LOWER THRESHOLD GROUND STATE ENERGY AND TESTABILITY OF MINIMAL BALANCED CUT DENSITY

András Krámli (Szeged, Hungary)

Roland Markó (Bonn, Germany)

Dedicated to András Benczúr on the occasion of his 70th birthday

Communicated by László Lakatos

Abstract. Lovász and his coauthors in [4] defined the notion of microcanonical ground state energy $\hat{\mathcal{E}}_{\mathbf{a}}(G, J)$ – borrowed from the statistical physics – for weighted graphs G, where $\mathbf{a} \in \mathrm{Pd}_q$ is a probability distribution on $\{1, ..., q\}$ and J is a symmetric $q \times q$ matrix with real entries. We define a new version of the ground state energy $\hat{\mathcal{E}}^c(G, J) = \inf_{\mathbf{a} \in A_c} \hat{\mathcal{E}}_{\mathbf{a}}(G, J)$, called lower threshold ground state energy, where $A_c = \{\mathbf{a} \in \mathrm{Pd}_q \colon a_i \geq c, i = 1, \ldots, q\}$. Both types of energies can be extended for graphons W, the limit objects of convergent sequences of simple graphs. The main result of the paper is Theorem 3.2 stating that if $0 \leq c_1 < c_2 \leq 1$, then the convergence of the sequences $(\hat{\mathcal{E}}^{c_2/q}(G_n, J))_{n \geq 1}$ for each $J \in \mathrm{Sym}_q$ implies convergence of the sequences $(\hat{\mathcal{E}}^{c_1/q}(G_n, J))_{n \geq 1}$ for each $J \in \mathrm{Sym}_q$. As a byproduct one can derive in a natural way the testability of minimum balanced multiway cut densities – one of the fundamental problems of cluster analysis – proved in [2].

1.   Preliminaries and notation

The main goal of this paper is to introduce and to reveal the properties of an intermediate object between the microcanonical ground state energy (MGSE) and ground state energy (GSE) of weighted graphs defined in [4]. For this purpose we need to define the fundamental notions used in [3], [4], and [5] and to cite the main results therein necessary for us, these will be given below. The main contribution of the paper is that we give a convergence hierarchy with respect to the aforementioned intermediate objects that are Hamiltonians subject to certain conditions. This can be regarded as a refined version of Theorem 2.9 (ii) from [4] combined with the equivalence assertion of Theorem 2.8 (v) from the same paper. In short, these state that MGSE convergence implies GSE convergence. Counterexamples are provided indicating that the implication is strict. We also reprove with the aid of the established hierarchy one of the main results of [2]. Our motivation comes from cluster analysis, where the minimal cut problem is a central subject of research. The graph limit theory of Lovász et al. sheds new light on this, especially their statistical physics correspondence suits for application in the cluster analysis setting.

We consider both unweighted simple graphs G (graphs without loops and multiple edges) and weighted graphs. We denote the node and edge sets of G by $V(G)$ and $E(G)$, respectively. Usually we denote by $\alpha_i = \alpha_i(G) > 0$ the weight associated with the node i and $\beta_{ij} = \beta_{ij}(G) \in \mathbb{R}$ the weight associated with the edge ij. We set $\alpha_G = \sum_i \alpha_i(G)$.

Definitions 1.1 - 1.9 (except 1.7) are borrowed from [3] and [4]. In order to facilitate the reading, at every definition we indicate its exact place of occurrence.

Definition 1.1. *([3] Definition 3.1.) Let $\mathcal{W}$ denote the space of bounded symmetric measurable functions $W \colon [0,1]^2 \to \mathbb{R}$, that is $W(x,y) = W(y,x)$. Assume that the functions $W \in \mathcal{W}$ take their values in an interval I and call them graphons.*

Usually $I = [0,1]$. We can think of the interval $[0,1]$ as the set of nodes of graph with a node set that has cardinality continuum, and of the values $W(x,y)$ as the weight of the edge xy.

Definition 1.2. *([4] Definition 2.4.) Let G and G' be two weighted graphs with node set V and V', respectively. For $i \in V$ and $u \in V'$ set $\mu_i = \alpha_i(G)/\alpha_G$ and $\mu'_u = \alpha_u(G')/\alpha_{G'}$. Then we define the set of fractional overlays $\chi(G, G')$ as the set of probability distributions X on $V \times V'$ (or couplings of μ and μ')*

such that

$$\sum_{u \in V'} X_{iu} = \mu_i \text{ for all } i \in V, \text{ and } \sum_{i \in V} X_{iu} = \mu'_u \text{ for all } u \in V',$$

and set

$$(1.1) \quad \delta_\square(G, G') = \min_{X \in \chi(G,G')} \max_{S,T \subset V \times V'} \left| \sum_{\substack{(i,u) \in S \\ (j,v) \in T}} X_{iu} X_{jv} \left(\beta_{ij}(G) - \beta_{uv}(G') \right) \right|.$$

The distance defined in Definition 1.2 can be extended to graphons in terms of the cut norm.

Definition 1.3. *([3] formula (3.3)) The cut norm of a graphon W is given by*

$$(1.2) \quad \|W\|_\square = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} W(x,y) dx dy \right|$$

$$= \sup_{f,g\colon [0,1] \to [0,1]} \left| \int W(x,y) f(x) g(y) dx dy \right|,$$

where the suprema go over measurable subsets and functions, respectively.

Definition 1.4. *([4] formula (3.3)) The cut distance of two graphons U and W is defined as*

$$(1.3) \quad \delta_\square(U, W) = \inf_\phi \|U - W^\phi\|_\square,$$

where the infimum goes over all measure preserving permutations of $[0,1]$, and W^ϕ is given by $W^\phi(x,y) = W(\phi(x), \phi(y))$.

Now we define three versions of the *ground state energies* (GSE) borrowed from the statistical physics that are the objects investigated in this paper (for the mathematical treatment of statistical physics, see, e.g. Sinai's book [6]). They are defined in terms of a finite set of states $[q] = \{1, \ldots, q\}$, and a symmetric $q \times q$ matrix J with entries in $\mathbb{R}$, the set of these matrices is denoted by Sym_q. A *spin configuration* on a simple or weighted graph G is given by a map $\phi\colon V(G) \to [q]$.

