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ON THE WEIGHTED GRÜNWALD–ROGOSINSKI PROCESS

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Abstract. It is known that for every set of interpolation nodes, there exists a continuous function for which the sequence of Lagrange interpolation polynomials is not uniformly convergent. In the case of the Chebyshev abscissas, G. Grünwald constructed a process that is uniformly convergent for all continuous functions on the whole interval $[-1,1]$. However, M.S. Webster showed that for the roots of the Chebyshev polynomials of the second kind, the analogous construction is uniformly convergent only in closed subintervals of $(-1,1)$. Our aim is to improve this result by using weighted Lagrange interpolation. We shall prove that the weighted Grünwald–Rogosinski process is uniformly convergent on the whole interval $[-1,1]$ in suitable weighted function spaces. Order of convergence will also be investigated.

1. Introduction

In 1914, G. Faber [1] proved the following fundamental result (see also e.g. [15]): For any fixed interpolation matrix

$$X = \{x_{k,n} \mid k = 1,2,\ldots,n;\ n = 1,2,\ldots\} \subset \mathbb{R}$$

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there exists a continuous function f for which the sequence of Lagrange interpolation polynomials $L_n(f, X, \cdot)$ $(n \in \mathbb{N} = \{1, 2, \ldots\})$ is not uniformly convergent. In 1935 and 1936, for the Chebyshev matrix

$$T = \left\{ x_{k,n} = \cos \frac{2k-1}{2n}\pi \;\middle|\; k = 1, 2, \ldots, n;\; n = 1, 2, \ldots \right\}$$

(where the order of the Lebesgue function is the smallest possible) G. Grünwald [2], [3] and J. Marcinkiewicz [5], [6] proved the following theorem: there exists a continuous function f such that the sequence $L_n(f, T, x)$ $(n \in \mathbb{N})$ is divergent for all points x of the interval $[-1, 1]$. (See the survey paper [12].)

Therefore the following result, which was proved by G. Grünwald [4] in 1941, is interesting.

Theorem A. *Let f be a continuous function on $[-1, 1]$. Then*

$$\begin{gathered} \lim_{n\to\infty} \tfrac{1}{2} \left\{ L_n\left[f, T, \theta - \varphi_n\right] + L_n\left[f, T, \theta + \varphi_n\right] \right\} = f(x), \\ x = \cos\theta \in [-1, 1], \end{gathered} \tag{1.1}$$

and the convergence is uniform on the whole interval $[-1, 1]$. Here $L_n[f, T, \theta]$ denotes the Lagrange interpolation polynomial $L_n(f, T, x)$ after the substitution $x = \cos\theta$ and $\varphi_n = \frac{\pi}{2n}$ $(n = 1, 2, \ldots)$.

It is known that between the interpolation polynomials $L_n(f, T, \cdot)$ $(n \in \mathbb{N})$ and the partial sums of the trigonometric Fourier series of the even function f there is a far reaching analogy. Moreover the above theorem of G. Grünwald is analogous with the well known theorem of Rogosinski in the theory of the trigonometric Fourier series. The process (1.1) will be called *Grünwald–Rogosinski process.*

In 1943, M.S. Webster [19] obtained a similar result for the roots of the Chebyshev polynomials of the second kind, i.e. when the interpolation matrix is given by

$$U = \left\{ x_{k,n} = \cos \frac{k}{n+1}\pi \;\middle|\; k = 1, 2, \ldots, n;\; n = 1, 2, \ldots \right\}. \tag{1.2}$$

He proved the following result.

Theorem B. *Let f be a continuous function on $[-1, 1]$ and $\varphi_n = \frac{\pi}{2(n+1)}$ $(n = 1, 2, \ldots)$. Then*

$$\begin{gathered} \lim_{n\to\infty} \tfrac{1}{2} \left\{ L_n\left[f, U, \theta - \varphi_n\right] + L_n\left[f, U, \theta + \varphi_n\right] \right\} = f(x), \\ x = \cos\theta \in (-1, 1), \end{gathered} \tag{1.3}$$

and the convergence is uniform only on any interval $[a,b] \subset (-1,1)$. *In general, convergence does not hold for* $x = 1$ *or* $x = -1$. *In addition,* φ_n *may be replaced by* $p\pi/2(n+1)$, *where* p *is any fixed odd integer.*

In 1975, P. Vértesi [16] generalized Webster's result for the roots of Jacobi polynomials $P_n^{(\alpha,\beta)}$ with arbitrary parameters $\alpha, \beta > -1$.

The aim of this paper is to improve Webster's result. We shall consider the point system (1.2) and modify the process (1.3) by using the weighted Lagrange interpolation polynomials (see (2.4)). We shall prove that the weighted Grünwald–Rogosinski process (2.6) is uniformly convergent on the whole interval $[-1,1]$ in suitable weighted spaces C_{w_γ} (see (2.1)). Order of convergence will also be investigated.

