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SOME IDENTITIES WITH APPLICATIONS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

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Abstract. For positive integers r computable formulas for the sums of the doubly infinite series $\sum_{k\in\mathbb{Z}} 1/(z-k\pi)^r$ and $\sum_{k\in\mathbb{Z}}(-1)^k/(z-k\pi)^r$ will be presented. As applications exact lower and upper bounds for the derivatives of the functions cot and $1/\sin$ will be also shown.

1. The sums of the series $\sum\limits_{k\in\mathbb{Z}} \frac{1}{(z-k\pi)^r}$ and $\sum\limits_{k\in\mathbb{Z}} \frac{(-1)^k}{(z-k\pi)^r}$

Let us denote the set of integers, positive integers and complex numbers by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$, respectively.

1.1. It is clear that the doubly infinite series $\sum_{k\in\mathbb{Z}} \frac{1}{(z-k\pi)^r}$ is absolutely convergent on the domain

$$D := \mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\}$$

and uniformly convergent in every compact subset of D, if $r = 2, 3, \ldots$. Its sum function

$$A_r(z) := \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} \qquad (z \in D)$$

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is a π-periodic meromorphic function on $\mathbb{C}$ with rth order poles at the points $k\pi$ $(k \in \mathbb{Z})$.

If $r = 1$, then the above doubly infinite series does not converge (see for example $z = -1/2$), but the sequence of its symmetric partial sums is absolutely convergent and

$$\lim_{n\to+\infty} \sum_{k=-n}^{n} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right) = \cot z \qquad (z \in D),$$

moreover the convergence is uniform in every compact subset of D, see [3, p. 310]. Therefore let us define the function $A_1(z)$ by the following way

$$A_1(z) := \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{1}{z - k\pi} = \cot z \qquad (z \in D). \tag{1}$$

By the partial fraction decomposition of the function $1/\sin^r$ explicit formulas can be obtained for A_{2r} (see [7]). Using another method we give formulas for *all* functions A_r $(r \in \mathbb{N})$.

Theorem 1. *Let $r = 2, 3, \ldots$. The function A_r can be written in the following form*

$$A_r(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z - k\pi)^r} = \frac{1}{\sin^r z} S_r(\cos z) \qquad (z \in D), \tag{2}$$

where S_r are algebraic polynomials of degree $\le (r-2)$ $(S_r \in \mathcal{P}_{r-2}$ shortly). They satisfy the recursive relation:

$$\begin{aligned} S_2(z) &= 1, \\ S_{r+1}(z) &= zS_r(z) + \frac{1 - z^2}{r} S_r'(z) \\ &\qquad (z \in \mathbb{C},\ r = 2, 3, \ldots). \end{aligned} \tag{3}$$

Proof. If $r = 2$ then

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(z - k\pi)^2} = \frac{1}{\sin^2 z} = \frac{1}{\sin^2 z} S_2(\cos z) \qquad (z \in D)$$

(see [5, p. 246]). In general case, we prove the statement by induction. Suppose that for an $r \in \mathbb{N}$ we have

$$\sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z - k\pi)^r} = S_r(\cos z) \qquad (z \in D).$$

After derivation, we get

$$r\sin^{r-1} z \cdot \cos z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} - r\sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} =$$

$$= -S_r'(\cos z)\cdot \sin z,$$

thus

$$\sin^{r+1} z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} = \cos z \cdot \sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} +$$

$$+\frac{1}{r} S_r'(\cos z)\cdot \sin^2 z =$$

$$= \cos z \cdot S_r(\cos z) + \frac{1-\cos^2 z}{r} S_r'(\cos z) = S_{r+1}(\cos z),$$

which means that the statement is true for $(r+1)$, too. ■

The first few S_r polynomials obtained from the above recursive relation are as follows:

$$\begin{aligned}
S_2(z) &= 1,\\
S_3(z) &= z,\\
S_4(z) &= \tfrac{1}{3}(2z^2+1),\\
S_5(z) &= \tfrac{1}{3}(z^3+2z),\\
S_6(z) &= \tfrac{1}{15}(2z^4+11z^2+2),\\
S_7(z) &= \tfrac{1}{45}(2z^5+26z^3+17z),\\
S_8(z) &= \tfrac{1}{315}(4z^6+114z^4+180z^2+17),\\
S_9(z) &= \tfrac{1}{315}(z^7+60z^5+192z^3+62z),\\
S_{10}(z) &= \tfrac{1}{2835}(2z^8+247z^6+1452z^4+1072z^2+62).
\end{aligned} \tag{4}$$

Consequently for A_r we obtain the following formulas which are valid at all points $z \in D$. The convergence is absolute in every $z \in D$ and uniform in every compact subset of D.

$$A_2(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^2} = \frac{1}{\sin^2 z},$$

$$A_3(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^3} = \frac{1}{\sin^3 z}\cos z,$$

$$A_4(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^4} = \frac{1}{\sin^4 z}\left[\tfrac{1}{3} + \tfrac{2}{3}\cos^2 z\right],$$

$$A_5(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^5} = \frac{1}{\sin^5 z}\left[\tfrac{2}{3}\cos z + \tfrac{1}{3}\cos^3 z\right],$$

$$A_6(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^6} = \frac{1}{\sin^6 z}\left[\tfrac{2}{15} + \tfrac{11}{15}\cos^2 z + \tfrac{2}{15}\cos^4 z\right],$$

$$A_7(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^7} = \frac{1}{\sin^7 z}\left[\tfrac{17}{45}\cos z + \tfrac{26}{45}\cos^3 z + \tfrac{2}{45}\cos^5 z\right].$$

1.2. Now we consider the doubly infinite series $\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(z-k\pi)^r}$, which is absolutely convergent on D and uniformly convergent in every compact subset of D for every $r = 2, 3, \ldots$. The sum

$$A_r^{\pm}(z) := \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^r} \qquad (z \in D, \quad r = 2, 3, \ldots).$$

is a π-periodic meromorphic function on $\mathbb{C}$ with rth order poles at the points $k\pi$ $(k \in \mathbb{Z})$.

If $r = 1$, then

$$\lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} = \frac{1}{\sin z} \qquad (z \in D),$$

and the convergence is absolute in D and uniform in every compact subset of D, see [5, p. 246]. Let

$$A_1^{\pm}(z) := \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} = \frac{1}{\sin z} \qquad (z \in D). \tag{5}$$

By the partial fraction decomposition of the function $1/\sin^r$ explicit formulas can be obtained for $A_{2r-1}^{\pm}$ (see [7]). Using another method we give formulas for *all* functions $A_r^{\pm}$ $(r \in \mathbb{N})$.

Theorem 2. *Let $r = 1, 2, \ldots$. The function $A_r^{\pm}$ can be written in the following form*

$$A_r^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^r} = \frac{1}{\sin^r z} Q_r(\cos z) \qquad (z \in D),$$

where $Q_r \in \mathcal{P}_{r-1}$, satisfies the recursive relation:

$$\begin{gathered} Q_1(z) = 1, \\ Q_{r+1}(z) = zQ_r(z) + \frac{1-z^2}{r} Q_r'(z) \\ (z \in \mathbb{C}, \quad r = 1, 2, \ldots). \end{gathered} \tag{6}$$

The proof of this statement is similar to the proof of Theorem 1, so we omit the details.