Definition 1.5. *([4] formulae (2.8) and (2.11)) The energy density of a spin configuration of G with respect to J is given by*

$$(1.4) \quad \mathcal{E}_\phi(G, J) = -\frac{2}{|V(G)|^2} \sum_{uv \in E(G)} J_{\phi(u)\phi(v)}.$$

The ground state energy (GSE) of G with respect to J is

$$\hat{\mathcal{E}}(G, J) = \min_{\phi \colon V(G) \to [q]} \mathcal{E}_\phi(G, J).$$

For a graphon W and a $\rho = (\rho_1, \ldots, \rho_q)$ with $\rho_i \colon [0,1] \to [0,1]$ being measurable and satisfying $\sum_i \rho(x) = 1$ for each $x \in [0,1]$ called q-fractional partition the energy is defined as

$$\mathcal{E}_\rho(W, J) = -\sum_{i,j=1}^{q} J_{ij} \int_{[0,1]^2} \rho_i(x)\rho_j(y)W(x,y)\mathrm{d}x\mathrm{d}y,$$

and the GSE is

$$\mathcal{E}(W, J) = \min_\rho \mathcal{E}_\rho(W, J).$$

Let Pd_q be the set of all probability distributions on $[q]$.

Imposing some restrictions on the set where the minimum is taken in the above definition we can define another version of energies that are important in graph limit theory.

Definition 1.6. *([4] formula (2.14)) The microcanonical ground state energy (MGSE) of G with respect to $J \in \mathrm{Sym}_q$ and a probability distribution $\mathbf{a} = (a_1, \ldots, a_q) \in \mathrm{Pd}_q$ is defined using the set*

$$(1.5) \quad \Omega_{\mathbf{a}}(G) = \left\{ \phi \colon V(G) \to [q] \colon \left| |\phi^{-1}(\{i\})| - a_i|V(G)| \right| \leq 1 \text{ for all } i \in [q] \right\},$$

and is the quantity

$$(1.6) \qquad\qquad \hat{\mathcal{E}}_{\mathbf{a}}(G, J) = \min_{\phi \in \Omega_{\mathbf{a}}(G)} \mathcal{E}_\phi(G, J).$$

Let $\omega_{\mathbf{a}} = \left\{ \rho \colon \int_0^1 \rho_i(x)\mathrm{d}x = a_i \text{ for all } i \in [q] \right\}$ be a subset of q-fractional partitions. Then the MGSE of a graphon W is defined as

$$\mathcal{E}_{\mathbf{a}}(W, J) = \min_{\rho \in \omega_{\mathbf{a}}} \mathcal{E}_\rho(W, J).$$

Next we introduce the central object of our current investigation.

Definition 1.7. *Let G be a simple graph, $q \geq 1$, $J \in \mathrm{Sym}_q$, and $0 \leq c \leq 1/q$. We define the set $A_c = \{\mathbf{a} \in \mathrm{Pd}_q \colon \quad a_i \geq c, i = 1, \ldots, q\}$, and with its help the lower threshold ground state energy (LTGSE):*

$$(1.7) \qquad\qquad \hat{\mathcal{E}}^c(G, J) = \inf_{\mathbf{a} \in A_c} \hat{\mathcal{E}}_{\mathbf{a}}(G, J).$$

In a similar manner we introduce the LTGSEs for a graphon W for $q \geq 1$, $J \in \mathrm{Sym}_q$, and $0 \leq c \leq 1/q$ lower threshold:

$$(1.8) \qquad \mathcal{E}^c(W, J) = \inf_{\mathbf{a} \in A_c} \mathcal{E}_{\mathbf{a}}(W, J).$$

We remind the reader of the definition of testability of simple graph parameters. Before doing it, we should define the randomization procedure for graphs, used here.

Definition 1.8. *([3] Introduction of Section 2.5.3.) For a graph G and a positive integer k let $\mathbb{G}(k, G)$ denote the random induced subgraph $G[S]$ where S is chosen uniformly from all subsets of $V(G)$ of cardinality k.*

Definition 1.9. *([3] Definition 2.11.) A real function f defined on the set of simple graphs is a testable simple graph parameter, if for every $\varepsilon > 0$ there exists a $k = k(\varepsilon) \in \mathbb{N}$ such that for every simple graph G on at least k vertices*

$$P(|f(G) - f(\mathbb{G}(k, G))| > \varepsilon) < \varepsilon.$$

All the above definitions can be extended to weighted graphs in an analogous way.

This paper is organized as follows. In the second section we prove yet another equivalent condition to left-convergence of a graph sequence relying on a subclass of MGSE, the reasoning will be instrumental for the proof of our main result in the subsequent section. In the third section we study the convergence of LTGSEs, see (1.8) for their definition. We will consider $c \colon \mathbb{N} \to [0, 1]$ threshold functions with the property that $c(q)q$ is constant as a function of q. For this case we will prove that if $0 \leq c_1(q) < c_2(q) \leq 1/q$ (for all q), then the convergence of $(\mathcal{E}^{c_2(q)}(W_n, J))_{n \geq 1}$ for all $q \geq 1$, $J \in \mathrm{Sym}_q$ implies the convergence of $(\mathcal{E}^{c_1(q)}(W_n, J))_{n \geq 1}$ for all $q \geq 1$, $J \in \mathrm{Sym}_q$.

In the fourth section we provide some examples of graphs and graphons which support the fact, that the implication of convergence in the third section is strict in the sense that convergence of LTGSE sequences with smaller threshold do not imply convergence of LTGSE sequences with larger threshold in general. We also present a one-parameter family of block-diagonal graphons whose elements can be distinguished by LTGSEs for any threshold $c > 0$, but not by GSEs.

2.   Microcanonical convergence

We start by showing that for each discrete probability distribution with rational probabilities there exists a uniform probability distribution, such that

the microcanonical ground state energies (MGSE) of it can be expressed as MGSEs corresponding to the uniform distribution.

