is not everywhere dense in $U$?

**Conjecture 6.1.** *Let* $f \in \mathcal{A}_T^*, \quad S_f = T, \quad e_n := (f(n), \dots, f(n+k-1))$. *Then either* $\{e_n | n \in \mathbb{N}\}$ *is everywhere dense in* $T_k = T \times \dots \times T$, *or* $f(n) = \lambda \log n \pmod{\mathbb{Z}}$ *with some* $\lambda \in \mathbb{R}$.

**Conjecture 6.2.** *Let* $f, g \in \mathcal{A}_T^*, \quad S_f = S_g = T, \quad e_n := (f(n), g(n+1))$. *If* $e_n$ *is not everywhere dense in* $T^2$, *then* $f$ *and* $g$ *are rationally dependent continuous characters, i.e. there exists* $\lambda \in \mathbb{R}, \quad s \in \mathbb{Q}$ *such that* $g(n) = sf(n) \pmod{\mathbb{Z}}$, $f(n) = \lambda \log n \pmod{\mathbb{Z}}$.

## 7. A conjecture on primes

**Conjecture 7.1.** *For every integer* $k \geq 1$ *there exists a constant* $c_k$ *such that for every prime* $p$ *greater than* $c_k$

$$\min_{j=1,\dots,p-1} \max_{\substack{l=-k,\dots,k \\ l \neq 0}} P(jp+l) < p. \tag{7.1}$$

Here $P(n)$ is the largest prime factor of $n$. This problem is unsolved even in the case $k = 2$. Some heuristic arguments support our opinion that Conjecture 7.1 is true. Hence Conjecture 5.1 it would follow.

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