The first few Q_r polynomials obtained from the above recursive relation are as follows:

$$
\begin{aligned}
Q_1(z) &= 1,\\
Q_2(z) &= z,\\
Q_3(z) &= \tfrac{1}{2}(z^2+1),\\
Q_4(z) &= \tfrac{1}{6}(z^3+5z),\\
Q_5(z) &= \tfrac{1}{24}(z^4+18z^2+5),\\
Q_6(z) &= \tfrac{1}{120}(z^5+58z^3+61z),\\
Q_7(z) &= \tfrac{1}{720}(z^6+179z^4+479z^2+61),\\
Q_8(z) &= \tfrac{1}{5040}(z^7+543z^5+3111z^3+1385z),\\
Q_9(z) &= \tfrac{1}{40320}(z^8+1636z^6+18270z^4+19028z^2+1385).
\end{aligned}
\tag{7}
$$

Consequently for $A_r^{\pm}$ we obtain the following formulas which are valid at all points $z \in D$. The convergence is absolute in every $z \in D$ and uniform in every compact subset of D.

$$A_1^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)} = \frac{1}{\sin z},$$

$$A_2^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^2} = \frac{1}{\sin^2 z}\cos z,$$

$$A_3^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^3} = \frac{1}{\sin^3 z}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos^2 z\right],$$

$$A_4^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^4} = \frac{1}{\sin^4 z}\left[\tfrac{5}{6}\cos z+\tfrac{1}{6}\cos^3 z\right],$$

$$A_5^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^5} = \frac{1}{\sin^5 z}\left[\tfrac{5}{24}+\tfrac{3}{4}\cos^2 z+\tfrac{1}{24}\cos^4 z\right],$$

$$A_6^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^6} = \frac{1}{\sin^6 z}\left[\tfrac{61}{120}\cos z+\tfrac{29}{60}\cos^3 z+\tfrac{1}{120}\cos^5 z\right],$$

$$A_7^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^7} = \frac{1}{\sin^7 z}\left[\tfrac{61}{720}+\tfrac{479}{720}\cos^2 z+\tfrac{179}{720}\cos^4 z+\tfrac{1}{720}\cos^6 z\right].$$

2. Inequalities

In this section we shall consider the functions A_r and $A_r^{\pm}$ on $\mathbb{R}$ and shall give exact lower and upper estimates.

2.1. Let us define the functions*:

$$U_r(x) := (\sin^r x) \cdot A_r(x) = \sum_{k=-\infty}^{+\infty} \frac{\sin^r x}{(x-k\pi)^r} \tag{8}$$
$$(x \in \mathbb{R}, \quad r = 1, 2, \ldots).$$

Theorem 1 states that U_r is an algebraic polynomial of the cos function:

$$U_r(x) = S_r(\cos x) \qquad (x \in \mathbb{R}, r = 1, 2, 3, \ldots),$$

where S_r ($\in \mathcal{P}_{r-2}$) satisfies the recursive relation (3).

Theorem 3. (i) *Let r be any positive and even integer. Then U_r is a π-periodic even function and*

$$m(U_r) \le U_r(x) \le 1 \qquad (x \in \mathbb{R}), \tag{9}$$

where

$$m(U_r) = S_r(0) = \frac{2^r(2^r-1)}{r!}|B_r| \quad (r = 2, 4, 6, \ldots). \tag{10}$$

The values of $S_r(0)$ $(r = 2, 4, 6, \ldots)$ can be computed using the recursive relation (3), B_r denotes the rth Bernoulli number. On the interval $[0, \pi/2]$ the upper (lower) bound is attained exactly at the point $x = 0$ $(x = \pi/2)$.

(ii) *Let r be any positive and odd integer. Then U_r is a 2π-periodic even function and*

$$-1 \le U_r(x) \le 1 \qquad (x \in \mathbb{R}).$$

On the interval $[0, \pi]$ the upper (lower) bound is attained exactly at the point $x = 0$ $(x = \pi)$.

Remark 1. By (4) the first few values of $S_r(0)$ are

$$S_2(0) = 1, \quad S_4(0) = \tfrac{1}{3}, \quad S_6(0) = \tfrac{2}{15}, \quad S_8(0) = \tfrac{17}{315}, \quad S_{10} = \tfrac{62}{2835}.$$

*At points $x \in \mathbb{R}$ for which the function is formally undefined but has a finite limit, it is defined to be its limit.