Lemma 2.1. *Let $q, q' \geq 1$, and $\mathbf{a} \in \mathrm{Pd}_q$ be such that $\mathbf{a} = (a_1, \ldots, a_q)$, and $a_i q'$ is a positive integer for all $i \in [q]$. Then for all $J \in \mathrm{Sym}_q$ there exists a $J' \in \mathrm{Sym}_{q'}$, such that for $\mathbf{b} = (1/q', \ldots, 1/q') \in \mathrm{Pd}_{q'}$ and every graphon W it holds that*

$$\mathcal{E}_{\mathbf{a}}(W, J) = \mathcal{E}_{\mathbf{b}}(W, J').$$

Proof. If the i-th component of $\mathbf{a}$ is 0, then erase this component from $\mathbf{a}$, and also erase the i-th row and column of J. This transformation clearly will have no effect on the value of the GSE. Set $k_0 = 0$, and $k_i = q' \sum_{j=1}^{i} a_j$ for each $i \in [q]$. Let us define the $q' \times q'$ matrix J' by blowing up rows and columns of J in the following way. For each $u, v \in [q']$ let $J'_{uv} = J_{ij}$, where $k_{i-1} < u \leq k_i$ and $k_{j-1} < v \leq k_j$. The matrix J' defined this way is clearly symmetric.

Now we will show that for every q-fractional partition with distribution $\mathbf{a}$ there exists a q'-fractional partition ρ' with distribution $\mathbf{b}$ such that $\mathcal{E}_{\rho}(W, J) = \mathcal{E}_{\rho'}(W, J')$, and vice versa. On one hand, for $1 \leq u \leq q'$ let $\rho'_u = \frac{\rho_i}{k_i - k_{i-1}}$, where $k_{i-1} < u \leq k_i$. Then

$$\mathcal{E}_{\rho'}(W, J') = -\sum_{u,v=1}^{q'} J'_{u,v} \int_{[0,1]^2} \rho'_u(x) \rho'_v(y) W(x, y) \mathrm{d}x \mathrm{d}y$$

$$= -\sum_{i,j=1}^{q} J_{i,j} \sum_{l=k_{i-1}+1}^{k_i} \sum_{h=k_{j-1}+1}^{k_j} \int_{[0,1]^2} \rho'_l(x) \rho'_h(y) W(x, y) \mathrm{d}x \mathrm{d}y$$

$$= \mathcal{E}_{\rho}(W, J).$$

On the other hand, for $1 \leq i \leq q$ let $\rho_i := \sum_{l=k_{i-1}+1}^{k_i} \rho'_l$. Then

$$\mathcal{E}_{\rho}(W, J) = -\sum_{i,j=1}^{q} J_{i,j} \int_{[0,1]^2} \rho_i(x) \rho_j(y) W(x, y) \mathrm{d}x \mathrm{d}y$$

$$= -\sum_{i,j=1}^{q} J_{i,j} \sum_{l=k_{i-1}+1}^{k_i} \sum_{h=k_{j-1}+1}^{k_j} \int_{[0,1]^2} \rho'_l(x) \rho'_h(y) W(x, y) \mathrm{d}x \mathrm{d}y$$

$$= -\sum_{u,v=1}^{q'} J'_{u,v} \int_{[0,1]^2} \rho'_u(x) \rho'_v(y) W(x, y) \mathrm{d}x \mathrm{d}y = \mathcal{E}_{\rho'}(W, J').$$

So we conclude that

$$\mathcal{E}_{\mathbf{a}}(W, J) = \inf_{\rho \in \omega_{\mathbf{a}}} \mathcal{E}_{\rho}(W, J) = \inf_{\rho' \in \omega_{\mathbf{b}}} \mathcal{E}_{\rho'}(W, J') = \mathcal{E}_{\mathbf{b}}(W, J').$$

∎

First we show that the MGSE with fixed parameters W, J are close, whenever their corresponding probability distribution parameters are close to each other.

Lemma 2.2. *Let $q \geq 1$, $J \in \mathrm{Sym}_q$, and W be an arbitrary graphon. Then for $\mathbf{a}, \mathbf{b} \in \mathrm{Pd}_q$ we have that*

$$|\mathcal{E}_{\mathbf{a}}(W, J) - \mathcal{E}_{\mathbf{b}}(W, J)| < 2\|\mathbf{a} - \mathbf{b}\|_1 \|W\|_\infty \|J\|_\infty.$$

Proof. Let $\rho \in \omega_{\mathbf{a}}$, let us construct according to this a $\rho' \in \omega_{\mathbf{b}}$ the following way. First let us line up those i's, for which $b_i \geq a_i$, for simplicity index them by integers from 1 to k. Let ρ'_1 be such, that $\rho_1(x) \leq \rho'_1(x) \leq 1$ for each $x \in [0, 1]$ and $\int_0^1 \rho'_1(x)\mathrm{d}x = b_1$. It is clear that such a function exists. We define ρ'_2 in similar fashion: let $\rho_2(x) \leq \rho'_2(x) \leq 1 - \rho'_1(x)$ for all $x \in [0, 1]$ and $\int_0^1 \rho'_2(x)\mathrm{d}x = b_2$, the existence is again clear. We define subsequently ρ'_i for i's obeying $b_i \geq a_i$ by taking care that

$$\rho_i(x) \leq \rho'_i(x) \leq 1 - \left[\sum_{j=1}^{i-1} \rho'_j(x)\right]$$

holds at each step. In the other case, when $b_i < a_i$, we reverse the inequality we wish to be satisfied by the functions ρ_i and ρ'_i, and define ρ'_i accordingly. For the constructed ρ'_i either $\rho'_i(x) \leq \rho_i(x)$ for all $x \in [0, 1]$, or $\rho'_i(x) \geq \rho_i(x)$ for all $x \in [0, 1]$, and additionally $\sum_i \rho_i(x) = 1$. Hence

$$\|\rho - \rho'\|_1 = \sum_{i=1}^{q} \int_0^1 |\rho_i(x) - \rho'_i(x)|\,\mathrm{d}x = \sum_{i=1}^{q} \left|\int_0^1 \rho_i(x) - \rho'_i(x)\mathrm{d}x\right|$$

$$= \sum_{i=1}^{q} |a_i - b_i| = \|\mathbf{a} - \mathbf{b}\|_1.$$

Now we give an upper bound on the deviation of MGSEs.