2. Result

Let $C(I)$ be the linear space of real valued continuous functions defined on the interval $I \subset \mathbb{R}$. For a parameter $\gamma \geq 0$ we consider the Jacobi weight

$$w_\gamma(x) := (1-x^2)^\gamma \qquad \big(x \in [-1,1]\big),$$

and define the weighted function space

$$C_{w_\gamma} := \Big\{ f \in C(-1,1) : \lim_{|x|\to 1} (fw_\gamma)(x) = 0 \Big\} \tag{2.1}$$

if $\gamma > 0$. If $\gamma = 0$ (i.e. $w_\gamma \equiv 1$) then let $C_{w_\gamma} = C[-1,1]$.

Then

$$\|f\|_{w_\gamma} := \|fw_\gamma\|_\infty := \max_{x\in[-1,1]} \big|(fw_\gamma)(x)\big| \quad \big(f \in C_{w_\gamma}\big)$$

is a norm on C_{w_γ} and $\big(C_{w_\gamma}, \|\cdot\|_{w_\gamma}\big)$ is a Banach space.

Let us consider the Chebyshev polynomials of the second kind:

$$\begin{gathered} U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \\ x = \cos\theta,\ x \in [-1,1],\ \theta \in [0,\pi],\ n = 0,1,2,\dots. \end{gathered} \tag{2.2}$$

They are orthogonal on the interval $[-1,1]$ with respect to the weight $w_{1/2}(x) = \sqrt{1-x^2}$ $(x \in [-1,1])$. The roots of U_n $(n = 1,2,\dots)$ are

$$x_{k,n} = \cos\theta_{k,n} = \cos\frac{k}{n+1}\pi \quad (k = 1,2,\dots,n). \tag{2.3}$$

The *weighted Lagrange interpolation polynomials* of a function $f \in C_{w_\gamma}$ on the point system (2.3) are defined by

$$L_n\big(f, U, w_\gamma, x\big) := w_\gamma(x) \sum_{k=1}^{n} f(x_{k,n}) \ell_{k,n}(U, x) \tag{2.4}$$
$$\big(x \in [-1, 1],\ n = 1, 2, \ldots\big),$$

where

$$\ell_{k,n}(x) := \ell_{k,n}(U, x) = \frac{U_n(x)}{U_n'(x_{k,n})(x - x_{k,n})} \tag{2.5}$$
$$\big(x \in [-1, 1];\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots\big)$$

are the Lagrange fundamental polynomials. It is clear that

$$L_n\big(f, U, w_\gamma, x_{j,n}\big) = w_\gamma\big(x_{j,n}\big) f\big(x_{j,n}\big)$$
$$\big(j = 1, 2, \ldots, n;\ n = 1, 2, \ldots\big).$$

Let

$$\varphi_n := \frac{\pi}{2(n+1)} \qquad (n = 1, 2, \ldots)$$

and for a given $\cos\theta = x \in [-1, 1]$ and $n \in \mathbb{N}$ we define

$$x_+ := \cos\theta_+ := \cos\left(\theta + \varphi_n\right) = x \cos\varphi_n - \sqrt{1 - x^2} \sin\varphi_n,$$
$$x_- := \cos\theta_- := \cos\left(\theta - \varphi_n\right) = x \cos\varphi_n + \sqrt{1 - x^2} \sin\varphi_n.$$

For the point system (2.3) the *weighted Grünwald–Rogosinski process* will be defined by

$$(A_n f)(x) := A_n(f, U, w_\gamma, x) := \tfrac{1}{2} \left\{L_n(f, U, w_\gamma, x_+) + L_n(f, U, w_\gamma, x_-)\right\} \tag{2.6}$$
$$(n \in \mathbb{N},\ x \in [-1, 1],\ f \in C_{w_\gamma}).$$

Recall that the number

$$E_n(f, w_\gamma) := \inf_{p \in \mathcal{P}_n} \|(f - p) w_\gamma\|_\infty \qquad (n \in \mathbb{N})$$

is called the best nth degree weighted polynomial approximation of the function $f \in C_{w_\gamma}$, where $\mathcal{P}_n$ denotes the linear space of algebraic polynomials of degree at most n. It is known that

$$\lim_{n \to \infty} E_n(f, w_\gamma) = 0,$$

i.e. the set of the weighted polynomials $w_\gamma p$ (p is an arbitrary algebraic polynomial) is dense in the Banach space $\big(C_{w_\gamma}, \|\cdot\|_{w_\gamma}\big)$ (see e.g. [14, Section 3]).

For a function $f \in C[-1,1]$ the second order modulus of smoothness is defined by

$$\omega_2(f,t) := \sup_{0<h\le t} \|\Delta_h(f,\cdot)\|_\infty \quad (t>0),$$

where

$$\Delta_h(f,x) = f(x+h) + f(x-h) - 2f(x) \quad (x \in [-1+h, 1-h]).$$

Theorem. *The weighted Grünwald–Rogosinski process* (2.6) *is uniformly convergent in the function space C_{w_γ}, i.e.*

$$\lim_{n\to\infty} |A_n(f,U,w_\gamma,x) - (fw_\gamma)(x)| = 0 \tag{2.7}$$

holds uniformly on the whole interval $[-1,1]$ for every function $f \in C_{w_\gamma}$ if and only if the parameter γ satisfies the relation

$$\frac{1}{2} \le \gamma \le 2. \tag{2.8}$$

For the order of convergence we have

$$\begin{gathered} |A_n(f,U,w_\gamma,x) - (fw_\gamma)(x)| = O(1)\Big(E_{n-1}(f,w_\gamma) + \omega_2(fw_\gamma,\varphi_n)\Big) \\ (x\in[-1,1],\ n\in\mathbb{N},\ f\in C_{w_\gamma}). \end{gathered} \tag{2.9}$$

Remarks. 1. From the proof of Theorem it follows that the process (2.6) is uniformly convergent on arbitrary closed interval $[a,b] \subset (-1,1)$ for *every* parameter $\gamma \ge 0$. If $\gamma = 0$ then we obtain Webster's result (see Theorem B), also Vértesi's result for Jacobi parameters $\alpha = \beta = 1/2$.

