

DIVISOR FUNCTION $\tau_3(\omega)$ IN ARITHMETIC PROGRESSION

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*Dedicated to Professors Zoltán Daróczy and Imre Kátai
on their 75th anniversary*

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Abstract. We constructed the asymptotic formula over the ring of the Gaussian integers for summatory function of the divisor function $d_3(\omega)$ in an arithmetic progression $N(\omega) \equiv \ell \pmod{q}$ which is non-trivial for $q \leq x^{\frac{2}{7}-\varepsilon}$

1. Introduction

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers and $k \geq 2$, $k \in \mathbb{N}$. We define the divisor function $\tau_k(w)$, $w \in \mathbb{Z}[i]$ as the coefficient of $N(w)^s$ in the Dirichlet series

$$Z^k(s) = \sum_w^* \frac{\tau_k(w)}{N(w)^s}, \quad \Re s > 1.$$

(Here, as usually, $N(w) = |w|^2$ is a norm of a Gaussian integer w , $Z(s)$ is the Hecke Z -function).

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The main point of this paper is to consider the case $k = 3$ and to construct an asymptotic formula of the sum

$$D_k(x; \ell, q) = \sum_{\substack{N(w) \equiv \ell \pmod{q}, \\ N(w) \leq x}} \tau_k(\alpha), \quad (1 \leq \ell < q, (\ell, q) = 1)$$

in particular to investigate the ranges of q and x for which this formula is non-trivial.

The similar problem was considered over $\mathbb{Z}$ in the works of Deshoulieres and Iwaniec[3] and Heath-Brown[4]. Over the ring of integer elements of the quadratic extension of $\mathbb{Q}$ in [6] it has been obtained an asymptotic formula for

$$T_k(x, w_0, \gamma) = \sum_{\substack{w \equiv w_0 \pmod{\gamma}, \\ |N(w)| \leq x}} \tau_k(w), \quad (w_0, \gamma \in \mathbb{Q}(\sqrt{d}), d \in \mathbb{Z}),$$

where γ is a fix integer from $\mathbb{Q}(\sqrt{d})$.

In rational case Deshoulieres and Iwaniec and also Heath-Brown used Deligne's bound for the k -fold Kloosterman sum to estimate $D_3(x; \ell, q)$.

In our paper we use the norm Kloosterman sums for which it is obtained a non-trivial estimate (see, section 3). This let us to construct an asymptotic formula for $D_3(x; \ell, q)$ over $\mathbb{Z}[i]$ that can be applied for investigation of asymptotic behavior for the sum

$$\sum_{N(w) \leq x} \tau(N(w)) \tau_3(N(w)).$$

Notations. We denote $G := \mathbb{Z}[i]$ the ring of the Gaussian integers

$$G = \{a + bi \mid a, b, \in \mathbb{Z}, i^2 = -1\}.$$

For the designation of the Gaussian integers we shall use the Greek letters $\alpha, \beta, \gamma, \xi, \eta$; a Gaussian prime number denote through $\mathfrak{p}$ if $\mathfrak{p} \notin \mathbb{Z}$. For $\alpha \in \mathbb{Z}[i]$ we put $Sp(\alpha) = \alpha + \bar{\alpha} = 2\Re(\alpha)$, $N(\alpha) = \alpha \cdot \bar{\alpha}$, where $\bar{\alpha}$ denotes a complex conjugate with α ; $Sp(\alpha)$ and $N(\alpha)$ we name a trace and a norm (respectively) of α from $\mathbb{Q}(i)$ into $\mathbb{Q}$.

The writing $a \in \mathbb{Z}_q$ (respectively, $a \in G_\gamma$) denotes that $a \in \mathbb{Z}$ (respectively, $\alpha \in G$) and a (respectively, α) runs a complete residue system modulo q (modulo γ). Analogous, $a \in \mathbb{Z}_q^*$ (respectively, $a \in G_\gamma^*$) denotes $a \in \mathbb{Z}$ (respectively, $\alpha \in G$) and runs a reduced residue system modulo q (respectively, modulo γ).

The writing $\sum_{S(C)}$ denotes that the summation runs over the region C which describe extra.

For $A \in \mathbb{N}$ (or $\alpha \in G$) put $\nu_p(A) = a$, (or $\nu_{\mathfrak{p}(\alpha)} = a$) if $p^a \parallel A$ (or $\mathfrak{p}^a \parallel \alpha$).