$$|\mathcal{E}_\rho(W, J) - \mathcal{E}_{\rho'}(W, J)|$$

$$= \left| \sum_{i,j=1}^{q} J_{i,j} \int_{[0,1]^2} (\rho_i(x)\rho_j(y) - \rho_i'(x)\rho_j'(y))W(x, y)\mathrm{d}x\mathrm{d}y \right|$$

$$\leq \|W\|_\infty \|J\|_\infty \sum_{i,j=1}^{q} \int_{[0,1]^2} |\rho_i(x)\rho_j(y) - \rho_i(x)\rho_j'(y)|$$

$$+ |\rho_i(x)\rho_j'(y) - \rho_i'(x)\rho_j'(y)| \, \mathrm{d}x\mathrm{d}y$$

$$\leq \|W\|_\infty \|J\|_\infty \sum_{i,j=1}^{q} a_i \|\rho_j - \rho_j'\|_1 + b_j \|\rho_i - \rho_i'\|_1$$

$$= 2\|\mathbf{a} - \mathbf{b}\|_1 \|W\|_\infty \|J\|_\infty.$$

The second inequality follows by Fubini's theorem. From the definition of MGSE the statement of the lemma follows. ∎

With the aid of the two previous lemmas we are able to prove the main assertion of the section. In the statement of the following theorem the LTGSE expression $\mathcal{E}^{1/q}(W, J)$ (which is equal to $\mathcal{E}_{\mathbf{b}}(W, J)$, with $\mathbf{b} = (1/q, \ldots, 1/q)$) appears, the notion will further be generalized in what follows later on.

Theorem 2.1. *Let I be a bounded interval, and $(W_n)_{n\geq 1}$ a sequence of graphons from $\mathcal{W}_I$. If for all $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{1/q}(W_n, J))_{n\geq 1}$ converge, then for all $q \geq 1$, $\mathbf{a} \in \mathrm{Pd}_q$, and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}_{\mathbf{a}}(W_n, J))_{n\geq 1}$ converge.*

Proof. Let $q \geq 1$, $\mathbf{a} \in \mathrm{Pd}_q$ and $J \in \mathrm{Sym}_q$ be arbitrary and fixed. We will prove that whenever the conditions of the theorem are satisfied, then $(\mathcal{E}_{\mathbf{a}}(W_n, J))_{n\geq 1}$ is Cauchy convergent. Fix an arbitrary $\varepsilon > 0$. Let q' be such that $4\frac{q}{q'}\|I\|_\infty\|J\|_\infty < \frac{\varepsilon}{3}$, and let $\mathbf{b} \in \mathrm{Pd}_q$ be such that $b_i = [a_i/q']$ ($i = 1, \ldots, q-1$), $b_q = 1 - \sum_{i=1}^{q-1} b_i$ (where $[x]$ is the lower integer part x). Then

$$\|\mathbf{a} - \mathbf{b}\|_1 = \sum_{i=1}^{q} |a_i - b_i| \leq 2\frac{q-1}{q'} < 2\frac{q}{q'}.$$

$\mathbf{b}$ is a q'-rational distribution, so by Lemma 2.1 there exists $J' \in \mathrm{Sym}_{q'}$, such that for all $n \geq 1$

$$\mathcal{E}_{\mathbf{b}}(W_n, J) = \mathcal{E}^{1/q'}(W_n, J').$$

It follows from the conditions of the theorem that there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ it is true that $\left| \mathcal{E}^{1/q'}(W_n, J') - \mathcal{E}^{1/q'}(W_m, J') \right| < \frac{\varepsilon}{3}$. Applying Lemma 2.2 to all $m, n \geq n_0$ we get that

$$
\begin{aligned}
|\mathcal{E}_{\mathbf{a}}(W_n, J) - \mathcal{E}_{\mathbf{a}}(W_m, J)| &\leq |\mathcal{E}_{\mathbf{a}}(W_n, J) - \mathcal{E}_{\mathbf{b}}(W_n, J)| \\
&\quad + |\mathcal{E}_{\mathbf{b}}(W_n, J) - \mathcal{E}_{\mathbf{b}}(W_m, J)| + |\mathcal{E}_{\mathbf{b}}(W_m, J) - \mathcal{E}_{\mathbf{a}}(W_m, J)| \\
&\leq 2\|\mathbf{a} - \mathbf{b}\|_1 \|I\|_\infty \|J\|_\infty + \left| \mathcal{E}^{1/q'}(W_n, J') - \mathcal{E}^{1/q'}(W_m, J') \right| \\
&\quad + 2\|\mathbf{a} - \mathbf{b}\|_1 \|I\|_\infty \|J\|_\infty \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

∎

We remark that Theorem 2.1 also appears in [4] as Corollary 7.4, but its proof follows a different line of thought in the present paper.

3. Weaker convergence, lower threshold microcanonical ground state energies

In various cases of testing, for certain cuts of graphs neither the notion of ground state energies, nor the notion of microcanonical ground state energies is satisfactory. For example when investigating clusteredness of a graph in a certain sense the GSE notion becomes useless, because the partition for which energies attain the minimal value are trivial partitions. On the other hand, in many applications one only asks for a lower bound on the size of these classes to keep a grade of freedom of the ground state case and at the same time achieve a balance with respect to the sizes of classes. Our setting can be regarded as an intermediate energy notion that manages to get rid of values corresponding to trivial partitions. Recall Definition 1.7 of the lower threshold ground state energies.

The next theorem will deliver an upper bound on the difference of the MGSEs of G and W_G for fixed $\mathbf{a}$ and J, W_G is the graphon constructed from the adjacency matrix of G in the natural way. A straightforward consequence of this will be the analogous statement for the LTGSEs.

Theorem 3.1. *[4] Let G be a weighted graph, $q \geq 1$, $\mathbf{a} \in \mathrm{Pd}_q$, and $J \in \mathrm{Sym}_q$. Then*

$$
\left| \hat{\mathcal{E}}_{\mathbf{a}}(G, J) - \mathcal{E}_{\mathbf{a}}(W_G, J) \right| \leq 6q^3 \frac{\alpha_{\max}(G)}{\alpha_G} \beta_{\max}(G) \|J\|_\infty.
$$

Since the upper bound in the theorem for a given q does not depend on $\mathbf{a}$, it is easily possible to apply it to the LTGSEs.

Corollary 3.1. *Let G be a weighted graph, $q \geq 1$, $0 \leq c \leq 1/q$, and $J \in \mathrm{Sym}_q$. Then*

$$\left| \hat{\mathcal{E}}^c(G, J) - \mathcal{E}^c(W_G, J) \right| \leq 6q^3 \frac{\alpha_{\max}(G)}{\alpha_G} \beta_{\max}(G) \|J\|_\infty.$$

Based on the preceding facts we are able to perform analysis on the LTGSEs in the same way as the authors of [4] did in the case of MGSE.