2. From a general summation theorem Zs. Németh [8, Theorem 3.2] obtained the uniform convergence of the process (2.6) on the whole interval $[-1,1]$, but only for parameter $\gamma = 1/2$.

3. There is an other possibility to achieve the uniform convergence of the Grünwald–Rogosinski process on the whole interval $[-1,1]$. This is the so-called *additional nodes method* (see e.g. [7, 4.2.2] and [9]). L. Szili [11, 7.3] proved the following statement: If to the point system (2.3) we add the endpoints ± 1, then the (unweighted) Grünwald–Rogosinski process is uniformly convergent on $[-1,1]$ for every continuous function $f \in C[-1,1]$.

4. L. Szili and P. Vértesi defined the weighted Grünwald–Rogosinski process in another way, using suitable summation of Lagrange interpolation (see [13, Example 7, p. 328]). In [14, Corollary 3.4 (d)] they obtained that that process on the point system (1.2) is uniformly convergent in C_{w_γ} if (cf. (2.8))

$$0 < \gamma < 2. \tag{2.10}$$

3. A Lemma

It is clear that for all fixed $n \in \mathbb{N}$ (see (2.6))

$$A_n : \big(C_{w_\gamma}, \|\cdot\|_{w_\gamma}\big) \to \big(C[-1,1], \|\cdot\|_\infty\big)$$

is a bounded linear operator and its norm is defined by

$$\|A_n\| := \sup\big\{ \|A_n f\|_\infty \ : \ \|f\|_{w_\gamma} \le 1\big\}.$$

Lemma. *There exists a constant $c > 0$ independent of n such that*

$$\|A_n\| \le c \qquad (n \in \mathbb{N})$$

if and only if the weight parameter γ satisfies (2.8).

Remark. It is interesting to compare the operator norms $\|A_n\|$ $(n \in \mathbb{N})$ with the norms of the weighted Lagrange interpolation operators (they are also called *the weighted Lebesgue constants*) on the roots of the Chebyshev polynomials of the second kind. It is known that (see [7, p. 271]) they are given by

$$\big\|\mathcal{L}_n(w_\gamma)\big\| = \max_{x\in[-1,1]} \sum_{k=1}^{n} \frac{w_\gamma(x)}{w_\gamma(x_{k,n})}\,\big|\ell_{k,n}(x)\big| \qquad \big(n \in \mathbb{N},\ \gamma \ge 0\big).$$

G. Szegő's classical result states that (see [7, Theorem 4.2.1] and [10, Theorem 14.4]) in the unweighted case (i.e. when $\gamma = 0$) $\big\|\mathcal{L}_n(w_0)\big\| \sim n$ $(n \in \mathbb{N})$, where the constants in "$\sim$"* are independent of n. G. Faber [1] proved that *for any interpolation matrix X* the order of the Lebesgue constants is at least $\log n$. P. Vértesi [17] showed the analogous result for weighted Lagrange interpolation for arbitrary Jacobi weights. It is also known that (see [7, Theorem 4.3.1]) on the point system U (see (1.2)) this optimal order can be attained if and only if the Jacobi parameter γ satisfies the conditions (cf. (2.8))

$$\frac{1}{2} \le \gamma \le \frac{3}{2}\,. \tag{3.1}$$

*$a_n \sim b_n$ means that $c_1 \le a_n/b_n \le c_2$ $(n \in \mathbb{N})$, where $0 < c_1 \le c_2$.

Proof. We prove the Lemma in several steps.

Step 1. First we write the norm of the operator A_n in trigonometric form. Since the weighted Grünwald–Rogosinski process (2.6) can be written as

$$
\begin{aligned}
(A_n f)(x) &= \sum_{k=1}^{n} f\big(x_{k,n}\big)\, \frac{(w_\gamma \ell_{k,n})(x_-) + (w_\gamma \ell_{k,n})(x_+)}{2} = \\
&= \sum_{k=1}^{n} (f w_\gamma)\,(x_{k,n})\, \frac{(w_\gamma \ell_{k,n})(x_-) + (w_\gamma \ell_{k,n})(x_+)}{2 w_\gamma (x_{k,n})} =: \\
&=: \sum_{k=1}^{n} (f w_\gamma)\,(x_{k,n})\, S_{k,n}(x)
\end{aligned}
\tag{3.2}
$$

$$\big(x \in [-1,1];\ k = 1,2,\ldots,n;\ n \in \mathbb{N};\ f \in C_{w_\gamma}\big),$$