Remark 2. We recall that the Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ satisfy the recurrence relation

$$B_0 = 1,$$
$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \cdots + \binom{n}{n-1}B_{n-1} = 0 \qquad (n = 2, 3, \ldots)$$

(see [9] or [6, I, p. 682]). The first few Bernoulli numbers B_n are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}$$

with $B_{2n+1} = 0$ for $n \in \mathbb{N} \setminus \{1\}$.

Proof of Theorem 3. Let r be an arbitrary positive integer. Then by (1) and (2) we have $U_r(x) = S_r(\cos x)$ $(x \in \mathbb{R})$. From the recursive relation (3) it follows that (by induction)

- the polynomial S_r is even (odd) if r is even (odd),
- every coefficients of S_r are nonnegative,
- $S_r(1) = 1$ and $S_r(-1) = (-1)^r$.

Using these facts we obtain every statements of Theorem 3. We show only the assertion with respect to the minimum of U_r, if $r = 2, 4, 6, \ldots$. Using (2) we have

$$\min_{x\in\mathbb{R}} U_r(x) = \min_{x\in\mathbb{R}} S_r(\cos x) = S_r\left(\cos\frac{\pi}{2}\right) = S_r(0) =$$
$$= \sum_{k=-\infty}^{+\infty} \frac{\sin^r \frac{\pi}{2}}{\left(\frac{\pi}{2} - k\pi\right)^r} = \frac{2^r}{\pi^r} \sum_{k=-\infty}^{+\infty} \frac{1}{(2k-1)^r}.$$

Since

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(2k-1)^r} = \frac{2(2^r-1)}{2^r} \sum_{k=1}^{+\infty} \frac{1}{k^r} \qquad (r = 2, 4, 6, \ldots),$$

and (see [6, p. 684] or [9, (40)])

$$B_r = 2(-1)^{r/2-1} \frac{r!}{(2\pi)^r} \sum_{k=1}^{+\infty} \frac{1}{k^r} \qquad (r = 2, 4, 6, \ldots), \tag{11}$$

thus we have

$$\min_{x\in\mathbb{R}} U_r(x) = m(U_r) = S_r(0) = \frac{2^r(2^r-1)}{r!}|B_r| \qquad (r = 2, 4, 6, \ldots) \qquad \blacksquare$$

The first few values of $m(U_r)$ $(r = 2, 4, \dots)$ are

$$m(U_2) = 1, \quad m(U_4) = \frac{1}{3}, \quad m(U_6) = \frac{2}{15}, \quad m(U_8) = \frac{17}{315}, \quad m(U_{10}) = \frac{62}{2835}.$$

Since for every $r = 2, 4, 6, \dots$

$$0 < m(U_r) = \frac{2^r(2^r-1)}{r!}|B_r| = 2\,\frac{2^r-1}{\pi^r}\sum_{k=1}^{+\infty}\frac{1}{k^r} < 4\,\frac{2^r-1}{\pi^r},$$

thus we have

$$\lim_{r\to+\infty} m(U_r) = 0.$$

Remark 3. From (11) we obtain the exact values of the Rieman's zeta-function at positive even integers:

$$\zeta(r) = \sum_{k=1}^{+\infty}\frac{1}{k^r} = (-1)^{r/2-1}\frac{(2\pi)^r}{2\cdot r!}B_r \quad (r = 2, 4, 6, \dots).$$

Thus $\zeta(r)$ can be computed recursively using the Bernoulli numbers or the values $S_r(0)$.

Remark 4. In the theory of wavelet analysis the exact lower and upper bounds for the functions $U_r(x)$ $(x \in \mathbb{R})$ have important applications (see [2], [4, p. 24]). For positive even integers the inequality (9) is known, see [2, p. 90]. There the following explicit form for the lower bound $m(U_r)$ is proved:

$$m(U_r) = \frac{1}{(r-1)!}\prod_{k=1}^{r/2-1}\frac{(1+\lambda_k)^2}{|\lambda_k|} \qquad (r = 2, 4, 6, \dots),$$

where λ_k's are the roots of the Euler–Frobenius polynomials. But the exact values of λ_k's are not known. The explicit forms (10) are more simple and they are computable, moreover the above proof of (9) is also simpler than in [2].