Corollary 3.2. *Let G_n be a sequence of weighted graphs with uniformly bounded edge weights, and $\frac{\alpha_{\max}(G_n)}{\alpha_{G_n}} \to 0$ ($n \to \infty$). Then for all $q \geq 1$, $0 \leq c \leq 1/q$, and $J \in \mathrm{Sym}_q$ the sequences $(\hat{\mathcal{E}}^c(G_n, J))_{n \geq 1}$ converge if, and only if $(\mathcal{E}^c(W_{G_n}, J))_{n \geq 1}$ converge, and in that case*

$$\lim_{n \to \infty} \hat{\mathcal{E}}^c(G_n, J) = \lim_{n \to \infty} \mathcal{E}^c(W_{G_n}, J).$$

Recall the definition of testability, Definition 1.9. It was shown in [3], among presenting other characterizations, that the testability of a graph parameter f is equivalent to the existence of a $\delta_\square$-continuous extension $\hat{f}$ of f to the space $\mathcal{W}_I$, where extension here means that $f(G_n) - \hat{f}(W_{G_n}) \to 0$ whenever $|V(G_n)| \to \infty$ (see [3], Theorem 6.1, the equivalence of (a) and (d)). Using this we are able to present yet another consequence of Theorem 3.1, that was verified earlier using a different approach in [2] (see also [1], Chapter 4).

Corollary 3.3. *For all $q \geq 1$, $0 \leq c \leq 1/q$, and $J \in \mathrm{Sym}_q$ the simple graph parameter $f(G) = \hat{\mathcal{E}}^c(G, J)$ is testable. Choosing J appropriately, $f(G)$ can be regarded as a type of balanced multiway minimal cut in [2].*

Proof. Let $q \geq 1$, $0 \leq c \leq 1/q$ and $J \in \mathrm{Sym}_q$ be fixed, and we define $\hat{f}(W) = \mathcal{E}^c(W, J)$. It follows from Corollary 3.1 that $f(G_n) - \hat{f}(W_{G_n}) \to 0$ whenever $|V(G_n)| \to \infty$. It remains to show that $\hat{f}$ is $\delta_\square$-continuous. To elaborate on this issue, let $U, W \in \mathcal{W}_I$ and ϕ be a measure-preserving permutation of $[0, 1]$ such that $\delta_\square(U, W) = \|U - W^\phi\|_\square$, and let $\rho = (\rho_1, \ldots, \rho_q)$ be an arbitrary fractional partition. Then

$$\left| \mathcal{E}_\rho(U, J) - \mathcal{E}_\rho(W^\phi, J) \right| \leq \sum_{i,j=1}^{q} |J_{ij}| \left| \int_{[0,1]^2} (U - W^\phi)(x, y) \rho_i(x) \rho_j(y) \mathrm{d}x \mathrm{d}y \right|$$

$$(3.1) \qquad\qquad\qquad \leq q^2 \|J\|_\infty \|U - W^\phi\|_\square = q^2 \|J\|_\infty \delta_\square(U, W).$$

This implies our claim, as $\mathcal{E}_{\mathbf{a}}(W, J) = \mathcal{E}_{\mathbf{a}}(W^\phi, J)$ for any $\mathbf{a} \in \mathrm{Pd}_q$, and any ϕ that is a measure preserving permutation, and the fact that the right-hand side of (3.1) does not depend on $\mathbf{a}$, and that by definition $\mathcal{E}^c(W, J) = \inf_{\mathbf{a} \in A_c} \mathcal{E}_{\mathbf{a}}(W, J)$. ∎

In order to analyze the convergence relationship of LTGSEs with different thresholds for a given graph sequence it is sensible to consider c as a function of q. We restrict our attention to lower threshold functions c with $c(q)q$ being constant, which means that in the case of graphons the total size of the thresholds stays the same relative to the size of the interval $[0, 1]$ (in the case of graphs relative to the cardinality of the vertex set). The main statement of the current section informally asserts that the convergence of LTGSEs with larger lower threshold imply convergence of all LTGSEs with smaller ones. By the results of the previous section we know that in the case of $c(q) = 1/q$ the convergence of these LTGSEs is equivalent convergence of the MGSEs for all probability distributions, and by this, according to [4], to left convergence of graphs. Moreover, in the case of $c(q) = 0$ it is equivalent to the convergence of the unrestricted GSEs, that property is known to be strictly weaker than left convergence. For technical purposes we introduce *general* LTGSEs and will refer to the previously presented notion in all that follows as *homogeneous* LTGSEs.

Definition 3.1. *Let $q \geq 1$, $\mathbf{x} = (x_1, \ldots, x_q)$, $x_1, \ldots x_q \geq 0$, and $\sum_{i=1}^{q} x_i \leq 1$, and let $A_{\mathbf{x}} = \{\mathbf{a} \in \mathrm{Pd}_q \colon a_i \geq x_i, i = 1, \ldots, q\}$. For a graphon W and $J \in \mathrm{Sym}_q$ we call the following expression the lower threshold ground state energy corresponding to $\mathbf{x}$:*

$$\mathcal{E}^{\mathbf{x}}(W, J) = \inf_{\mathbf{a} \in A_{\mathbf{x}}} \mathcal{E}_{\mathbf{a}}(W, J).$$

The definition of $\hat{\mathcal{E}}^{\mathbf{x}}(G, J)$ for graphs is analogous.

Similarly to Lemma 2.1, the convergence of homogeneous LTGSEs is equivalent to the convergence of certain general LTGSEs.

Lemma 3.1. *Let I be a bounded interval, $(W_n)_{n \geq 1}$ a sequence of graphons in $\mathcal{W}_I$. Let c be a lower threshold function, so that $c(q)q = h$ for all $q \geq 1$ and for some $0 \leq h \leq 1$. If for all $q \geq 1$, and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{c(q)}(W_n, J))_{n \geq 1}$ converge, then for all $q \geq 1$, all*

$$(3.2) \qquad \mathbf{x} = (x_1, \ldots, x_q) \quad x_1, \ldots, x_q \geq 0 \qquad \sum_{i=1}^{q} x_i = h,$$

and $J \in \mathrm{Sym}_q$, the sequences $(\mathcal{E}^{\mathbf{x}}(W_n, J))_{n \geq 1}$ also converge.