thus the norm of the operator A_n is given by

$$\|A_n\| = \max_{x \in [-1,1]} \sum_{k=1}^{n} |S_{k,n}(x)|\,.$$

If $x = \cos\theta \in [-1,1]$ then

$$w_\gamma(x) = (1-x^2)^\gamma = \sin^{2\gamma}\theta.$$

By (2.2) and (2.5)

$$
\begin{aligned}
U_n'(x_{k,n}) &= (-1)^{k+1}\, \frac{n+1}{1-x_{k,n}^2} = (-1)^{k+1}\, \frac{n+1}{\sin^2\theta_{k,n}}, \\
\ell_{k,n}(x) &= (-1)^{k+1}\, \frac{\sin^2\theta_{k,n}}{n+1} \cdot \frac{\sin(n+1)\theta}{\sin\theta(\cos\theta - \cos\theta_{k,n})}, \\
(w_\gamma \ell_{k,n})\,(x) &= (-1)^{k+1}\, \frac{\sin^2\theta_{k,n}}{n+1} \cdot \sin^{2\gamma-1}\theta \cdot \frac{\sin(n+1)\theta}{\cos\theta - \cos\theta_{k,n}},
\end{aligned}
$$

thus we get

$$
\begin{aligned}
S_{k,n}(x) = {} & \frac{(-1)^{k+1}}{2(n+1)} \cdot \sin^{2-2\gamma}\theta_{k,n} \cdot \\
& \cdot \left(\frac{\sin^{2\gamma-1}\theta_+ \sin(n+1)\theta_+}{\cos\theta_+ - \cos\theta_{k,n}} + \frac{\sin^{2\gamma-1}\theta_- \sin(n+1)\theta_-}{\cos\theta_- - \cos\theta_{k,n}} \right).
\end{aligned}
$$

Since

$$\sin(n+1)\theta_\pm = \sin[(n+1)(\theta \pm \tfrac{\pi}{2(n+1)})] = \pm\cos(n+1)\theta$$

thus using the notation

$$K(\theta) := K_\gamma(\theta) := \sin^{2\gamma-1}\theta \qquad \big(\theta \in (0,\pi)\big)$$

the above form of $S_{k,n}(x)$ may be simplified as

$$S_{k,n}(x) = \frac{(-1)^{k+1}}{2(n+1)} \sin^{2-2\gamma}\theta_{k,n} \cdot \cos(n+1)\theta \cdot \\ \cdot \Big(\frac{K(\theta_+)}{\cos\theta_+ - \cos\theta_{k,n}} - \frac{K(\theta_-)}{\cos\theta_- - \cos\theta_{k,n}}\Big).$$

Finally we get

$$\|A_n\| = \max_{x\in[-1,1]} \sum_{k=1}^{n} |S_{k,n}(x)| = \tag{3.3}$$

$$= \max_{\theta\in[0,\pi]} \frac{|\cos(n+1)\theta|}{2(n+1)} \sum_{k=1}^{n} \sin^{2-2\gamma}\theta_{k,n} \Big| \frac{K(\theta_+)}{\cos\theta_+ - \cos\theta_{k,n}} - \frac{K(\theta_-)}{\cos\theta_- - \cos\theta_{k,n}} \Big|.$$

Step 2. Since the functions $K(\theta_\pm)/(\cos\theta_\pm - \cos\theta_{k,n})$ are defined only for $\theta_\pm \neq \theta_{k,n}$ (i.e. $\theta \neq \theta_{k,n} \pm \varphi_n$) therefore we have to split the sum in (3.3) into two parts. For this purpose, let us introduce the following notation.

For a fixed point θ of the interval $[0,\pi]$ let us denote by $\theta_{j,n}$ (one of) the closest node(s) to θ, i.e.

$$\min_{1\le k\le n} \big|\theta - \theta_{k,n}\big| = \big|\theta - \theta_{j,n}\big|. \tag{3.4}$$

Moreover let

$$\sum_{k=1}^{n} |S_{k,n}(x)| = \sum_{k=j-1,j,j+1} |S_{k,n}(x)| + \sum_{\substack{1\le k\le n \\ k\neq j-1,j,j+1}} |S_{k,n}(x)| =: \\ =: \sum\nolimits_I + \sum\nolimits_{II} . \tag{3.5}$$

Since

$$|\ell_{k,n}(x)| < 2 \quad (-1 \le x \le 1;\ k = 1,2,\ldots,n;\ n = 1,2,\ldots), \tag{3.6}$$

(see [18]) thus we have

$$\sum_{k=j-1,j,j+1} |S_{k,n}(x)| = \sum_{k=j-1,j,j+1} \Big| \frac{(w_\gamma\ell_{k,n})(x_-) + (w_\gamma\ell_{k,n})(x_+)}{2w_\gamma(x_{k,n})} \Big| \le c \\ \big(x\in[-1,1],\ n\in\mathbb{N},\ \gamma\ge 0\big). \tag{3.7}$$