2.2. Let

$$V_r(x) := (\sin^r x)\cdot A_r^{\pm}(x) = \sum_{k=-\infty}^{+\infty}(-1)^k\frac{\sin^r x}{(x-k\pi)^r} \tag{12}$$

$$(x \in \mathbb{R}, \quad r = 1, 2, \dots).$$

Theorem 2 states that V_r is an algebraic polynomial of the cos function:

$$V_r(x) = Q_r(\cos x) \qquad (x \in \mathbb{R}, r = 1, 2, 3, \dots),$$

where Q_r $(\in \mathcal{P}_{r-1})$ satisfies the recursive relation (6).

Theorem 4. (i) *Let r be an arbitrary positive odd integer. Then V_r is a π-periodic even function and*

$$m(V_r) \leq V_r(x) = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{\sin^r x}{(x-k\pi)^r} \leq 1 \qquad (x \in \mathbb{R},\ r = 1,3,5,\ldots),$$

where

$$m(V_r) = Q_r(0) \quad (r = 1,3,5,\ldots). \tag{13}$$

The values of $Q_r(0)$ $(r = 1,3,5,\ldots)$ can be computed by the recursive relations (6)*. On the interval $[0,\pi/2]$ the upper* (*lower*) *bound is attained exactly at the point $x = 0$ $(x = \pi/2)$.*

(ii) *Let r be any positive and even integer. Then V_r is a 2π-periodic even function and*

$$-1 \leq V_r(x) \leq 1 \qquad (x \in \mathbb{R},\ r = 2,4,6,\ldots).$$

On the interval $[0,\pi]$ the upper (*lower*) *bound is attained exactly at the point $x = 0$ $(x = \pi)$.*

Proof. Theorem 2 states that $V_r(x) = Q_r(\cos x)$ $(x \in \mathbb{R})$, where Q_r is an even (odd) algebraic polynomial if r odd (even); every coefficients of Q_r are nonnegative; $Q_r(1) = 1$ and $Q_r(-1) = (-1)^{r+1}$. From these facts we obtain every statements of Theorem 4. ■

By (7) and (13) the first few values of $m(V_r)$ are

$$m(V_1) = 1, \quad m(V_3) = \tfrac{1}{2}, \quad m(V_5) = \tfrac{5}{24}, \quad m(V_7) = \tfrac{61}{720}, \quad m(V_9) = \tfrac{277}{8064}. \tag{14}$$

Remark 5. Since the algebraic polynomial Q_r is even if r is odd, thus

$$\min_{x\in\mathbb{R}} V_r(x) = \min_{x\in\mathbb{R}} Q_r(\cos x) = Q_r\left(\cos\tfrac{\pi}{2}\right) = Q_r(0) =$$

$$= \sum_{k=-\infty}^{+\infty} (-1)^k \frac{\sin^r \frac{\pi}{2}}{\left(\frac{\pi}{2} - k\pi\right)^r} = \frac{2^r}{\pi^r} \sum_{k=-\infty}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r} = \frac{2^{r+1}}{\pi^r} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r}.$$

Consequently we obtain the following exact values for the sum of the series $\sum_{k\in\mathbb{N}} (-1)^{k+1}/(2k-1)^r$:

$$F_r := \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r} = \frac{\pi^r}{2^{r+1}} Q_r(0) \qquad (r = 1,3,5,\ldots).$$

By (14) we have

$$F_1 = \frac{1}{4}\pi, \quad F_3 = \frac{1}{32}\pi^3, \quad F_5 = \frac{5}{1536}\pi^3, \quad F_7 = \frac{61}{184320}\pi^7, \quad F_9 = \frac{277}{8257536}\pi^9.$$

3. Derivatives

Not so easy to find explicit forms for the derivatives of the cot or of the $1/\sin$ functions. Starting from their series representation different computable formulas may be given. Exact lower and upper bounds for the derivatives will be also presented.