Moreover, $\exp(z) = e^z$, $e_q(z) = e^{2\pi i \frac{z}{q}}$ for $q \in \mathbb{N}$; the Vinogradov symbol as in $f(x) \ll g(x)$ means that $f(x) = O(g(x))$. The abbreviations " $e_q(\cdot)$ " and " $e\left(\frac{2\pi i \dots}{q}\right)$ " are equal and use depend on the length of certain formula.

For $z \in \mathbb{Z}$ (respectively, $z \in G$), $(z, p) = 1$ let z^{-1} be the multiplicative inverse modulo p^m .

The signs $\sum^*$ and $\prod^*$ mean that the summation (product) conducts by all the non-associated integer (respectively, prime) Gaussian numbers.

2. Preliminary results

We begin this section with a few background definitions and facts.

Let q be a positive integer, $q > 1$, and let χ_q be a Dirichlet character modulo q . Determine the function χ on the ring of Gaussian integers G as $\chi(\alpha) : G \rightarrow \mathbb{C}$, $\chi(\alpha) := \chi_q(N(\alpha))$ for any $\alpha \in G$. It is clear that χ is a character of the group G_q^* .

Lemma 1. *Let ℓ, q be the positive integers, $q > 1$, $\ell \not\equiv 0 \pmod{q}$, and let $J(\ell, q)$ denote the number of the solutions of $x^2 + y^2 \equiv \ell \pmod{q}$. Then we have*

$$J(\ell, q) = E(\ell, q) q \prod_{\substack{p^a \parallel q \\ p \text{ is odd}}} \left\{ 1 - \frac{(\chi_4(p))^{\nu_p((\ell, p^a))+1} - \chi_4(p^{a+1})}{p} + \left(1 - \frac{1}{p}\right) \sum_{b=a-\nu_p((\ell, p^a))}^{a-1} \chi_4(p^{a-b}) \right\},$$

where

$$E(\ell, q) = \begin{cases} 1 & \text{if } q \text{ is odd or } q \equiv 2 \pmod{4}; \\ 1 & \text{if } q \equiv 0 \pmod{4}, \nu_2(\ell) > \nu_2(q) - 2; \\ 2 & \text{if } q \equiv 0 \pmod{4}, \ell \cdot 2^{-\nu_2(\ell)} \equiv 1 \pmod{4}; \\ 0 & \text{if } q \equiv 0 \pmod{4}, \ell \cdot 2^{-\nu_2(\ell)} \equiv 3 \pmod{4}, \end{cases}$$

moreover, the sign ' in production $\prod$ denotes that if $\nu_p(\ell) \geq \nu_p(q)$ then an appropriate multiplier in this production have to be substituted on $1 + \left(1 - \frac{1}{p}\right) \times$

$$\times \sum_{b=0}^{a-1} \chi_4(p^{a-b}).$$

Proof. The statement of Lemma follows by multiplication at q of the function $J(\ell, q)$ and the equation

$$J(\ell, p^a) = \frac{1}{p^a} \sum_{x, y \in \mathbb{Z}_{p^a}} \sum_{z=1}^{p^a} e_{p^a}(z(x^2 + y^2 - \ell))$$

with the subsequent application the formula for the square of Gaussian sum $G(h, p^a) = \sum_{x \in \mathbb{Z}_{p^a}} e_{p^a}(hx)$. ■

Lemma 2. *A non-principal character χ_q produces a non-principal character χ if $q \not\equiv 0 \pmod{4}$. For $q \equiv 0 \pmod{4}$ it has only one non-principal character χ_q producing the principal character χ_0 of group G_q^* .*

Proof. Using the standard representation of a Dirichlet character, it's enough to prove only the case $q = p^n$, p is a prime number. Every rational number a has a norm in $\mathbb{Q}(i)$ that equal to a^2 , and hence, $\chi(a) = \chi_q(a^2)$, and therefore the equation $\chi(a) = 1$, $(a, q) = 1$ can be hold only if $\chi_q(a) = \pm 1$. Consequently, only the real characters χ_q can produce the principal character χ_0 . The basic modulo of the Dirichlet character is an equal to $2^m P$, where $m = 0, 1, 2, 3$, and $P = 1$ or is a square-free odd number. So we should consider only the case of real characters with the basic modulus 2, 4, 8, p . If $p > 2$ be a prime number, for every $b \in \mathbb{Z}$, b is a quadratic non-residue mod p , there exists $\alpha \in G$ such that $N(\alpha) \equiv b \pmod{p}$ (see, Lemma 1), and hence, for only one non-principal real character $\chi_p(x) := \left(\frac{x}{p}\right)$ ($\left(\frac{x}{p}\right)$ is the Legendre symbol) we have $\chi(\alpha) = \chi_p(b) = -1$, i.e. χ is a non-principal character.