Step 3. For the estimation of $\sum_{II}$ in (3.5) we introduce the functions: if $\theta \in [0,\pi]$ and $\theta_\pm \neq \theta_{k,n}$ (i.e. $\theta \neq \theta_{k,n} \pm \varphi_n$) then for $k = 1, 2, \dots, n$ and $n \in \mathbb{N}$ let

$$F_{k,n}(\theta) := \frac{K(\theta_+)}{\cos\theta_+ - \cos\theta_{k,n}} - \frac{K(\theta_-)}{\cos\theta_- - \cos\theta_{k,n}}. \tag{3.8}$$

Then by (3.3) and (3.5) we have for $\theta \in [0,\pi]$ and $n \in \mathbb{N}$

$$\sum\nolimits_{II} \leq \frac{c}{n} \sum_{\substack{1\leq k\leq n \\ k\neq j-1,j,j+1}} \sin^{2-2\gamma}\theta_{k,n}|F_{k,n}(\theta)|.$$

Using that $\theta_\pm = \theta \pm \varphi_n$ we get

$$\begin{gathered}
K(\theta_+)\big(\cos\theta_- - \cos\theta_{k,n}\big) - K(\theta_-)\big(\cos\theta_+ - \cos\theta_{k,n}\big) = \\
= -\big[K(\theta_+) - K(\theta_-)\big]\cos\theta_{k,n} + K(\theta_+)\cos\theta_- - K(\theta_-)\cos\theta_+ = \\
= -\big[K(\theta_+) - K(\theta_-)\big]\cos\theta_{k,n} + K(\theta_+)\big(\cos\theta\cos\varphi_n + \sin\theta\sin\varphi_n\big) - \\
- K(\theta_-)\big(\cos\theta\cos\varphi_n - \sin\theta\sin\varphi_n\big) = \\
= \tfrac{1}{2}\big[K(\theta_+) - K(\theta_-)\big]\big(\cos\theta_+ - \cos\theta_{k,n} + \cos\theta_- - \cos\theta_{k,n}\big) + \\
+ \big[K(\theta_+) + K(\theta_-)\big]\sin\theta\sin\varphi_n.
\end{gathered}$$

Since

$$|K(\theta_+) - K(\theta_-)| = 2\varphi_n|K'(\theta^*)| \leq \frac{c}{n}\sin^{2\gamma-2}(\theta^*), \tag{3.9}$$

where $\theta^* \in \big(\theta_-, \theta_+\big)$ thus we have

$$\begin{aligned}
\big|F_{k,n}(\theta)\big| \leq \frac{c}{n}\sin^{2\gamma-2}\theta^* \left(\frac{1}{|\cos\theta_+ - \cos\theta_{k,n}|} + \frac{1}{|\cos\theta_- - \cos\theta_{k,n}|}\right) + \\
+ \frac{c}{n}\frac{|K(\theta_+) + K(\theta_-)|\sin\theta}{|\cos\theta_+ - \cos\theta_{k,n}|\cdot|\cos\theta_- - \cos\theta_{k,n}|}.
\end{aligned} \tag{3.10}$$

If $\theta \neq \theta_{k,n} \pm \varphi_n$ then let

$$\begin{aligned}
G_{k,n}(\theta) &:= \frac{1}{n^2}\left(\frac{\sin\theta_{k,n}}{\sin\theta^*}\right)^{2-2\gamma}\left(\frac{1}{|\cos\theta_+ - \cos\theta_{k,n}|} + \frac{1}{|\cos\theta_- - \cos\theta_{k,n}|}\right), \\
H_{k,n}(\theta) &:= \frac{\sin^{2-2\gamma}\theta_{k,n}}{n^2}\frac{|K(\theta_+) + K(\theta_-)|\sin\theta}{|\cos\theta_+ - \cos\theta_{k,n}|\cdot|\cos\theta_- - \cos\theta_{k,n}|}, \\
M_n(\theta) &:= \sum_{\substack{1\leq k\leq n \\ k\neq j-1,j,j+1}} \big(G_{k,n}(\theta) + H_{k,n}(\theta)\big).
\end{aligned}$$

By (3.10) we get

$$\sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} |S_{k,n}(x)| \le c \sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} \big(G_{k,n}(\theta)+H_{k,n}(\theta)\big) = cM_n(\theta). \tag{3.11}$$

For the proof of the Lemma we have to show that the functions $M_n(\theta)$ $(\theta\in[0,\pi],\ n\in\mathbb{N})$ are uniformly bounded if and only if γ satisfies (2.8).

Step 4. Now we prove that if $\frac{1}{2}\le\gamma\le 2$ then there exists a constant $c>0$ independent of n such that

$$M_n(\theta)\le c \qquad \big(\theta\in[0,\pi],\ n\in\mathbb{N}\big). \tag{3.12}$$

We will distinguish three cases.

Case 1. Let $\theta\in(\frac{c}{n},\frac{\pi}{2}]$ with a positive fixed constant c. In this case for the index $j=j(n)$ defined by (3.4) we have $1\le j\le[Cn]$ with a constant $C>0$.

In order to estimate the sum in (3.11), we split it into four parts:

$$\sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} = \sum_{1\le k\le j/2} + \sum_{\substack{j/2<k\le 2j\\ k\ne j-1,j,j+1}} + \sum_{2j<k\le[cn]} + \sum_{[cn]<k\le n}, \tag{3.13}$$

where $c\in(0,1)$ is independent of $n\in\mathbb{N}$ such that $\theta_{k,n}\in[0,\frac{3\pi}{4}]$, if $1\le k\le[cn]$.

