ORTHOGONALITY RELATIONS FOR CONTINUOUS WAVELET TRANSFORMS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

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Abstract. Some orthogonality relations are proved for the continuous wavelet transform.

1. Introduction

It is well-known for Fourier transforms that $\langle f,g\rangle = \left\langle \widehat{f},\widehat{g}\right\rangle$ and $\|f\|_2 = \left\|\widehat{f}\right\|_2$, where $f,g\in L_2(\mathbb{R})$ (see e.g. [4] or [8]). The analogous result can also be found in the literature (e.g. Chui [2], Daubechies [3] or Gröchenig [5]) for continuous wavelet transforms,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_{g_1}f_1(x,s)\overline{W_{g_2}f_2(x,s)}\,\frac{dx\,ds}{s^2} = C_{g_1,g_2}\left\langle f_1,f_2\right\rangle,$$

where

$$C_{g_1,g_2} := \int_{\mathbb{R}} \overline{\widehat{g_1}(s)}\widehat{g_2}(s)\,\frac{ds}{|s|}$$

and $W_g f$ denotes the continuous wavelet transform of $f\in L_2(\mathbb{R})$ with respect to a wavelet $g\in L_2(\mathbb{R})$. However, the proofs are incomplete and superficial.

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Usually the following fact is used: if $\ell : \mathbb{X} \to \mathbb{Y}$ is a linear operator between the two Banach spaces $\mathbb{X}$, $\mathbb{Y}$ and $\ell : \mathbb{X}_0 \to \mathbb{Y}$ is bounded, then ℓ is also bounded from $\mathbb{X}$ to $\mathbb{Y}$, where $\mathbb{X}_0 \subset \mathbb{X}$ is dense in $\mathbb{X}$. However, this is not necessarily true (see Meyer, Taibleson and Weiss [6], Bownik [1] and also Weisz [7]). In this paper we give an exact proof of the orthogonality result for continuous wavelet transforms.

2. The continuous wavelet transform

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k \qquad \text{and} \qquad \|x\|_r := \left(\sum_{k=1}^{d} |x_k|^r \right)^{1/r}$$

with the usual modification for $r = \infty$.

We briefly write $L_p(\mathbb{R}^d)$ instead of $L_p(\mathbb{R}^d, \lambda)$ space equipped with the norm (or quasi-norm)

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \leq \infty),$$

where λ is the Lebesgue measure. A function f is in the space $L_p(\mathbb{R}^{d+1}, s^{-d-1} d\lambda)$ if for $d = 1$

$$\|f\|_{L_p(\mathbb{R}^2, s^{-2} d\lambda)} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,s)|^p \, \frac{dx\, ds}{s^2} \right)^{1/p} < \infty$$

and for $d > 1$

$$\|f\|_{L_p(\mathbb{R}^{d+1}, s^{-d-1} d\lambda)} := \left(\int_0^\infty \int_{\mathbb{R}^d} |f(x,s)|^p \, \frac{dx\, ds}{s^{d+1}} \right)^{1/p} < \infty.$$

Of course the space $L_p(\mathbb{R}^{d+1}, s^{-d-1} d\lambda)$ is a Banach space.

The *Fourier transform* of a tempered distribution is denoted by $\widehat{f}$. If $f \in L_1(\mathbb{R}^d)$ then

$$\widehat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi\imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$. *Translation* and *dilation* of a function f are defined, respectively, by

$$T_x f(t) := f(t - x), \qquad D_s f(t) := |s|^{-d/2} f(s^{-1} t),$$

where $t, x \in \mathbb{R}^d$, $s \in \mathbb{R}$, $s \neq 0$. In this paper we will investigate the continuous wavelet transform. The *continuous wavelet transform* of f with respect to a wavelet g is defined by

$$W_g f(x,s) := |s|^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(s^{-1}(t-x))}\, dt = \langle f, T_x D_s g\rangle,$$

$(x \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0)$ when the integral does exist.

Plancherel's theorem is well-known for Fourier transforms: if $f, g \in L_2(\mathbb{R})$, then

$$\langle f, g\rangle = \left\langle \widehat{f}, \widehat{g} \right\rangle \qquad \text{and} \qquad \|f\|_2 = \left\| \widehat{f} \right\|_2 .$$

In the next sections we will consider the analogues of these results for continuous wavelet transforms. Since the proofs of the analogues are superficial in the literature, we will give their exact proofs.

3. Orthogonality results

In this section we present the one-dimensional results.