For the modulo 2 we have only the principal character. For the modulo 4 a non-principal character χ_4 generates the principal character χ (because $N(\alpha) = 1 \pmod{4}$ if $(N(\alpha), p) = 1$). At last, modulo 8 we have only one the non-principal character χ_8 which inducing χ_4 . So, if $q \not\equiv 0 \pmod{4}$, only the principal character $\chi_{q,0}$ produces the principal character χ_0 , and for $q \equiv 0 \pmod{4}$ we have only one character χ_q inducing the principal character χ_0 . That completes the proof of lemma. ■

Lemma 3. *Let $p > 2$ be a prime number, $a_1, a_2, b_1, b_2, c, m \in \mathbb{Z}$, $(a_1, b_1, a_2, b_2, p) = 1$, $m \geq 1$. Then for the exponential sum*

$$S := \sum_{x, y \in \mathbb{Z}_{p^m}^*} e_{p^m}(a_1 x + a_2 y + b_1 x^2 + b_2 y^2 + pcxy)$$

we have

$$S = \begin{cases} p^m e_{p^m}(A_1 a_1 + A_2 a_2 + B_1 b_1 + B_2 b_2 + \overline{C}_1 c) & \text{if } m \text{ is even,} \\ p^{\frac{m+1}{2}} e_{p^m}(C_1 a_1 + C_2 a_2 + D_1 b_1 + D_2 b_2 + \overline{C}_2 c) & \text{if } m \text{ is odd} \end{cases}$$

where $A_i, B_i, C_i, D_i, \overline{C}_i \in \mathbb{Z}$, $i = 1, 2$.

Proof. First, we suppose that $m = 2n$. Putting

$$x = x_0 + p^n u, \quad y = y_0 + p^n v, \quad x_0, y_0 \in \mathbb{Z}_{p^n}^*, \quad u, v \in \mathbb{Z}_{p^n}$$

we infer

$$S = \sum_{x_0, y_0 \in \mathbb{Z}_{p^n}^*} e_{p^{2n}}(a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + p c x_0 y_0) \times \\ \times \left\{ \sum_{u, v \in \mathbb{Z}_{p^n}} e_{p^n}(a_1 u + a_2 v + 2(b_1 u x_0 + b_2 v y_0) + c p(u y_0 + v x_0)) \right\}.$$

The summation over u, v shows that the inner sum vanishes if at least one of congruence

$$(2.1) \quad \begin{aligned} a_1 + 2b_1 x_0 + p c y_0 &\equiv 0 \pmod{p^n} \\ a_2 + 2b_2 y_0 + p c x_0 &\equiv 0 \pmod{p^n} \end{aligned}$$

violates.

Since $(a_1, a_2, b_1, b_2, p) = 1$ we have only one pair $(x_0, y_0) \in \mathbb{Z}_{p^n}^{*2}$ such that (2.1) holds.

Thus we have

$$S = p^{2n} e_{p^{2n}}(a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + p c x_0 y_0),$$

moreover, x_0, y_0 are linear combinations at a_1, a_2, b_1, b_2 with the coefficients from $\mathbb{Z}_{p^{2n}}$.

Now let $m = 2n + 1$. Putting

$$x = x_0 + p^n u, \quad y = y_0 + p^n v, \quad x_0, y_0 \in \mathbb{Z}_{p^n}^*, \quad u, v \in \mathbb{Z}_{p^{n+1}},$$

modulo p^{2n+1} we have

$$\begin{aligned} x^2 &\equiv x_0^2 + 2p^n x_0 u + p^{2n} u^2, \\ y^2 &\equiv y_0^2 + 2p^n y_0 v + p^{2n} v^2, \\ xy &\equiv x_0 y_0 + p^n (x_0 v + y_0 u) + p^{2n} uv, \end{aligned}$$

and hence,

$$S = \sum_{x_0, y_0 \in \mathbb{Z}_{p^n}^*} e^{\frac{a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + p c x_0 y_0}{p^{2n}}} \times \\ \times \left\{ \sum_{u, v=0}^{p^{n+1}} e^{\frac{a_1 u + a_2 v + 2(b_1 x_0 u + b_2 y_0 v) + p^n (b_1 u^2 + b_2 v^2) + p c (x_0 v + y_0 u)}{p^{n+1}}} \right\}.$$