For $1\le k\le[cn]$ and $n\in\mathbb{N}$ we use the following estimations:

$$\sin\theta_{k,n}\sim\frac{k}{n},$$
$$\sin\theta,\ \sin\theta_\pm,\ \sin\theta^*\sim\frac{j}{n},$$
$$|\cos\theta_\pm-\cos\theta_{k,n}|\sim\frac{|j^2-k^2|}{n^2}.$$

So we have

$$\sum_{1\le k\le j/2} G_{k,n}(\theta)\sim\sum_{1\le k\le j/2}\left(\frac{k}{j}\right)^{2-2\gamma}\frac{1}{|j^2-k^2|}\sim$$
$$\sim j^{2\gamma-4}\sum_{1\le k\le j/2}\left(\frac{1}{k}\right)^{2\gamma-2}\sim\begin{cases}\dfrac{\log j}{j}, & \text{if } 2\gamma-2=1,\\ j^{2\gamma-4}, & \text{if } 2\gamma-2>1,\\ j^{-1}, & \text{if } 2\gamma-2<1,\end{cases}$$

where we used that $j\pm k\sim j$. Moreover,

$$\sum_{1\le k\le j/2} H_{k,n}(\theta) \sim \sum_{1\le k\le j/2} \left(\frac{j}{k}\right)^{2\gamma} \left(\frac{k}{j^2-k^2}\right)^2 \sim$$

$$\sim j^{2\gamma-4} \sum_{1\le k\le j/2} \left(\frac{1}{k}\right)^{2\gamma-2} \sim \begin{cases} \dfrac{\log j}{j}, & \text{if } 2\gamma-2=1, \\ j^{2\gamma-4}, & \text{if } 2\gamma-2>1, \\ j^{-1}, & \text{if } 2\gamma-2<1. \end{cases}$$

This means that these sums are bounded independently from j and n if $\gamma \le 2$.

For the second part of (3.13) we use that $k \sim j$, $j+k \sim j$. Then for every j and n we have

$$\sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} G_{k,n}(\theta) \sim \sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} \left(\frac{k}{j}\right)^{2-2\gamma} \frac{1}{|j^2-k^2|} \sim$$

$$\sim \frac{1}{j} \sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} \frac{1}{|j-k|} \le C\frac{\log j}{j} \le C$$

and

$$\sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} H_{k,n}(\theta) \sim \sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} \left(\frac{j}{k}\right)^{2\gamma} \left(\frac{k}{j^2-k^2}\right)^2 \sim$$

$$\sim \sum_{\substack{j/2<k\le 2j \\ k\ne j-1,j,j+1}} \frac{1}{(j-k)^2} \le C,$$

which means that this part is uniformly bounded for every parameter $\gamma \ge 0$.

For the third part using $\frac{k}{k-j} \le 2$, $\frac{j}{k} < 1$, $j+k>k$ we obtain that

$$\sum_{2j<k\le[cn]} G_{k,n}(\theta) \sim \sum_{2j<k\le[cn]} \left(\frac{k}{j}\right)^{2-2\gamma} \frac{1}{|j^2-k^2|} \le$$

$$\le Cj^{2\gamma-2} \sum_{2j<k\le[cn]} \left(\frac{1}{k}\right)^{2\gamma} \sim \begin{cases} \dfrac{1}{j}\log\dfrac{n}{j}, & \text{if } 2\gamma=1, \\ \dfrac{1}{n}\left(\dfrac{j}{n}\right)^{2\gamma-2} + \dfrac{1}{j}, & \text{if } 2\gamma\ne 1, \end{cases}$$

which is uniformly bounded for every j and n if $\gamma > \frac{1}{2}$. However, if $\gamma = \frac{1}{2}$ then $K(\theta) = 1$ ($\theta \in [0, \pi]$), which means that (see Step 3)

$$\sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} |S_{k,n}(x)| \le c \sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} H_{k,n}(\theta),$$

so the above sum does not need to be bounded in this case.

Moreover

$$\begin{aligned}\sum_{2j<k\le[cn]} H_{k,n}(\theta) &\sim \sum_{2j<k\le[cn]} \left(\frac{j}{k}\right)^{2\gamma} \left(\frac{k}{j^2-k^2}\right)^2 \le \\ &\le C \sum_{2j<k\le[cn]} \frac{1}{k^2} \le C,\end{aligned}$$

which means that this is uniformly bounded for every parameter $\gamma \ge 0$.

