

ON COMPOSITIONS WHOSE PARTS ARE POLYGONAL NUMBERS

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Abstract. For a given strictly increasing sequence $\{a_n\}$ of natural numbers, let $g(n)$ be the number of compositions of n all of whose parts belong to $\{a_n\}$. We derive a recurrence that enables the computation of $g(n)$, as well as an estimate for $g(n)$ for large n . We use these results to obtain useful data concerning compositions whose parts belong to the following 3 sequences: triangular, square, and pentagonal numbers.

1. Introduction

If n is a natural number, let $g(n)$ be the number of compositions of n all of whose parts belong to a given strictly increasing sequence of natural numbers, $\{a_n\}$, with $g(0) = 1$. We show how to compute $g(n)$ via a recurrence, and also how to estimate $g(n)$ for large n . In a previous effort, this was done for binary and Fibonacci numbers, that is, the cases $a_n = 2^{n-1}$ and $a_n = F_{n+1}$, where F_n denotes the n^{th} Fibonacci number. (See [1].) Here we obtain similar results concerning triangular, square, and pentagonal numbers. In these 3 cases, we tabulate $g(n)$ for $1 \leq n \leq 20$.

Let $p(n)$ denote the partition function. Let $\omega(k) = \frac{k(3k-1)}{2}$, where $k \in \mathbb{Z}$. ($\omega(k)$ is the $(k^{\text{th}}$ pentagonal number). Let x denote a complex variable such that $|x| < 1$.

$$(1) \quad p(n) = \sum_{k \neq 0} (-1)^{k-1} p(n - \omega(k))$$

Remark. Identity (1) is Euler's well-known recurrence for $p(n)$.

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2. General theorems

Our first theorem concerns a recurrence that enables computation of $g(n)$.

Theorem 1. *Let $\{a_n\}$ be a strictly increasing sequence of natural numbers. If n is a natural number, let $g(n)$ denote the number of compositions of n whose parts all belong to $\{a_n\}$. Define $g(0) = 1$, and $g(\alpha) = 0$ if α is not a non)-negative integer. Then for all $n \geq 1$, we have*

$$g(n) = \sum_{k \geq 1} g(n - a_k) .$$

Proof. Let the generating function for compositions with exactly k elements from $\{a_n\}$ be

$$g_k(x) = \left(\sum_{i \geq 1} x^{a_i} \right)^k .$$

Let $G(x)$ be the generating function for all compositions of n with summands from $\{a_n\}$. Then we have

$$G(x) = 1 + \sum_{k=1}^{\infty} g_k(x) = \left(1 - \sum_{i=1}^{\infty} x^{a_i} \right)^{-1} ,$$

and also

$$G(x) = \sum_{n=0}^{\infty} g(n)x^n \rightarrow \left(1 - \sum_{i=1}^{\infty} x^{a_i} \right) \left(\sum_{n=0}^{\infty} g(n)x^n \right) = 1 .$$

The conclusion now follows if we equate coefficients of like powers of x . ■

The next theorem provides a convenient way to estimate $g(n)$ for large n .

Theorem 2. *Let*

$$a(x) = \sum_{j=1}^{\infty} x^{a_j} .$$

Let ρ be the unique positive root of $a(x) = 1$. Then, as $n \rightarrow \infty$, we have

$$g(n) \sim \frac{\rho^{-n-1}}{a'(\rho)} .$$

Proof. The function $a(x)$ has the following properties; (i) $a(0) = 0$; (ii) $a'(0) \neq 0$; (iii) there exists ρ such that $0 < \rho < 1$ and $a(\rho) = 0$; (iv) $a(x)$ is analytic at $x = \rho$. Therefore the function

$$B(x) = \frac{1}{1 - a(x)}$$

has a simple pole at $x = \rho$, and from the local expansion of $B(x)$ at this dominant pole, we have

$$B(x) \sim \left(\frac{1}{\rho a'(\rho)} \right) \left(\frac{1}{1 - \frac{x}{\rho}} \right).$$