Theorem 3.1. *Suppose that $g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and*

$$C_g := \int_{\mathbb{R}} |\widehat{g}(s)|^2 \, \frac{ds}{|s|} < \infty. \tag{3.1}$$

If $f \in L_2(\mathbb{R})$, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |W_g f(x,s)|^2 \, \frac{dx\, ds}{s^2} = C_g \|f\|_2^2.$$

Proof. It is easy to see that

$$W_g f(x,s) = (f * D_s g^*)(x),$$

where

$$g^*(y) := \overline{g(-y)}$$

is the involution. By Young's inequality $W_g f(\cdot, s) \in L_2(\mathbb{R})$ for each fixed $s \neq 0$, because

$$\|W_g f(\cdot,s)\|_2 = \|(f * D_s g^*)(\cdot)\|_2 \le \|f\|_2 \|D_s g^*\|_1 = |s|^{1/2} \|f\|_2 \|g\|_1 .$$

So we can take the Fourier transform of $W_g f(\cdot, s)$ in the first variable for each fixed $s \neq 0$:

$$\widehat{W_g f}(\omega, s) = \widehat{f}(\omega)\widehat{D_s g^*}(\omega) = |s|^{1/2} \widehat{f}(\omega)\widehat{g^*}(s\omega) = |s|^{1/2} \widehat{f}(\omega)\overline{\widehat{g}(s\omega)}.$$

By Plancherel's theorem

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |W_g f(x,s)|^2 \,\frac{dx\,ds}{s^2} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |W_g f(x,s)|^2 \,dx\right) \frac{ds}{s^2} = \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|\widehat{W_g f}(\omega,s)\right|^2 \,d\omega\right) \frac{ds}{s^2} = \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |s| \left|\widehat{f}(\omega)\overline{\widehat{g}(s\omega)}\right|^2 \,d\omega\right) \frac{ds}{s^2} = \\
&= \int_{\mathbb{R}} \left|\widehat{f}(\omega)\right|^2 \left(\int_{\mathbb{R}} |\widehat{g}(s\omega)|^2 \,\frac{ds}{|s|}\right) d\omega.
\end{aligned}$$

Substituting $t = s\omega$ in the inner integral, we can see that

$$\int_{\mathbb{R}} |\widehat{g}(s\omega)|^2 \,\frac{ds}{|s|} = \int_{\mathbb{R}} |\widehat{g}(t)|^2 \,\frac{dt}{|t|} = C_g,$$

which proves the theorem. ■

Usually a similar proof can be found in the literature (see e.g. [2, 3, 5]), however, the theorem is stated for $g \in L_2(\mathbb{R})$. The preceding proof does not show the result for all $g \in L_2(\mathbb{R})$ with $C_g < \infty$. More generally, suppose that $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces, $\mathbb{X}_0 \subset \mathbb{X}$ is dense in $\mathbb{X}$. If a linear operator ℓ is defined on $\mathbb{X}$ and $\ell : \mathbb{X}_0 \to \mathbb{Y}$ is bounded, then ℓ is not necessarily bounded from $\mathbb{X}$ to $\mathbb{Y}$ (see Meyer, Taibleson and Weiss [6], Bownik [1] and also Weisz [7]). Of course, the unique extension of $\ell\big|_{\mathbb{X}_0}$ is bounded from $\mathbb{X}$ to $\mathbb{Y}$, however, it is not sure that the extension is equal to ℓ on the whole $\mathbb{X}$. Now we prove the preceding result for all $g \in L_2(\mathbb{R})$ precisely.

Theorem 3.2. *If $g \in L_2(\mathbb{R})$ with $C_g < \infty$ and $f \in L_2(\mathbb{R})$, then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |W_g f(x,s)|^2 \,\frac{dx\,ds}{s^2} = C_g \|f\|_2^2. \tag{3.2}$$

Proof. Supposing that $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, we can repeat the preceding proof to obtain (3.2), where $g \in L_2(\mathbb{R})$ with $C_g < \infty$. Let us fix such a function g and consider the linear operator

$$T(f) := W_g f \qquad (f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})).$$

Then by (3.2) T is bounded from the space $(L_1(\mathbb{R}) \cap L_2(\mathbb{R}), \|\cdot\|_2)$ to the Banach space $L_2(\mathbb{R}^2, s^{-2} d\lambda)$. Since $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, we can extend T to the whole $L_2(\mathbb{R})$ space uniquely. Let us denote this extension again by T. From this it follows that

$$T(f) = \lim_{n\to\infty} T(f_n) \qquad \text{in } L_2(\mathbb{R}^2, s^{-2} d\lambda)\text{-norm}, \tag{3.3}$$

where $f_n \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $f \in L_2(\mathbb{R})$ and $f = \lim_{n\to\infty} f_n$ in $L_2(\mathbb{R})$-norm. This implies

$$\begin{aligned}\|T(f)\|_{L_2(\mathbb{R}^2,s^{-2}d\lambda)} &= \lim_{n\to\infty} \|T(f_n)\|_{L_2(\mathbb{R}^2,s^{-2}d\lambda)} \\ &= C_g^{1/2} \lim_{n\to\infty} \|f_n\|_2 = C_g^{1/2} \|f\|_2.\end{aligned}$$

We have to prove only, that $T(f) = W_g f$ for all $f \in L_2(\mathbb{R})$. Notice that

$$|W_g f(x,s) - W_g f_n(x,s)| = |\langle f - f_n, T_x D_s g\rangle| \le \|f - f_n\|_2 \, \|g\|_2 \to 0$$

almost everywhere, i.e.

