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ON THE ITERATES OF SOME MULTIPLICATIVE FUNCTIONS

Hoang Thi Lan Giao (Hue, Vietnam)

Imre Kátai and Bui Minh Phong (Budapest, Hungary)

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Abstract. Let $f_k(n)$ be the k-th iterates of a function $f(n)$, i.e. $f_0(n) := n$, $f_1(n) := f(n), \dots, f_{k+1}(n) := f(f_k(n))$ $(k = 1, 2, \cdots)$. We prove that if $n \in \mathbb{N}$ and the function f is defined by $f(p) = p$ and $f(p^\alpha) = p + p^2$ for all primes p, $\alpha \geq 2$, then for some $k \in \mathbb{N}$ there are an

$$u \in \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2, 2^2.3, 2.3^2, 2^3.3\}$$

and a square-free $\mathcal{D} \in \mathbb{N}$, $(\mathcal{D}, 6) = 1$ such that $f_k(n) = u.\mathcal{D}$.

1. Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, non-negative integers, integers, real and complex numbers, respectively. (n, m) denotes the greatest common divisor of n and m. We say that $f : \mathbb{N} \to \mathbb{R}$ is an additive function, if $f(mn) = f(m) + f(n)$ for all $(m, n) = 1$. Let $\mathcal{A}$ denote the set of all additive functions. A function $g : \mathbb{N} \to \mathbb{C}$ is multiplicative, if $g(1) = 1$ and $g(mn) = g(m) \cdot g(n)$ for all $(m, n) = 1$. We denote by $\mathcal{M}$ the set of all multiplicative functions.

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For some n let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be its prime decomposition. We say that n is square-free, if $\alpha_1 = \cdots = \alpha_r = 1$, and n is square-full if $\min \alpha_j \geq 2$. The Möbius function $\mu(n)$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } \max \alpha_j \geq 2, \\ (-1)^r & \text{if } \alpha_1 = \cdots = \alpha_r = 1. \end{cases}$$

It is clear that $\mu \in \mathcal{M}$, $|\mu(n)| = 1$ if and only if n is square free.

We are interested in such multiplicative functions f for which $f(n) \in \mathbb{N}$ ($n \in \mathbb{N}$), and $f(p^\alpha) = Q_\alpha(p)$ for all primes p, $Q_\alpha(z) \in \mathbb{Z}[z]$, $Q_\alpha(z) > 0$ for every $z \in \mathbb{N}$. Many of interesting multiplicative functions is of that type, e.g.

$\sigma(n)$ = sum of divisors function,

$\sigma^\star(n)$ = sum of unitary divisors function,

$\sigma^{(e)}(n)$ = sum of exponential divisors function,

$\varphi(n)$ = Euler's totient function.

We have

$$\varphi(p^\alpha) = p^{\alpha-1}(p-1), \quad \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1},$$

$$\sigma^\star(p^\alpha) = 1 + p^\alpha, \quad \sigma^{(e)}(p^\alpha) = \sum_{\beta|\alpha} p^\beta.$$

The corresponding $Q_\alpha(0) \in \{-1, 1\}$ in the cases $f = \varphi, \sigma, \sigma^\star$, while in the case $f = \sigma^{(e)}$ we have $Q_\alpha(0) = 0$.

Let

$$f_0(n) := n, \ f_1(n) := f(n), \ldots, \ f_{k+1}(n) := f(f_k(n)), \quad (k = 1, 2, \cdots).$$

Some functions of $f_k(n)$, especially $\omega(f_k(n)), \omega(f_k(p+1))$ have been investigated in several papers. Here $\omega(n)$ is the number of prime factors of n. For the function $f(n) = \sigma^{(e)}(n)$ is was proved that

$$\lim_{x\to\infty} \frac{1}{x} \sharp\Big\{n \leq x \ \Big| \ \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j, \ j = 1, 2, \cdots, k\Big\} =$$
$$= F_k(\alpha_1, \cdots, \alpha_k)$$

exists, F_k is strictly monotonic in each variables in $(\alpha_1, \cdots, \alpha_k) \in (1, \infty)^k$. (See [1] and the previous papers [2], [3], [4]).

If $f = \sigma^{(e)}$, then $f(n) > n$ holds for every non square-free n, and $f(n) = n$ if n is square-free ($n = 1$ is included).

Conjecture 1. *For every $n \in \mathbb{N}$ there exists a constant K_n such that*

$$p^2 \nmid f_k(n) \quad (k = 0, 1 \cdots) \quad \textit{if} \quad p > K_n.$$

Conjecture 2. *There exists an absolute constant R such that for every n there is a $k_0(n)$ such that*

$$p^2 \nmid f_k(n) \qquad \textit{if} \quad p > R, \ k > k_0(n).$$

We are unable to prove our conjectures. We shall prove such theorem in the case when

$$f(p^\alpha) = \begin{cases} p & \text{if} \quad \alpha = 1, \\ p + p^2 & \text{if} \quad \alpha \geq 2. \end{cases}$$

2. Let $Q_\alpha(z) \in \mathbb{Z}[z]$ be such a sequence of polynomials ($\alpha = 1, 2, \cdots$) for which

$$Q_1(z) = z, \quad Q_\alpha(z) = z + \sum_{\ell=2}^{t(\alpha)} a_\ell(\alpha) z^\ell,$$

where

$$t(\alpha) \leq C\alpha, \ |a_\ell(\alpha)| \leq C \quad (\ell = 2, \cdots, t(\alpha), \alpha \geq 2),$$

C is a suitable constant, $Q_\alpha(n) \geq n$.