If $[cn] < k \le n$ and $n \in \mathbb{N}$ then we use the following estimations:

$$\begin{gathered}\sin\theta_{k,n} \sim \frac{n+1-k}{n},\\ \sin\theta,\ \ \sin\theta_{\pm},\ \ \sin\theta^* \sim \frac{j}{n},\\ |\cos\theta_{\pm} - \cos\theta_{k,n}| \ge c.\end{gathered}$$

So we have

$$\begin{aligned}\sum_{[cn]<k\le n} G_{k,n}(\theta) &\sim \sum_{[cn]<k\le n} \frac{j^{2\gamma-2}}{n^2}(n+1-k)^{2-2\gamma} \sim \\ &\sim \frac{j^{2\gamma-2}}{n^2}\sum_{l=1}^{n-[cn]}\left(\frac{1}{l}\right)^{2\gamma-2} \sim \begin{cases} \dfrac{j\log n}{n^2}, & \text{if } 2\gamma-2=1,\\ \dfrac{j^{2\gamma-2}}{n^2}, & \text{if } 2\gamma-2>1,\\ \dfrac{j^{2\gamma-2}}{n^{2\gamma-1}}, & \text{if } 2\gamma-2<1,\end{cases}\end{aligned}$$

which is bounded independently from j and n if $\frac{1}{2} \le \gamma \le 2$.

On the other hand,

$$\sum_{[cn]<k\le n} H_{k,n}(\theta) \sim \sum_{[cn]<k\le n} \frac{j^{2\gamma}}{n^4}(n+1-k)^{2-2\gamma} \sim$$

$$\sim \frac{j^{2\gamma}}{n^4}\sum_{l=1}^{n-[cn]}\left(\frac{1}{l}\right)^{2\gamma-2} \sim \begin{cases} \dfrac{j^3\log n}{n^4}, & \text{if } 2\gamma-2=1,\\ \dfrac{j^{2\gamma}}{n^4}, & \text{if } 2\gamma-2>1,\\ \dfrac{j^{2\gamma}}{n^{2\gamma+1}}, & \text{if } 2\gamma-2<1. \end{cases}$$

This is bounded independently from j and n if $\gamma \le 2$.

Collecting the above formulas we have (3.12) for $\theta \in (\frac{c}{n}, \frac{\pi}{2}]$ and $n \in \mathbb{N}$.

Case 2. Let $\theta \in [0, \frac{c}{n}]$ and $x = \cos\theta$. If $1 \le k \le [cn]$ then by (3.2) and (3.6) we get

$$|S_{k,n}(x)| \le C\,\frac{|w_\gamma(x_-)| + |w_\gamma(x_+)|}{w_\gamma(x_{k,n})} = C\,\frac{(\sin^2\theta_-)^\gamma + (\sin^2\theta_+)^\gamma}{\sin^{2\gamma}\theta_{k,n}} \le$$

$$\le C\left(\frac{1}{n}\right)^{2\gamma}\left(\frac{n}{k}\right)^{2\gamma} = C\left(\frac{1}{k}\right)^{2\gamma},$$

and if $[cn] < k \le n$ then (similarly)

$$|S_{k,n}(x)| \le C\left(\frac{1}{n+1-k}\right)^{2\gamma}.$$

So we have

$$\sum_{k=1}^{n}|S_{k,n}(x)| \le C\left(\sum_{k=1}^{[cn]}\left(\frac{1}{k}\right)^{2\gamma} + \sum_{l=1}^{n-[cn]}\left(\frac{1}{l}\right)^{2\gamma}\right).$$

This is bounded in x and $n \in \mathbb{N}$ if $\gamma > \frac{1}{2}$.

On the other hand, when $\gamma = \frac{1}{2}$, $k = 1, 2, \ldots, [cn]$ and $k \ne j-1, j, j+1$ then we have

$$|S_{k,n}(x)| = \frac{\sin\theta_{k,n}}{2(n+1)}\left|\frac{\sin(n+1)\theta_+}{\cos\theta_+ - \cos\theta_{k,n}} + \frac{\sin(n+1)\theta_-}{\cos\theta_- - \cos\theta_{k,n}}\right| =$$

$$= \frac{\sin\theta_{k,n}}{n+1}\,\sin\varphi_n\left|\frac{\sin\theta\cdot\cos(n+1)\theta}{\big(\cos\theta_+ - \cos\theta_{k,n}\big)\big(\cos\theta_- - \cos\theta_{k,n}\big)}\right| \le c\,\frac{1}{k^3},$$

where we used that

$$\sin\theta \le \frac{c}{n}, \quad \sin\theta_{k,n} \sim \frac{k}{n}, \quad \sin\varphi_n \sim \frac{1}{n} \quad \text{and} \quad |\cos\theta_\pm - \cos\theta_{k,n}| \sim \frac{k^2}{n^2}.$$

If $[cn] \le k \le n$ then using that $\sin\theta_{k,n} \sim \frac{n+1-k}{n}$ and $|\cos\theta_\pm - \cos\theta_k| \ge c$ we similarly get

$$|S_{k,n}(x)| \le c\,\frac{n+1-k}{n^4}.$$

Consequently we obtain that

$$\sum_{\substack{1\le k\le n\\ k\ne j-1,j,j+1}} |S_{k,n}(x)| \le C\left(\sum_{k=1}^{[cn]}\left(\frac{1}{k}\right)^3 + \sum_{k=[cn]+1}^{n}\frac{n+1-k}{n^4}\right),$$

which is uniformly bounded.

Thus we proved (3.12) in Case 2.