Proof. Fix an arbitrary graphon W from $\mathcal{W}_I$, $q \geq 1$, and $J \in \mathrm{Sym}_q$, and an arbitrary vector $\mathbf{x}$ that satisfies condition (3.2). Select for each of these vectors $\mathbf{x}$ a positive vector $\mathbf{x}'$ that obeys the condition (3.2), and that has components which are integer multiples of $c(q')$ (q' will be chosen later), so that

$$\|\mathbf{x} - \mathbf{x}'\|_1 \leq 2qc(q') = 2h\frac{q}{q'}.$$

The sets $A_{\mathbf{x}}$ and $A_{\mathbf{x}'}$ have Hausdorff distance in the L^1-norm at most $\|\mathbf{x} - \mathbf{x}'\|_1$, in particular for every $\mathbf{a} \in A_{\mathbf{x}}$ there exists a $\mathbf{b} \in A_{\mathbf{x}'}$, such that $\|\mathbf{a} - \mathbf{b}\|_1 \leq \|\mathbf{x} - \mathbf{x}'\|_1$, and vice versa. Let $\varepsilon > 0$ be arbitrary, and $\mathbf{a} \in A_{\mathbf{x}}$ be such that $\mathcal{E}^{\mathbf{x}}(W, J) + \varepsilon > \mathcal{E}_{\mathbf{a}}(W, J)$ holds. Then by applying Lemma 2.2 we have that

$$\begin{aligned}
\mathcal{E}^{\mathbf{x}'}(W, J) - \mathcal{E}^{\mathbf{x}}(W, J) &< \mathcal{E}^{\mathbf{x}'}(W, J) - \mathcal{E}_{\mathbf{a}}(W, J) + \varepsilon \\
&\leq \mathcal{E}_{\mathbf{b}}(W, J) - \mathcal{E}_{\mathbf{a}}(W, J) + \varepsilon \\
&\leq 2\|\mathbf{a} - \mathbf{b}\|_1 \|W\|_\infty \|J\|_\infty + \varepsilon \\
&\leq 2\|\mathbf{x} - \mathbf{x}'\|_1 \|W\|_\infty \|J\|_\infty + \varepsilon.
\end{aligned}$$

The lower bound of the difference can be handled similarly, and therefore by the arbitrary choice of ε it holds that

$$\left| \mathcal{E}^{\mathbf{x}'}(W, J) - \mathcal{E}^{\mathbf{x}}(W, J) \right| \leq 2\|\mathbf{x} - \mathbf{x}'\|_1 \|W\|_\infty \|J\|_\infty \leq 4h\frac{q}{q'}\|I\|_\infty \|J\|_\infty.$$

With completely analogous line of thought to the proof of Lemma 2.1, one can show that there exists a $J' \in \mathrm{Sym}_{q'}$ such that $\mathcal{E}^{\mathbf{x}'}(W, J) = \mathcal{E}^{c(q')}(W, J')$. Finally, choose q' small enough in order to satisfy $4h\frac{q}{q'}\|I\|_\infty \|J\|_\infty < \frac{\varepsilon}{3}$, and $n_0 > 0$ large enough, so that for all $m, n \geq n_0$ the relation

$$\left| \mathcal{E}^{c(q')}(W_n, J') - \mathcal{E}^{c(q')}(W_m, J') \right| < \frac{\varepsilon}{3}$$

holds.

Then for all $m, n \geq n_0$:

$$\begin{aligned}
|\mathcal{E}^{\mathbf{x}}(W_n, J) - \mathcal{E}^{\mathbf{x}}(W_m, J)| &< \left| \mathcal{E}^{\mathbf{x}}(W_n, J) - \mathcal{E}^{\mathbf{x}'}(W_n, J) \right| \\
&\quad + \left| \mathcal{E}^{c(q')}(W_n, J') - \mathcal{E}^{c(q')}(W_m, J') \right| + \left| \mathcal{E}^{\mathbf{x}'}(W_m, J) - \mathcal{E}^{\mathbf{x}}(W_m, J) \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

We did not only prove the statement of the lemma, but we also showed that the convergence is uniform in the sense that n_0 does not depend on $\mathbf{x}$ for fixed q and J. ∎

With the aid of the former lemma we can now prove that if all homogeneous LTGSEs with large thresholds converge, then all homogeneous LTGSEs with smaller ones also converge.

Theorem 3.2. *Let I be a bounded interval, $(W_n)_{n\geq 1}$ a sequence of graphons in $\mathcal{W}_I$. Let c_1, c_2 be two lower threshold functions, so that $c_1(q)q = h_1 < h_2 = c_2(q)q$ for all $q \geq 1$ for some $0 \leq h_1, h_2 \leq 1$. If for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{c_2(q)}(W_n, J))_{n\geq 1}$ converge, then for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{c_1(q)}(W_n, J))_{n\geq 1}$ also converge.*

Proof. From Lemma 3.1 it follows that if the conditions of the theorem are satisfied then for every $q \geq 1$, every

$$(3.3) \qquad \mathbf{x} = (x_1, \ldots, x_q) \quad x_1, \ldots x_q \geq 0 \qquad \sum_{i=1}^{q} x_i = h_2,$$

and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{\mathbf{x}}(W_n, J))_{n\geq 1}$ converge, for fixed q and J uniformly in $\mathbf{x}$.