The inner sum does not turn into zero only if

$$(2.2) \quad \begin{aligned} pcx_0 + a_2 + 2b_2y_0 &\equiv 0 \pmod{p^n}, \\ pcy_0 + a_1 + 2b_1x_0 &\equiv 0 \pmod{p^n}. \end{aligned}$$

From (2.2) it follows that there is only one pair $(x_0, y_0) \in \mathbb{Z}_{p^{n-1}}^2$ for which

$$\begin{aligned} x_0 &\equiv -a_1(2b_1)' + pX_0, \\ y_0 &\equiv -a_2(2b_2)' + pY_0 \end{aligned} \pmod{p^n}.$$

In such case the inner sums over u and v are the Gaussian sums $\pmod{p^{n+1}}$. Consequently, we obtain

$$S = p^{\frac{m+1}{2}} e_{p^m}(D_1x_0 + D_2y_0 + E_1x_0^2 + E_2y_0^2),$$

where D_j, E_j are the linear combinations at a_1, a_2, b_1, b_2 with the coefficients from $\mathbb{Z}_{p^{2n+1}}$. ■

Now, we remind the some necessary information about the Hecke Z-function.

Let $\delta_0, \delta_1 \in \mathbb{Q}(x)$ and let $s \in \mathbb{C}$, $\Re s > 1$. We define the Hecke Z-function by the absolute convergent series

$$Z_m(s; \delta_0, \delta_1) = \sum_{\omega} \frac{e^{4mi \arg \omega}}{N(\omega + \delta_0)^s} e^{2\pi i \Re(\delta_1 \omega)},$$

where $\delta_0, \delta_1 \in \mathbb{Q}(i)$, $m \in \mathbb{Z}$.

In the case $m = 0$, $\delta_0 = \delta_1 = 0$ we write $Z(s)$ instead $Z(s; 0, 0)$.

Lemma 4. *For $m \neq 0$ or $m = 0$ and δ_1 is not a Gaussian integer, $Z_m(s; \delta_0, \delta_1)$ is an entire function. If $m = 0$ and $\delta_1 \in G$, the function Z_m is a holomorphic except as $s = 1$, where it has simple pole with a residue π . Moreover, the functional equation*

$$\pi^{-s} \Gamma(2|m|+s) Z_m(s; \delta_0, \delta_1) = \pi^{-(1-s)} \Gamma(2|m|+1-s) Z_m(1-s; \delta_1, -\delta_0) e^{-2\pi i \Re(\delta_0, \delta_1)}$$

holds in all the cases.

For the proof in the case $\delta_0 = \delta_1 = 0$ see[5]. The proof in other cases is similar.

Corollary 1. *In the domain $-\frac{1}{4} \leq \Re s = \sigma \leq 2$, $|\Im s| = |t| \geq 3$ the following estimates hold:*

$$Z_m(s; 0, 0) \ll \begin{cases} \log^4(m^2 + t^2) & \text{if } 1 \leq \sigma \leq 2; \\ (m^2 + t^2)^{\frac{1-\sigma}{3}} \log^4(m^2 + t^2) & \text{if } \frac{1}{2} \leq \sigma \leq 1; \\ (m^2 + t^2)^{\frac{19-32\sigma}{18}} \log^4(m^2 + t^2) & \text{if } 0 \leq \sigma \leq \frac{1}{2}; \\ (m^2 + t^2)^{1-2\sigma} & \text{if } -\frac{1}{4} \leq \sigma \leq 0. \end{cases}$$

This assertion follows from Lemma 4, an estimate $Z_m(s, 0, 0)$ on half-line (see, (6.16), [1]) and the application of the Phragmen-Lindelöf theorem for an analytic functions in the strip $-\frac{1}{4} \leq \Re s \leq \frac{1}{2}$.