Let $f \in \mathcal{M}$ be defined by

$$f(p^\alpha) = Q_\alpha(p) \quad (p \in \mathcal{P}).$$

Let h be a fixed positive number. Let $n = f_0(n)$, $n_j = f_j(n)$ $(j = 1, \cdots, h)$. We can write n as $n = Km$, where K is square-full, m is square-free, $(K, m) = 1$. Let m_1 be the largest divisor of m which is coprime to $f(K)$. Let $m = m_1\nu_1$. Then $\nu_1 \mid f(K)$. Write $K_1 := f(K)\nu_1$. We have

$$n_1 = f(n) = f(Km) = f(K)m = f(K)m_1\nu_1 = K_1 m_1.$$

Then clearly $(K_1, m_1) = 1$ and m_1 is square-free.

We can continue this procedure. We have

$$n_2 = f(K_1)m_1 = f(K_1)\nu_2 m_2,$$

where

$$\nu_2 | f(K_1),\ (m_2, f(K_1)) = 1, \quad m_h = \nu_2 m_2, \quad K_2 = f(K_1)\nu_2.$$

In general we have

$$n_j = f(K_{j-1})m_{j-1} = K_j m_j,\ K_j = f(K_{j-1})\nu_j$$

and

$$m_{j-1} = \nu_j m_j,\ \nu_j | f(K_{j-1}),\ (m_j, K_j) = 1$$

for $j = 1, \cdots, k$.

Assume now that K is a fixed square-full integer, $K_1,\ \nu_1, \cdots, K_h,\ \nu_h$ are defined as follows:

(1) $(\nu_1, K) = 1$, ν_1 is square-free, $\nu_1 | f(K)$, $K_1 = f(K)\nu_1$,

(2) $(\nu_2, KK_1) = 1$, ν_2 is square-free, $\nu_2 | f(K_1)$, $K_2 = f(K_1)\nu_2$,

$\vdots$

(h) $(\nu_h, KK_1 \cdots K_{h-1}) = 1$, ν_h is square-free, $\nu_h | f(K_{h-1})$, $K_h = f(K_{h-1})\nu_h$.

Let us consider those integers n which can be written as $n = KTM$, where $T = \nu_1 \cdots \nu_h$, $(M, KK_1 \cdots K_h) = 1$, M is square-free.

For such an n we have

$$n = KTM, \quad n_1 = f(K)TM, \quad \frac{n_1}{n} = \frac{f(K)}{K} = \frac{K_1}{\nu_1 K},$$

$$n_2 = f\big(f(K)\nu_1\big)\nu_2 \cdots \nu_h M, \quad n_1 = f(K)\nu_1\nu_2 \cdots \nu_h M,$$

thus

$$\frac{n_2}{n_1} = \frac{f(K_1)}{K_1} = \frac{K_2}{\nu_2 K_1},$$

and in general

$$\frac{n_{j+1}}{n_j} = \frac{K_{j+1}}{\nu_{j+1}K_j} \qquad (j = 1, \cdots, h-1).$$

Let A be an arbitrary positive integer and

$$M\big(x|A\big) = \sum_{\substack{n \le x \\ (n,A)=1}} |\mu(n)|.$$

Let $\kappa(p) = \frac{1}{1+\frac{1}{p}}$ $(p \in \mathcal{P})$, κ be strongly multiplicative.

It can be proved easily that

$$M\big(x|A\big) = (1+o_x(1))\frac{6}{\pi^2}\kappa(A)x.$$

Consequently

$$\begin{aligned}&\natural\big\{n = KTM \le x,\ \ M\ \text{ is square-free},\ \ (M,\ KT) = 1\big\} = \\ &= (1+o_x(1))\frac{6}{\pi^2}\,\frac{\kappa(KT)}{KT}x\end{aligned}$$

for every fixed K and T.

Hence we obtain easily

Theorem 1. *There exists a sequence*

$$\Big(y_1^{(m)},\cdots,y_h^{(m)}\Big) \in (0,\infty)^h \quad (m = 1,2\cdots)$$

such that

$$\frac{1}{x}\natural\Big\{n \le x \ \ \Big|\ \ \frac{f_j(n)}{f_{j-1}(n)} = y_j^{(m)},\ \ j = 1,\cdots,h\Big\} \to d_j \quad (x\to\infty),$$

and

$$\sum_{j=1}^{\infty} d_j = 1.$$

Let

$$(*) \qquad R\big(x|A\big) = \sum_{\substack{p\le x\\ (p+1,A)=1}} |\mu(p+1)|.$$

By using the prime number theorem for the arithmetical progression, we obtain that