3.1. Let us consider first the derivatives of cot. From the identity

$$\cot z = \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{1}{z-k\pi} \qquad (z \in D)$$

(here the convergence is uniform in every compact subset of D) we have

$$\cot^{(r)} z = \frac{d^r}{dz^r} \cot z = (-1)^r r! \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} \qquad (z \in D, \quad r = 1, 2, \ldots).$$

The idea is that, the above sum can be written as algebraic polynomials of some trigonometric functions. We recall the following known result, which states that $\cot^{(r)}$ can be expressed as a polynomial of the cot function.

Theorem 5. (See [1, p. 161], [4, p. 23].) *Let r be a positive integer. Then we have*

$$\frac{d^r}{dz^r} \cot z = (-1)^r P_r(\cot z) \qquad (z \in D),$$

where the algebraic polynomial P_r of degree $(r+1)$ obeys the following recursive relation

$$P_0(z) = z, \qquad P_{r+1}(z) = (1+z^2)P_r'(z) \qquad (z \in \mathbb{C}).$$

Consequently for every $z \in D$ we have

$$\begin{aligned} \cot' z &= -(\cot^2 z + 1), \\ \cot'' z &= 2\cot^3 z + 2\cot z, \\ \cot^{(3)} z &= -(6\cot^4 z + 8\cot^2 z + 2), \\ \cot^{(4)} z &= 24\cot^5 z + 40\cot^3 z + 16\cot z. \end{aligned}$$

From Theorem 1 it follows that the derivatives of the cot function can be expressed also by polynomials of the cos function.

Theorem 6. *Let r be a positive integer. Then we have*

$$\frac{d^r}{dz^r}\cot z = (-1)^r\, r!\, A_{r+1}(z) = \frac{(-1)^r\, r!}{\sin^{r+1} z}\, S_{r+1}(\cos z) \qquad (z \in D), \tag{15}$$

where the algebraic polynomials S_r are given by the recursive relation (3).

Using (4) we have for every $z \in D$

$$\begin{aligned}
\cot' z &= \frac{-1}{\sin^2 z},\\
\cot'' z &= \frac{2}{\sin^3 z}\cos z,\\
\cot^{(3)} z &= \frac{-6}{\sin^4 z}\left[\tfrac{1}{3} + \tfrac{2}{3}\cos^2 z\right] = \frac{-2}{\sin^4 z}\left[1 + 2\cos^2 z\right],\\
\cot^{(4)} z &= \frac{4!}{\sin^5 z}\left[\tfrac{2}{3}\cos z + \tfrac{1}{3}\cos^3 z\right] = \frac{8}{\sin^5 z}\left[2\cos z + \cos^3 z\right].
\end{aligned}$$

The main advantage of the above representation of $\cot^{(r)}$ is that the their exact lower and upper bounds can be obtained more easily then by using Theorem 5. Indeed, from (8) and (15) we have:

$$\begin{aligned}
\mathrm{SC_r}(x) := \sin^{r+1} x \cdot \cot^{(r)} x &= (-1)^r\, r!\, \sin^{r+1} x \cdot A_{r+1}(x) =\\
&= (-1)^r\, r!\, U_{r+1}(x) = (-1)^r\, r!\, S_{r+1}(\cos x)\\
&\qquad (x \in \mathbb{R},\ \ r = 1, 2, 3, \ldots).
\end{aligned}$$

Thus by Theorem 3 we have

Theorem 7. (i) *Let r be an odd positive integer. Then the function $\mathrm{SC_r}$ is a π-periodic even function and*

$$0 < r!\, m(U_{r+1}) \le -\mathrm{SC_r}(x) = -\sin^{r+1} x \cdot \cot^{(r)} x \le r!$$
$$(x \in \mathbb{R},\ \ r = 1, 3, 5, \ldots),$$

where $m(U_{r+1})$ is given by (10). *On the interval* $[0, \pi/2]$ *the upper* (*lower*) *bound is attained exactly at the point $x = 0$* ($x = \pi/2$).