It therefore follows that

$$g(n) = [x^n]B(x) \sim \left(\frac{1}{\rho a'(\rho)} \right) \rho^{-n} \quad \text{as } n \rightarrow \infty. \quad \blacksquare$$

3. Compositions whose parts are polygonal numbers

First, let $g_s(n)$ denote the number of compositions of n whose parts are square numbers. In this case, we have

$$a_n = n^2, \quad a(x) = \sum_{j=1}^{\infty} x^{j^2}, \quad \rho = .70534668, \quad a'(\rho) = 3.046149872.$$

Theorem 3.

$$(a) \quad g_s(n) = \sum_{k \geq 1} g_s(n - k^2); \quad (b) \quad \text{as } n \rightarrow \infty, \quad g_s(n) \sim \frac{.70534668^{-n-1}}{3.046149872}.$$

Proof. (a) follows from Theorem 1; (b) follows from Theorem 2. \blacksquare

Next, let $g_t(n)$ denote the number of compositions of n whose parts are triangular numbers. In this case, we have

$$a_n = \frac{n(n+1)}{2}, \quad a(x) = \sum_{j=1}^{\infty} x^{\frac{j(j+1)}{2}}, \quad \rho = .645222707, \quad a'(\rho) = 3.149722816.$$

Theorem 4.

$$(a) \quad g_t(n) = \sum_{k \geq 1} g_t(n - \frac{k(k+1)}{2}); \quad (b) \quad \text{as } n \rightarrow \infty, \quad g_t(n) \sim \frac{.645222707^{-n-1}}{3.149722816}.$$

Proof. Same as proof of Theorem 3. \blacksquare

Finally, let $g_p(n)$ denote the number of compositions of n whose parts are pentagonal numbers. In this case, we have

$$a_n = \frac{n(3n \mp 1)}{2}, \quad a(x) = \sum_{k \neq 0} x^{\frac{k(3k-1)}{2}}, \quad \rho = .578044621, \quad a'(\rho) = 3.011573540.$$

Theorem 5.

$$(a) \quad g_p(n) = \sum_{k \neq 0} g_p(n - \omega(k)); \quad (b) \quad \text{as } n \rightarrow \infty, \quad g_p(n) \sim \frac{.578044621^{-n-1}}{3.0115735}.$$

Proof. Same as proof of Theorem 3. ■

The table below lists $g_s(n)$, $g_t(n)$, $g_p(n)$ for $1 \leq n \leq 20$.

n	$g_s(n)$	$g_t(n)$	$g_p(n)$
1	1	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	4	9
6	4	7	15
7	5	11	27
8	7	16	46
9	11	25	80
10	16	40	138
11	22	61	238
12	30	93	413
13	43	147	713
14	62	227	1235
15	88	351	2136
16	124	546	3695
17	175	845	6393
18	249	1308	11057
19	354	2029	19130
20	502	3145	33091

Glancing at the table above, it appears that $g_p(n)$ has the same parity as the partition function, $p(n)$. This is shown in Theorem 6 below:

Theorem 6. $g_p(n) \equiv p(n) \pmod{2}$ for all $n \geq 0$.

Proof. (Induction on n). We have $p(0) = g_p(0) = 1$. If $n \geq 1$, then by (1), induction, and Theorem 5, part (a), we have

$$p(n) \equiv \sum_{k \neq 0} p(n - \omega(k)) \equiv \sum_{k \neq 0} g_p(n - \omega(k)) \equiv g_p(n) \pmod{2}. \quad \blacksquare$$

The function $g_p(n)$ appears to share the following property with $p(n)$:

Conjecture. $g_p(5n + 4) \equiv 0 \pmod{5}$ for all $n \geq 0$.

References

- [1] **Knopfmacher, A. and N. Robbins**, On binary and Fibonacci compositions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **22** (2003), 193–206.

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