3. Norm Kloosterman sums

Let $\alpha_1, \dots, \alpha_n \in G$, $q > 1$ be a positive integer. Denote,

$$(3.1) \quad \tilde{K}_n := \tilde{K}_n(\alpha_0, \alpha_1, \dots, \alpha_n; q) = \sum_{S(C)} e^{2\pi i \Re\left(\frac{\alpha_0 \xi_0 + \dots + \alpha_n \xi_n}{q}\right)},$$

where $C := \{\xi_j \in G_q, i = 0, 1, \dots, n, N(\xi_0 \dots \xi_n) \equiv 1 \pmod{q}\}$.

$\tilde{K}_n$ we will call the n -fold norm Kloosterman sum.

It's easy to check that for $q_1 q_2 = q$, $(q_1, q_2) = 1$

$$(3.2) \quad \tilde{K}_n(\alpha_0, \dots, \alpha_n; q) = \tilde{K}_n(\alpha_0 \bar{q}'_2, \dots, \alpha_n \bar{q}'_2; q_1) \cdot \tilde{K}_n(\alpha_0 \bar{q}'_1, \dots, \alpha_n \bar{q}'_1; q_2),$$

where $q_1 \bar{q}'_1 \equiv 1 \pmod{q_2}$, $q_2 \bar{q}'_2 \equiv 1 \pmod{q_1}$.

This property reduces our problem to compute $\tilde{K}_n(a_0, \dots, a_n; q)$ for prime power modulus $q = p^m$.

For a prime number p and the Gaussian integer α denote

$$(3.3) \quad r_\alpha = \max_r(p^r | \alpha) = \min(\nu_p(\Re \alpha), \nu_p(\Im \alpha)).$$

In order to estimate $\tilde{K}_n(\alpha_0, \dots, \alpha_n; p^m)$ it is sufficient to study the case

$$\min(r_{\alpha_0}, \dots, r_{\alpha_n}) = 0,$$

i.e. at least one from $\alpha_0, \dots, \alpha_n$ does not divide on p . But then, putting $\alpha_j = a_j + ib_j$, $j = 0, 1, \dots, n$, we have $GCD(a_0, \dots, a_n, b_0, \dots, b_n, p) = 1$.

We consider the case $n = 2$.

First, we assume that $m = 1$.

Let $p \equiv 1 \pmod{4}$.

Denote $\xi_j = x_j + iy_j$, $j = 0, 1, 2$. Then from (3.1) it follows

$$(3.4) \quad \tilde{K}_2 = \sum_{S(C)} e_p(a_0 x_0 + a_1 x_1 + a_2 x_2 - b_0 y_0 - b_1 y_1 - b_2 y_2),$$

where $C := \left\{ x_j, y_j \in \mathbb{Z}_p : \prod_{j=0}^2 (x_j^2 + y_j^2) \equiv 1 \pmod{p} \right\}$.

Let ε_0 be a solution of congruence $x^2 \equiv -1 \pmod{p}$.

Put for every $j = 0, 1, 2$

$$u_j = x_j + \varepsilon_0 y_j, \quad v_j = x_j - \varepsilon_0 y_j.$$

In view of $\xi_j \in G_p^*$ it follows that $N(\xi_j) = x_j^2 + y_j^2 \not\equiv 0 \pmod{p}$ and thus $u_j v_j \equiv x_j^2 + y_j^2 \not\equiv 0 \pmod{p}$. Hence, $u_j, v_j \in \mathbb{Z}_p^*$.

Furthermore, $x_j = 2'(u_j + v_j)$, $y_j = -2'\varepsilon_0(u_j - v_j)$, where $2 \cdot 2' \equiv 1 \pmod{p^m}$.

Consider three cases:

- (i) $\alpha_j \not\equiv 0 \pmod{p}$, $j = 0, 1, 2$;
- (ii) $\alpha_0 \equiv 0 \pmod{p}$, $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0 \pmod{p}$;
- (iii) $\alpha_0 \equiv \alpha_1 \equiv 0 \pmod{p}$, $\alpha_2 \not\equiv 0 \pmod{p}$.

For the first case we have

$$(3.5) \quad \begin{aligned} \tilde{K}_2(\alpha_0, \alpha_1, \alpha_2; p) &= \sum_{S(C)} e_p \left(\sum_{j=0}^2 (A_j u_j + B_j v_j) \right), \\ C &:= \left\{ u_j, v_j \in \mathbb{Z}_p^*, \quad j = 0, 1, 2; \quad \prod_{j=0}^2 u_j v_j \equiv 1 \pmod{p} \right\}. \end{aligned}$$

It is obvious that $A_j \equiv 2'(a_j + \varepsilon_0 b_j) \pmod{p}$, $B_j \equiv 2'(a_j - \varepsilon_0 b_j) \pmod{p}$, and hence, $A_j B_j \equiv 4'(a_j^2 + b_j^2) \equiv 4'N(\alpha_j) \pmod{p}$. But then $(A_j, p) = (B_j, p) = 1$.