Fix q. Our aim is to find for all $\mathbf{a} \in A_{c_1(q)}$ an $\mathbf{x}$, so that the condition (3.3) is satisfied, $\mathbf{a} \in A_{\mathbf{x}}$, and $A_{\mathbf{x}} \subseteq A_{c_1(q)}$, where the last two properties are equivalent to saying that $c_1(q) \leq x_i \leq a_i$ for $i = 1, \ldots, q$. As $h_1 < h_2 \leq 1$, there exists such an $\mathbf{x}$ for all $\mathbf{a} \in A_{c_1(q)}$, let us denote it by $\mathbf{x_a}$, for convenience set $(\mathbf{x_a})_i = \frac{h_1}{q} + \frac{a_i - h_1}{1 - h_1}(h_2 - h_1)$. According to this correspondence we have $A_{c_1(q)} = \bigcup_{\mathbf{a} \in A_{c_1(q)}} A_{\mathbf{x_a}}$. So for an arbitrary graphon W and $J \in \mathrm{Sym}_q$ we have

$$\mathcal{E}^{c_1(q)}(W, J) = \inf_{\mathbf{a} \in A_{c_1(q)}} \mathcal{E}^{\mathbf{x_a}}(W, J).$$

We fix $\varepsilon > 0$, $J \in \mathrm{Sym}_q$, and apply Lemma 3.1 for the case that the conditions of the theorem are satisfied. Then there exists a $n_0 \in \mathbb{N}$, so that for all $n, m > n_0$, and for all $\mathbf{x}$ which satisfies (3.3), and we have that

$$|\mathcal{E}^{\mathbf{x}}(W_n, J) - \mathcal{E}^{\mathbf{x}}(W_m, J)| < \varepsilon.$$

Let $\varepsilon' > 0$ be arbitrary and $\mathbf{b} \in A_{c_1(q)}$ such that $\mathcal{E}^{c_1(q)}(W_m, J) + \varepsilon' > \mathcal{E}^{\mathbf{x_b}}(W_m, J)$. Then

$$\mathcal{E}^{c_1(q)}(W_n, J) - \mathcal{E}^{c_1(q)}(W_m, J) < \mathcal{E}^{c_1(q)}(W_n, J) - \mathcal{E}^{\mathbf{x_b}}(W_m, J) + \varepsilon'$$
$$\leq \mathcal{E}^{\mathbf{x_b}}(W_n, J) - \mathcal{E}^{\mathbf{x_b}}(W_m, J) + \varepsilon' < \varepsilon + \varepsilon'.$$

The lower bound of $\mathcal{E}^{c_1(q)}(W_n, J) - \mathcal{E}^{c_1(q)}(W_m, J)$ can be established similarly and as ε' was arbitrary, it follows that

$$\left| \mathcal{E}^{c_1(q)}(W_n, J) - \mathcal{E}^{c_1(q)}(W_m, J) \right| < \varepsilon,$$

which verifies the statement of the theorem. ∎

A direct consequence is the version of Theorem 3.2 for weighted graphs.

Corollary 3.4. *Let G_n be a sequence of weighted graphs with uniformly bounded edge weights, and $\frac{\alpha_{\max}(G_n)}{\alpha_{G_n}} \to 0$ $(n \to \infty)$. Let c_1 and c_2 be two lower threshold functions, so that $c_1(q)q = h_1 < h_2 = c_2(q)q$ for all $q \geq 1$ for some $0 \leq h_1, h_2 \leq 1$. If for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\hat{\mathcal{E}}^{c_2(q)}(G_n, J))_{n \geq 1}$ converge, then for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\hat{\mathcal{E}}^{c_1(q)}(G_n, J))_{n \geq 1}$ also converge.*

The proof of Corollary 3.4 can be easily given through the combination of the results of Theorem 3.1 and Theorem 3.2.

Concluding this section we would like to mention a natural variant of the LTGSEs, the upper threshold ground state energies (UTGSE). Here we will only give an informal description of the definition and the results and leave the details to the reader, everything carries through analogously to the above. The homogeneous UTGSE, denoted by $\hat{\mathcal{E}}^{c\uparrow}(G, J)$, is determined by a formula similar to (1.7) with the set A_c replaced by A^c, that is the set of probability distributions whose components are at most c, the general variant of the UTGSE is defined in the same manner. The equivalence corresponding to the one stated in Lemma 3.1 between the convergence of the general and the homogeneous version follows by the same blow-up trick as there, here for $c(q)q = h \geq 1$. The counterpart of Theorem 3.2 also holds true in the following form for $1 \leq c_2(q)q \leq c_1(q)q \leq q$: If for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{c_2(q)\uparrow}(W_n, J))_{n \geq 1}$ converge, then for every $q \geq 1$ and $J \in \mathrm{Sym}_q$ the sequences $(\mathcal{E}^{c_1(q)\uparrow}(W_n, J))_{n \geq 1}$ also converge. This conclusion comes not unexpected, it says, as in the LTGSE case, that less restriction on the set A^c weakens the convergence property of a graph sequence.

4. Counterexamples

In this section we provide an example of a graphon family whose elements can be distinguished for a larger $c_2(q)$ lower threshold function for some pair of $q_0 \geq 1$ and $J_0 \in \mathrm{Sym}_q$ by looking at $\mathcal{E}^{c_2(q_0)}(W, J_0)$, but whose LTGSEs are identical for some smaller $c_1(q)$ lower threshold function for all $q \geq 1$ and $J \in \mathrm{Sym}_q$. Based on this it is possible to construct a sequence of graphs, whose $c_1(q)$-LTGESs converge for every $q \geq 1$ and $J \in \mathrm{Sym}_q$, but not the $c_2(q)$-LTGSEs through the same randomized method presented in [4] to show a non-convergent graph sequence with convergent ground state energies.

In the second part of the section we demonstrate that there exist a family of graphons, where elements can be distinguished from each other by looking only at their LTGSEs for an arbitrary small, but positive $c(q)$ lower threshold function, but whose corresponding GSEs without any threshold are identical.

Example 4.1. An example which can be treated relatively easily are block-diagonal graphons which are defined for the parameters $0 \leq \alpha \leq 1$, $0 \leq \beta_1, \beta_2$ as

$$W(x,y) = \begin{cases} \beta_1, & \text{if } 0 \leq x, y \leq \alpha, \\ \beta_2, & \text{if } \alpha < x, y \leq 1, \\ 0, & \text{else.} \end{cases}$$

In the case of $c(q)q = h$, $1 - \alpha \geq h$, $\beta_2 = 0$, for arbitrary $q \geq 1$ and $J \in \text{Sym}_q$ we have $\mathcal{E}(W, J) = \mathcal{E}^{c(q)}(W, J)$. Choosing $\beta_1 = \frac{1}{\alpha^2}$, we get a one parameter family of graphons which have identical $c(q)$-LTGSEs parametrized by α with $0 < \alpha \leq 1 - h$ for every $W \in \mathcal{W}_{[0,1]}$. This means that $\mathcal{E}^{c(q)}(W(\alpha), J) = \mathcal{E}(\mathbb{I}, J)$, where $\mathbb{I}$ stands for the constant 1 graphon.