Case 3. Let $\theta \in \left(\frac{\pi}{2}, \pi\right]$ and $\theta' := \pi - \theta$. Then with the notations

$$w_\gamma[\theta] := w_\gamma(\cos\theta), \quad \ell_{k,n}[\theta] := \ell_{k,n}(\cos\theta) \quad S_{k,n}[\theta] := S_{k,n}(\cos\theta)$$

we have

$$w_\gamma[\theta+\varphi_n] = w_\gamma[\theta'-\varphi_n], \quad w_\gamma[\theta-\varphi_n] = w_\gamma[\theta'+\varphi_n],$$

and

$$l_{k,n}[\theta+\varphi_n] = l_{n+1-k,n}[\theta'-\varphi_n], \quad l_{k,n}[\theta-\varphi_n] = l_{n+1-k,n}[\theta'+\varphi_n],$$

which means that $S_{n+1-k,n}[\theta'] = S_{k,n}[\theta]$. So we have

$$\sum_{k=1}^{n}|S_{k,n}[\theta]| = \sum_{k=1}^{n}|S_{n+1-k,n}[\theta']| = \sum_{l=1}^{n}|S_{l,n}[\theta']|,$$

which is uniformly bounded if $\frac{1}{2} \le \gamma \le 2$, as we have shown before.

Thus (3.12) is completely proved.

Step 5. To prove that the conditions for the parameter γ are also necessary, first let us assume that $\gamma < \frac{1}{2}$, and for the point $\theta = 0$ we have

$$\sum_{k=1}^{n}|S_k[0]| \ge \sum_{k=1}^{[cn]}|S_k[0]| = \frac{1}{n+1}\sum_{k=1}^{[cn]}\frac{\sin^{2-2\gamma}\theta_{k,n}\sin^{2\gamma-1}\varphi_n}{|\cos\varphi_n - \cos\theta_{k,n}|} \sim$$

$$\sim \frac{1}{n}\sum_{k=1}^{[cn]}\left(\frac{k}{n}\right)^{2-2\gamma}\left(\frac{1}{n}\right)^{2\gamma-1}\frac{n^2}{k^2}=\sum_{k=1}^{[cn]}\left(\frac{1}{k}\right)^{2\gamma},$$

which is not uniformly bounded in $n \in \mathbb{N}$.

Now, if $\gamma > 2$, we consider the point $\theta = \frac{\pi}{2}$:

$$\sum_{k=1}^{n}\left|S_k\left[\tfrac{\pi}{2}\right]\right| \geq \sum_{k=1}^{[cn]}\left|S_k\left[\tfrac{\pi}{2}\right]\right| =$$

$$=\frac{|\cos{(n+1)\frac{\pi}{2}}||\cos^{2\gamma-1}\varphi_n|}{n+1}\sum_{k=1}^{[cn]}\frac{\sin^{2-2\gamma}\theta_{k,n}\sin\varphi_n}{|\cos^2\theta_{k,n}-\sin^2\varphi_n|}\sim$$

$$\sim\frac{|\cos{(n+1)\frac{\pi}{2}}|}{n^2}\sum_{k=1}^{[cn]}\left(\frac{k}{n}\right)^{2-2\gamma}\frac{n^2}{n^2-(k^2+1)}\geq$$

$$\geq|\cos{(n+1)\tfrac{\pi}{2}}|n^{2\gamma-4}\sum_{k=1}^{[cn]}\left(\frac{1}{k}\right)^{2\gamma-2},$$

which is also not uniformly bounded in $n \in \mathbb{N}$.

Thus the Lemma is completely proved. ∎

4. Proof of the Theorem

Since the operator norms $\|A_n\|$ $(n \in \mathbb{N})$ are not uniformly bounded if $0 \leq \gamma < 1/2$ or $\gamma > 2$ thus by the Banach–Steinhaus theorem we obtain that the limit relation (2.7) is not true for every function $f \in C_{w_\gamma}$.

Now, let $1/2 \leq \gamma \leq 2$. For the proof of (2.7) let us denote by $P_n \in \mathcal{P}_n$ the best weighted approximating polynomial of f of order at most n, that is

$$E_n(f,w_\gamma)=\|fw_\gamma-P_nw_\gamma\|_\infty.$$

Then we have

$$\begin{aligned}&\left|A_n(f,w_\gamma,x)-(fw_\gamma)(x)\right|=\\&=\Big|A_n(f,w_\gamma,x)-\tfrac{1}{2}\left\{(P_{n-1}w_\gamma)(x_+)+(P_{n-1}w_\gamma)(x_-)\right\}+\\&+\tfrac{1}{2}\left\{(P_{n-1}w_\gamma)(x_+)-(fw_\gamma)(x_+)+(P_{n-1}w_\gamma)(x_-)-(fw_\gamma)(x_-)\right\}+\\&+\tfrac{1}{2}\left\{(fw_\gamma)(x_+)-2(fw_\gamma)(x)+(fw_\gamma)(x_-)\right\}\Big|=\\&=O(1)\left(A_n(f-P_{n-1},w_\gamma,x)+E_{n-1}(f,w_\gamma)+\omega_2(fw_\gamma,\varphi_n)\right)=\\&=O(1)\left(E_{n-1}(f,w_\gamma)+\omega_2(fw_\gamma,\varphi_n)\right),\end{aligned}$$

which proves (2.7). ∎

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