By (3.5) and the estimate of P. Deligne[2], we have

$$(3.6) \quad |\tilde{K}| \leq 4p^{\frac{5}{2}}.$$

For the second case we obtain

$$\tilde{K}_2(\alpha_0, \alpha_1, \alpha_2; p) = \sum_{u_0, v_0 \in \mathbb{Z}_p^*} \sum_{S(C)} e_p \left(\sum_{j=1}^2 (A_j u_j + B_j v_j) \right),$$

where

$$C := \left\{ u_j, v_j \in \mathbb{Z}_p^*, \quad j = 1, 2; \quad \prod_{j=1}^2 u_j v_j \equiv (u_0 v_0)' \pmod{p} \right\}.$$

For every collection of (u_0, v_0) (it is $(p-1)^2$ such collections at all) we have the norm Kloosterman sum $\tilde{K}_1$ (which has been estimated in [9]). Thus, in the case $\alpha_0 \equiv 0 \pmod{p}$, we obtain

$$(3.7) \quad |\tilde{K}_2(0, \alpha_1, \alpha_2; p)| \leq 2(p-1)^2 p^{\frac{3}{2}} \leq 2p^{\frac{7}{2}}.$$

At last, in the third case, we find at once

$$(3.8) \quad |\tilde{K}_2(0, 0, \alpha_2; p)| \leq (p-1)^4 \max_{a \in \mathbb{Z}_p^*} \left| \sum_{\substack{x \in G_p^* \\ N(x) \equiv a \pmod{p}}} e_p \left(\Re \frac{x}{p} \right) \right| \leq 2p^{\frac{9}{2}}.$$

Let now $p \equiv 3 \pmod{4}$.

In this case G_p is a finite field $\mathbb{F}_{p^2}$, $|G_p| = p^2$.

Moreover, for any $u, v \in \mathbb{Z}$

$$(u + iv)^p \equiv u^p - iv^p \equiv u - iv \pmod{p},$$

and so $2\Re(u + iv) \equiv \text{Tr}(u + iv) \pmod{p}$, where $\text{Tr}(\cdot)$ is a trace from $\mathbb{F}_{p^2}$ into $\mathbb{F}_p$.

So, the investigated sum $\tilde{K}_2$ coincide with the sum

$$S_1(V, \alpha) = \sum_{X \in V_1} e_p \left(m \frac{\text{Tr}(\alpha \cdot X)}{p} \right), \quad (m, p) = 1,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{F}_{p^2}^3$, $X = (x_1, x_2, x_3) \in \mathbb{F}_p$, $\alpha \cdot X = \sum_{i=0}^2 \alpha_i x_i$, V_1 is an algebraic manifold produced by a polynomial $x_0 x_1 x_2 - 1$ over $\mathbb{F}_{p^2}$.

Hence, applying to above the result of P. Deligne, we obtain

$$|S_2(V, \alpha)| \leq (n+1-\ell)(p^2)^{\frac{5}{2}+\ell},$$

where ℓ is a number of α_i under condition $\alpha_i \equiv 0 \pmod{p}$.

Moreover, from the representation $S_2(V, \alpha)$ by way of characteristic roots of the Riemann zeta-function of algebraic manifold over finite field we conclude $S_1^2(V, \alpha) \sim 3S_2(V, \alpha)$ and thus we have

$$(3.9) \quad |\tilde{K}_2| \leq 3p^{\frac{5}{2}+\ell}.$$

The case $p = 2$, $m = 1$ is trivial.

Let $m > 1$.

There is no loss of generality in assuming that $\alpha_0 \not\equiv p$. Thus it follows that $(a_0, b_0, p) = 1$.