$$R\big(x|A\big) = (1+o_x(1))\frac{6}{\pi^2}\mathcal{D}(A)\mathrm{li}x,$$

where

$$\mathcal{D}(A) = \prod_{p|A}\Big(1-\frac{1}{p}\Big).\prod_{p\nmid A}\Big(1-\frac{1}{p(p-1)}\Big) = \prod_{p|A}\Big(\frac{p^2-2p}{p^2-p-1}\Big)E$$

and

$$E := \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p(p-1)}\Big).$$

Hence we can deduce

Theorem 2. *There exists a sequence*

$$\left(z_1^{(n)}, \cdots, z_h^{(n)}\right) \in (0,\infty)^h \quad (n = 1, 2\cdots)$$

such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \natural \left\{ p \le x \;\Big|\; \frac{f_j(p+1)}{f_{j-1}(p+1)} = z_j^{(n)}, \quad j = 1, \cdots, h \right\} \to \mathcal{D}_j \quad (x \to \infty)$$

and

$$\sum_{j=1}^{\infty} \mathcal{D}_j = 1.$$

Remark. The relation $(*)$ is even uniform as $1 \le A \le C_1(\log x)^{C_2}$, where $C_1,\ \ C_2$ are arbitrary constants. This easily follows from the Siegel-Walfisz theorem.

3. Let

$$f(p^\alpha) = \begin{cases} p & \text{if } \alpha = 1, \\ p + p^2 & \text{if } \alpha \ge 2. \end{cases}$$

Theorem 3. *Let $n \in \mathbb{N}$. Then for some $k \in \mathbb{N}$ there are an*

$$u \in \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2, 2^2.3, 2.3^2, 2^3.3\}$$

and a square-free $\mathcal{D} \in \mathbb{N}$, $(\mathcal{D}, 6) = 1$ such that

$$f_k(n) = u.\mathcal{D}.$$

Furthermore, the following possibilities can be occur:

a) If $u \in \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2\}$, then

$$f_{k+\ell}(n) = u.\mathcal{D} \quad \textit{for all} \quad \ell = 0, 1, 2, \cdots.$$

b) If $u \in \{2^2.3,\ 2.3^2,\ 2^3.3\}$, then

$$f_{k+\ell}(n) \in \{2^2.3.\mathcal{D},\ 2.3^2.\mathcal{D},\ 2^3.3.\mathcal{D}\} \quad \textit{for all} \quad \ell = 0, 1, 2, \cdots.$$

Proof of Theorem 3. For each $n \in \mathbb{N}, n > 1$ let

$$\kappa(n) := \max_{\substack{p^\alpha \parallel n \\ \alpha \geq 2}} \{\ p \in \mathcal{P}\ \}.$$

Let us observe that if $\kappa(n) \geq 5$, then $\kappa(f(n)) < \kappa(n)$. This is obvious. Let

$$n = p_1^{\alpha_1} \cdots p_j^{\alpha_j} p_{j+1} \cdots p_r,$$

where $p_1 < \cdots < p_j,\ \alpha_1 \geq 2, \cdots, \alpha_j \geq 2$ and $(p_1 \cdots p_j,\ p_{j+1} \cdots p_r) = 1$. Since $\kappa(n) = p_j \geq 5$, $\alpha_j \geq 2$, therefore $f(p_j^{\alpha_j}) = p_j(1 + p_j)$, $1 + p_j$ is even, thus the largest prime factor of $1+p_j$ is less than p_j. Consequently $\kappa(f(n)) < p_j = \kappa(n)$.

Hence it follows that for every n in the sequence $f_0(n), f_1(n), \cdots$, there is a k for which $p^2 \mid f_k(n)$ implies that $p = 2$ or $p = 3$. Then $f_k(n) = 2^\alpha 3^\beta \mathcal{D}$, $(\mathcal{D}, 6) = 1$, $\mathcal{D}$ is square-free. It is clear that

$$f(2^\alpha 3^\beta) \in \{1, f(2), f(3), f(2^2), f(3^2), f(2.3), f(2^2.3), f(2.3^2), f(2^2.3^2)\} =$$
$$= \{1,\ 2,\ 3,\ 2.3,\ 2^2.3,\ 2.3^2,\ 2^3.3,\ 2^3.3,\ 2^3.3^2\} := \mathcal{U}.$$

Let

$$\mathcal{U}_1 = \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2\} \quad \text{and} \quad \mathcal{U}_2 = \{2^2.3, 2.3^2, 2^3.3\}.$$

Let us consider the graph

$$2^\alpha.3^\beta \to 2^{\alpha_1}.3^{\beta_1} \quad \text{if} \quad f(2^\alpha.3^\beta) = 2^{\alpha_1}.3^{\beta_1}.$$

It is clear that

$$a \to a \quad \text{for all} \quad a \in \mathcal{U}_1.$$

Furthermore

$$2^2.3 \to\ 2.3^2 \to\ 2^3.3 \to\ 2.3^2.$$

Theorem 3 is proved.

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Hoang Thi Lan Giao
Hue University
77 Nguyen Hue
Hue City, Vietnam
htlgao@hueuni.edu.vn

Imre Kátai and Bui Minh Phong
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu
bui@compalg.inf.elte.hu