(ii) *Let r be an even positive integer. Then the function $\mathrm{SC_r}$ is a 2π-periodic even function and*

$$-r! \le \mathrm{SC_r}(x) = \sin^{r+1} x \cdot \cot^{(r)} x \le r!$$
$$(x \in \mathbb{R},\ \ r = 2, 4, 6, \ldots).$$

On the interval $[0, \pi]$ *the upper* (*lower*) *bound is attained exactly at the point $x = 0$* ($x = \pi$).

3.2. The derivatives of the csc $:= 1/\sin$ function can be obtained from the identity

$$\csc z = \frac{1}{\sin z} = \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} \qquad (z \in D)$$

(here the convergence is uniform in every compact subset of D):

$$\csc^{(r)} z = \left(\frac{1}{\sin}\right)^{(r)}(z) = (-1)^r r! \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^{r+1}}$$
$$(z \in D, \quad r = 1, 2, \ldots).$$

From Theorem 2 it follows that the derivatives of the $1/\sin$ function can be expressed also by polynomials of the cos function.

Theorem 8. *Let r be a positive integer. Then we have*

$$\csc^{(r)} z = (-1)^r\, r!\, A^{\pm}_{r+1}(z) = \frac{(-1)^r\, r!}{\sin^{r+1} z}\, Q_{r+1}(\cos z) \qquad (z \in D), \tag{16}$$

where the algebraic polynomials Q_r are given by the recursive relation (6).

Using (7) we have for every $z \in D$

$$\begin{aligned}
\csc' z &= \frac{-1}{\sin^2 z} \cdot \cos z, \\
\csc'' z &= \frac{1}{\sin^3 z}[1 + \cos^2 z], \\
\csc^{(3)} z &= \frac{-1}{\sin^4 z}\left[5\cos z + \cos^3 z\right], \\
\csc^{(4)} z &= \frac{1}{\sin^5 z}\left[5 + 18\cos^2 z + \cos^4 z\right].
\end{aligned}$$

From (12) and (16) we have:

$$\begin{aligned}
\mathrm{SCS_r}(x) := \sin^{r+1} x \cdot \csc^{(r)} x &= (-1)^r\, r!\, \sin^{r+1} x \cdot A^{\pm}_{r+1}(x) = \\
&= (-1)^r\, r!\, V_{r+1}(x) = (-1)^r\, r!\, Q_{r+1}(\cos x)
\end{aligned}$$
$$(x \in \mathbb{R}, \quad r = 1, 2, 3, \ldots).$$

Thus by Theorem 4 we have

Theorem 9. (i) *Let r be an even positive integer. Then the function* $\mathrm{SCS_r}$ *is a π-periodic even function and*

$$0 < r!\, m(V_{r+1}) \le -\mathrm{SCS_r}(x) = -\sin^{r+1} x \cdot \csc^{(r)} x \le r!$$
$$(x \in \mathbb{R}, \ r = 1, 3, 5, \ldots),$$

where $m(V_{r+1})$ is given by (14). On the interval $[0, \pi/2]$ the upper (lower) bound is attained exactly at the point $x = 0$ ($x = \pi/2$).

(ii) *Let r be an odd positive integer. Then the function $\mathrm{SCS_r}$ is a 2π-periodic even function and*

$$-r! \le \mathrm{SCS_r}\,(x) = \sin^{r+1} x \cdot \csc^{(r)} x \le r!$$
$$(x \in \mathbb{R},\ r = 2, 4, 6, \ldots).$$

On the interval $[0, \pi]$ the upper (lower) bound is attained exactly at the point $x = 0$ ($x = \pi$).

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