Let $\xi_0 = x_0 + iy_0$. By the definition of $\tilde{K}_2$, we may write

$$(3.10) \quad \begin{aligned} \tilde{K}_2 &= \sum_{\substack{\xi_j \in G_{p^m} \\ j=0,1,2}} e_{p^m} \left(\sum_{j=0}^2 \Re(\alpha_j \xi_j) \frac{1}{p^m} \sum_{k=0}^{p^m-1} e_{p^m}(k(N(\xi_0, \xi_1 \xi_2) - 1)) \right) = \\ &= \frac{1}{p^m} \sum_{S(C)} e_{p^m} \left(k(N(\xi_0 \xi_1 \xi_2) - 1) + \sum_{j=0}^2 (a_j \Re \xi_j - b_j \Im \xi_j) \right), \end{aligned}$$

where

$$C := \{k \in \mathbb{Z}_{p^m}, \xi_j \in G_{p^m}, j = 0, 1, 2\}.$$

Note, that the summation over k in (3.10) gives zero if $N(\xi_1)$, $N(\xi_2)$ are not coprimes to p .

Therefore, we have

$$\tilde{K}_2 = \frac{1}{p^m} \sum_{\xi_1, \xi_2 \in G_{p^m}^*} \sum_{k=0}^{p^m-1} \sum_{x_0, y_0 \in \mathbb{Z}_{p^m}} e^{2\pi i \frac{k((x_0^2 + y_0^2)N(\xi_1)N(\xi_2) - 1) + \sum_{j=0}^2 (a_j \Re \xi_j - b_j \Im \xi_j)}{p^m}}.$$

Now, taking into account that the summation over x_0 (or y_0) gives zero if $k \equiv 0 \pmod{p}$, we deduce

$$(3.11) \quad \begin{aligned} \tilde{K}_2 &= \frac{1}{p^m} \sum_{k \in \mathbb{Z}_{p^m}^*} e_{p^m}(-k) \sum_{\xi_1, \xi_2 \in G_{p^m}^*} e_{p^m}(\Re(\alpha_1 \xi_1 + \alpha_2 \xi_2)) \times \\ &\quad \times \sum_{x_0, y_0 \in \mathbb{Z}_p} e_{p^m}(F(k, \xi)), \end{aligned}$$

where $F(k, \xi) = kN(\xi_1 \xi_2)(x_0^2 + y_0^2) + a_0 x_0 + b_0 y_0$.

The inner sum over x_0, y_0 in (3.11) is the product of two classical Gaussian sums, and, hence

$$(3.12) \quad \begin{aligned} &\sum_{x_0, y_0 \in \mathbb{Z}_{p^m}} e_{p^m}(kN(\xi_1 \xi_2)(x_0^2 + y_0^2) + a_0 x_0 - b_0 y_0) = \\ &= e_{p^m}(-(4kN(\xi_1 \xi_2))' \cdot (a_0^2 + b_0^2))(-1)^{\frac{p-1}{2}} p^m. \end{aligned}$$

(Here, as always A' denotes the multiplicate inversive for A modulo p^m if $(A, p) = 1$).

We continue the calculation of $\tilde{K}_2$.

From (3.11), (3.12) we infer

$$(3.13) \quad \tilde{K}_2 = \sum_{k \in \mathbb{Z}_{p^m}^*} (-1)^{\frac{p-1}{2}} e_{p^m}(-k) \sum_{\xi_1, \xi_2 \in G_{p^m}^*} e_{p^m}(F_1(k, \xi)),$$

where

$$F_1(k, \xi) = \sum_{j=1}^2 (a_j \Re \xi_j - b_j \Im \xi_j) - 4' k' (a_0^2 + b_0^2) N(\xi_1 \xi_2)'.$$

Put $m_1 = \left[\frac{m}{2}\right]$, $\xi_j = \eta_j + p^{m_1} \zeta_j$, $\eta_j \in G_{p^{m_1}}^*$, $\zeta_j \in G_{p^{m-m_1}}$,

$$(3.14) \quad \eta_j = x_j + iy_j, \quad \zeta_j = u_j + iv_j, \quad x_j, y_j \in \mathbb{Z}_{p^{m_1}}, \quad u_j, v_j \in \mathbb{Z}_{p^{m-m_1}}, \quad j = 1, 2.$$

Then

$$(3.15) \quad N(\xi_j)' = N(\eta_j)'(1 - 2p^{m_1}(x_j u_j + y_j v_j)N(\zeta_j)').$$