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} W_g f_n = W_g f \qquad \text{a.e.}$$

This and (3.3) complete the proof. ■

We can easily give conditions such that (3.1) is satisfied. If $g \in L_2(\mathbb{R})$ and $\widehat{g} = 0$ on the interval $(-\epsilon,\epsilon)$, or $|\widehat{g}(\omega)| \le C|\omega|^\alpha$ on $(-\epsilon,\epsilon)$ $(\epsilon, \alpha > 0)$, then C_g is finite. If $g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, then $\widehat{g}$ is continuous, so (3.1) imply $\widehat{g}(0) = \int_{\mathbb{R}} g(x)\,dx = 0$. If this condition is satisfied and we impose a slightly stronger condition than integrability on g, namely

$$\int_{\mathbb{R}} (1 + |x|)\,|g(x)|\; dx < \infty,$$

then $|\widehat{g}(\omega)| \le C\,|\omega|$, hence C_g is finite.

Now we formulate the preceding result for the scalar product of two wavelet transforms.

Theorem 3.3. *Suppose that $g_1, g_2 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and*

$$C_{g_1,g_2} := \int_{\mathbb{R}} \overline{\widehat{g_1}(s)}\widehat{g_2}(s)\,\frac{ds}{|s|}.$$

is a finite number. If $f_1, f_2 \in L_2(\mathbb{R})$, then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} W_{g_1} f_1(x,s)\overline{W_{g_2} f_2(x,s)}\,dx \right) \frac{ds}{s^2} = C_{g_1,g_2} \left\langle f_1, f_2 \right\rangle.$$

Proof. As in Theorem 3.1,

$$\begin{aligned}&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} W_{g_1} f_1(x,s)\overline{W_{g_2} f_2(x,s)}\,dx \right) \frac{ds}{s^2} = \\ &\quad= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (f_1 * D_s g_1^*)(x)\overline{(f_2 * D_s g_2^*)(x)}\,dx \right) \frac{ds}{s^2} = \\ &\quad= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\widehat{f_1 * D_s g_1^*}\right)(\omega)\overline{\left(\widehat{f_2 * D_s g_2^*}\right)(\omega)}\,d\omega \right) \frac{ds}{s^2} = \\ &\quad= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |s|\,\widehat{f_1}(\omega)\overline{\widehat{g_1}(s\omega)}\,\overline{\widehat{f_2}(\omega)\overline{\widehat{g_2}(s\omega)}}\,d\omega \right) \frac{ds}{s^2}.\end{aligned}$$

By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)}\, dx \right) \frac{ds}{s^2} &= \\ &= \int_{\mathbb{R}} \widehat{f_1}(\omega) \overline{\widehat{f_2}(\omega)} \left(\int_{\mathbb{R}} \overline{\widehat{g_1}(s\omega)} \widehat{g_2}(s\omega)\, \frac{ds}{|s|} \right) d\omega = \\ &= \int_{\mathbb{R}} \widehat{f_1}(\omega) \overline{\widehat{f_2}(\omega)} \left(\int_{\mathbb{R}} \overline{\widehat{g_1}(t)} \widehat{g_2}(t)\, \frac{dt}{|t|} \right) d\omega = \\ &= C_{g_1,g_2} \int_{\mathbb{R}} \widehat{f_1}(\omega) \overline{\widehat{f_2}(\omega)}\, d\omega,\end{aligned}$$

which proves the theorem. ■

The preceding result is stated often for all $g_1, g_2 \in L_2(\mathbb{R})$. However, in this case we have to suppose that $C_{g_1} < \infty$ and $C_{g_2} < \infty$.

Theorem 3.4. *Suppose that $g_1, g_2 \in L_2(\mathbb{R})$, $C_{g_1} < \infty$ and $C_{g_2} < \infty$. If $f_1, f_2 \in L_2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)}\, \frac{dx\, ds}{s^2} = C_{g_1,g_2} \langle f_1, f_2 \rangle . \tag{3.4}$$

Proof. By Hölder's inequality C_{g_1,g_2} is a finite number. We can see as in the previous proof that (3.4) holds for all $f_1, f_2 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and $g_1, g_2 \in \in L_2(\mathbb{R})$, $C_{g_1} < \infty$ and $C_{g_2} < \infty$. Here we can apply Fubini's theorem, because $W_{g_j} f_j \in L_2(\mathbb{R}^2, s^{-2} d\lambda)$ $(j = 1, 2)$ by Theorem 3.2.