For every $\alpha_0 > 1 - h$ there are $q \geq 1$ and $J \in \text{Sym}_q$, so that the former equality does not hold anymore. Let $J_q \in \text{Sym}_q$ be the $q \times q$ matrix, whose diagonal entries are 0, all other entries being -1 (this is the q-partition mincut problem). Then $\mathcal{E}(\mathbb{I}, J_q) = 0$ for all $q \geq 1$, but for q_0 large enough $\mathcal{E}^{c(q_0)}(W(\alpha_0), J_{q_0}) > 0$, we leave the details to the reader.

With the aid of the previous example it is possible to construct a sequence of graphs which verify that in Theorem 3.2 the implication of the convergence property of the sequence is strictly one-way. This example is degenerate in the sense that the graphs consist of a quasi-random part and a sub-dense part with the bipartite graph spanned between the two parts also being sub-dense.

Example 4.2. Let us consider block-diagonal graphons with $0 < \alpha < 1$, $\beta_1, \beta_2 > 0$. It was shown in [4] that if we restrict our attention to a subfamily of block-diagonal graphons, where $\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2$ is constant, then in these subfamilies the corresponding GSEs are identical. Let $c(q)$ be an arbitrarily small positive threshold function. Next we will show that the $c(q)$-LTGSEs determine the parameters of the block-diagonal graphon at least for a one-parameter family (up to graphon equivalence, since $(\alpha, \beta_1, \beta_2)$ belongs to the same equivalence class as $(1 - \alpha, \beta_2, \beta_1)$). The constant δ_{ij} is 1, when $i = j$, and 0 otherwise.

The value of the expression $\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2$ is determined by the MAXCUT problem by $\mathcal{E}(W, J)$ with $q = 2$ and $J_{ij} = 1 - \delta_{ij}$.

In the second step let q_0 be as large so that $c(q_0) < \min\{\alpha, 1 - \alpha\}$ holds, and let J be the $q_0 \times q_0$ matrix with entries $J_{ij} = -\delta_{i1}\delta_{j1}$. In this case $\mathcal{E}(W, J) = 0$, but simple calculus gives $-\mathcal{E}^{c(q_0)}(W, J) = -\frac{\beta_1 \beta_2}{\beta_1 + \beta_2} c(q_0)^2$. Hence $\frac{1}{\beta_1} + \frac{1}{\beta_2}$ is determined by the LTGSEs.

The extraction of a third dependency of the parameters from $c(q)$-LTGSEs requires little more effort, we will only sketch details here. First consider α's with $\min\{\alpha, 1 - \alpha\} \geq c(2)$. Let for $q = 2$ and $k \geq 1$ be

$$J_k = \begin{pmatrix} 1 & -k \\ -k & 2 \end{pmatrix}.$$

For every α with $\min\{\alpha, 1 - \alpha\} \geq c(2)$ we have

$$\lim_{k \to \infty} -\mathcal{E}^{c(2)}(W, J_k) = \alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 + \max\{\alpha^2 \beta_1, (1 - \alpha)^2 \beta_2\}.$$

Now apply the notion of the general lower threshold. For $q = 2$ let $c_1(n) = 2c(2)/n$ and $c_2(n) = 2c(2)(n - 1)/n$ two threshold functions, let us first consider the threshold $\mathbf{x}_n = (c_1(n), c_2(n))$. If $\alpha \geq c_1(n)$ or $1 - \alpha \geq c_1(n)$, then analogously to the case of the homogeneous lower thresholds

$$\lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}_n}(W, J_k) = 2\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 \quad \text{or} \quad \alpha^2 \beta_1 + 2(1 - \alpha)^2 \beta_2.$$

If for example $\alpha < c_1(n)$, then it is easy to see that the LTGSE tends to infinity, because for some k_0, for all $k > k_0$ we have

$$-\mathcal{E}^{\mathbf{x}_n}(W, J_k) < -k(c_1(n) - \alpha)c_2(n) \min\{\beta_1, \beta_2\} + 2(\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2).$$

So for fixed n then
$$\lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}_n}(W, J_k) = -\infty.$$

In order to extract the expression $\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 + \max\{\alpha^2 \beta_1, (1 - \alpha)^2 \beta_2\}$, we only have to consider the lower threshold obtained by swapping the bounds, $\mathbf{x}'_n = (c_2(n), c_1(n))$.

Then, if $\alpha \geq c_1(n)$ or $1 - \alpha \geq c_1(n)$, we have

$$\max\left\{ \lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}'_n}(W, J_k), \lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}'_n}(W, J_k) \right\}$$
$$= \alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 + \max\{\alpha^2 \beta_1, (1 - \alpha)^2 \beta_2\},$$

otherwise

$$\max\left\{ \lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}_n}(W, J_k), \lim_{k \to \infty} -\mathcal{E}^{\mathbf{x}'_n}(W, J_k) \right\} = -\infty.$$

For every α there is a minimal n_0 so that one of the conditions $\alpha \geq c_1(n)$ and $1 - \alpha \geq c_1(n)$ is satisfied, and for $n < n_0$ the LTGSEs corresponding to $\mathbf{x}_n$ and $\mathbf{x}_{n'}$ tend to infinity when k goes to infinity. Therefore the expression $\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 + \max(\alpha^2 \beta_1, (1 - \alpha)^2 \beta_2)$ is determined by $c(q)$-LTGSEs.

Consider the one-parameter block-diagonal graphon family analyzed in [4], that is $W(\alpha) = W(\alpha, \frac{1}{\alpha}, \frac{1}{1-\alpha})$, where $0 < \alpha < 1$. In this case the values of our first two expressions are constant, for every $0 < \alpha < 1$ we have

$$\frac{1}{\beta_1} + \frac{1}{\beta_2} = 1,$$
$$\alpha^2 \beta_1 + (1 - \alpha)^2 \beta_2 = 1.$$

But by applying the third expression for $c(q)$-LTGSEs, we extract $\max\{\alpha^2\beta_1, (1-\alpha)^2\beta_2\} = \max\{\alpha, 1-\alpha\}$, which determines the graphon uniquely in this family up to equivalence.

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A. Krámli
Bolyai Institute
University of Szeged
Szeged, Hungary
kramli@math.u-szeged.hu

R. Markó
Hausdorff Center for Mathematics
University of Bonn
Bonn, Germany
roland.marko@hcm.uni-bonn.de