Let us fix the functions $f_2 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and $g_1, g_2 \in L_2(\mathbb{R})$ and consider the linear operators

$$T_1(f_1) := \int_{\mathbb{R}} \int_{\mathbb{R}} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)}\, \frac{dx\, ds}{s^2} \qquad (f_1 \in L_2(\mathbb{R}))$$

and

$$U_1(f_1) := C_{g_1,g_2} \langle f_1, f_2 \rangle \qquad (f_1 \in L_2(\mathbb{R})).$$

We have seen before that

$$T_1(f_1) = U_1(f_1) \qquad (f_1 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})).$$

By Hölder's inequality U_1 is a bounded functional from the whole $L_2(\mathbb{R})$ space to $\mathbb{C}$, and because of Theorem 3.2, the same holds for T_1. By the unique extension theorem $U_1 = T_1$, i.e., (3.4) holds for all functions $f_1, g_1, g_2 \in L_2(\mathbb{R})$ and $f_2 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

Similarly, for fixed $f_1, g_1, g_2 \in L_2(\mathbb{R})$ consider the linear operators

$$T_2(f_2) := \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{W_{g_1} f_1(x,s)} W_{g_2} f_2(x,s)\, \frac{dx\, ds}{s^2} \qquad (f_2 \in L_2(\mathbb{R}))$$

and

$$U_2(f_2) := \overline{C_{g_1,g_2}\langle f_1, f_2\rangle} \qquad (f_2 \in L_2(\mathbb{R})).$$

We can show as before that $U_2 = T_2$, hence (3.4) is satisfied for all functions $f_1, f_2, g_1, g_2 \in L_2(\mathbb{R})$. ■

4. Higher dimensional results

In this section we formulate the analogous theorems of the preceding section for higher dimensions. A function f is radial with respect to the norm $\|\cdot\|_r$ (briefly r-radial) if there exists a one-variable function η such that $f(t) = \eta(\|t\|_r)$. If $r = 2$ we call the function simply radial. If $\widehat{g_j}$ is r-radial, then let $\mu_j(\|x\|_r) := \widehat{g_j}(x)$, $j = 1, 2$ and

$$C_{g_j} = \int_0^\infty |\mu_j(s)|^2 \,\frac{ds}{s}, \qquad C_{g_1,g_2} = \int_0^\infty \overline{\mu_1}(s)\mu_2(s)\,\frac{ds}{s}.$$

This yields that

$$\begin{aligned} \int_0^\infty \overline{\widehat{g_1}}(s\omega)\widehat{g_2}(s\omega)\,\frac{ds}{s} &= \int_0^\infty \overline{\widehat{g_1}}\left(s\frac{\omega}{\|\omega\|_r}\right)\widehat{g_2}\left(s\frac{\omega}{\|\omega\|_r}\right)\frac{ds}{s} \\ &= \int_0^\infty \overline{\mu_1}(s)\mu_2(s)\,\frac{ds}{s} \end{aligned}$$

for almost all $\omega \in \mathbb{R}^d$. The following results can be proved as in the one-dimensional case.

Theorem 4.1. *Suppose that $g \in L_2(\mathbb{R}^d)$ and $\widehat{g}$ is an r-radial function such that $C_g < \infty$. If $f \in L_2(\mathbb{R}^d)$, then*

$$\int_0^\infty \int_{\mathbb{R}^d} |W_g f(x,s)|^2 \,\frac{dx\,ds}{s^{d+1}} = C_g \|f\|_2^2.$$

Theorem 4.2. *Suppose that $g_1, g_2 \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $\widehat{g_1}$ and $\widehat{g_2}$ are r-radial functions such that C_{g_1,g_2} is a finite number. If $f_1, f_2 \in L_2(\mathbb{R}^d)$, then*

$$\int_0^\infty \left(\int_{\mathbb{R}^d} W_{g_1} f_1(x,s)\overline{W_{g_2} f_2(x,s)}\,dx\right)\frac{ds}{s^{d+1}} = C_{g_1,g_2}\langle f_1, f_2\rangle.$$

Theorem 4.3. *Suppose that $g_1, g_2 \in L_2(\mathbb{R}^d)$, $\widehat{g}$ and $\widehat{\gamma}$ are r-radial functions such that $C_{g_1} < \infty$ and $C_{g_2} < \infty$. If $f_1, f_2 \in L_2(\mathbb{R}^d)$, then*

$$\int_0^\infty \int_{\mathbb{R}^d} W_{g_1} f_1(x,s)\overline{W_{g_2} f_2(x,s)}\,\frac{dx\,ds}{s^{d+1}} = C_{g_1,g_2}\langle f_1, f_2\rangle.$$

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