Consequently, by (3.13)-(3.15), we obtain

$$(3.16) \quad \tilde{K}_2 = \sum_{k \in \mathbb{Z}_{p^m}^*} e_{p^m}(-k) \prod_{j=1}^2 \left(\sum_{\substack{x_j, y_j \in \mathbb{Z}_{p^{m_1}} \\ (x_j^2 + y_j^2, p)=1}} e_{p^{m_1}}(\Re(\alpha_j \eta_j)) \times \right. \\ \left. \times \sum_{u_j, v_j \in \mathbb{Z}_{p^{m-m_1}}} e_{p^m}(F_2(k, x, y)) \right),$$

where

$$(3.17) \quad \begin{aligned} F_2(k, x, y) &= H_0 + p^{m_1} H_1, \\ H_0 &= 4' k' (a_0^2 + b_0^2) ((x_1^2 + y_1^2)(x_2^2 + y_2^2))' := 4' k' (a_0^2 + b_0^2) D, \\ H_1 &= \sum_{j=1}^2 (-2' k' (a_0^2 + b_0^2) D(x_j u_j + y_j v_j) + a_j u_j - b_j v_j). \end{aligned}$$

The summation over u_j, v_j in (3.16) gives zero if it disturbs at least one of the congruences

$$(3.18) \quad \begin{cases} 2ka_j - D(a_0^2 + b_0^2)x_j \equiv 0 \pmod{p^{m-m_1}} \\ 2kb_j + D(a_0^2 + b_0^2)y_j \equiv 0 \pmod{p^{m-m_1}} \end{cases}, \quad j = 1, 2$$

If (3.18) holds, then the summation over u_j, v_j gives $p^{2(m-m_1)}$.

Let $\min(r_{\alpha_1}, r_{\alpha_2}) = 0$ (see, notation (3.3)). In such case we have $(a_1, a_2, b_1, b_2, p) = 1$. Let $(a_1, p) = 1$. From the congruence

$$ka_1 - D(a_0^2 + b_0^2)x_1 \equiv 0 \pmod{p}$$

it follows that $(x_1, p) = 1$ and $D(a_0^2 + b_0^2) \equiv ka_1 x_1 \pmod{p^{m-m_1}}$. Thus from (3.18) we infer

$$(3.19) \quad x_2 \equiv a'_1 a_2 x_1, \quad y_1 \equiv -a'_1 b_1 x_1, \quad y_2 \equiv -a'_1 b_2 x_1 \pmod{p^{m-m_1}}.$$

By (3.16), (3.17), (3.19) and Lemma 1, we lead the estimate for $\tilde{K}_2$ to an exponential sum over k .

$$(3.20) \quad |\tilde{K}_2| \leq p^{2m} \left| \sum_{k \in \mathbb{Z}_{p^m}^*} e_{p^m}(-k + Ak') \right| \leq 2p^{\frac{5m}{2}}.$$

In the case $\min(r_{\alpha_1}, r_{\alpha_2}) = r > 0$ the system (3.18) has no solutions under assumption $(x_j^2 + y_j^2, p) = 1$, $j = 1, 2$ and, consequently, the sum $\tilde{K}_2$ is zero.

The same estimates hold for $\tilde{K}_2$ if $p = 1 + i$.

So, we proved our statement.

Lemma 5. *Let $\alpha_0, \alpha_1, \alpha_2$ be the Gaussian integers, $\min(r_{\alpha_0}, r_{\alpha_1}, r_{\alpha_2}) = 0$, and $q > 1$ be a positive integer, $q = q_1 q_2$, $(q_1, q_2) = 1$, q_1 is square-free, q_2 is square-full.*

Then

$$\tilde{K}_2(\alpha_0, \alpha_1, \alpha_2; q) = \tilde{K}_2(\alpha_0 \bar{q}_2, \alpha_1 \bar{q}_2, \alpha_2 \bar{q}_2; q_1) \cdot \tilde{K}_2(\alpha_0 \bar{q}_1, \alpha_1 \bar{q}_1, \alpha_2 \bar{q}_1; q_2),$$

moreover,

$$|\tilde{K}_2(\alpha_0 \bar{q}_2, \alpha_1 \bar{q}_2, \alpha_2 \bar{q}_2; q_1)| \leq 2^{\omega(q_1)} \prod_{p|q_1} p^{\frac{5}{2} + \ell_p(\alpha_1, \alpha_2)},$$

where

$$\ell_p(\alpha_1, \alpha_2) = \begin{cases} 0 & \text{if } r_{\alpha_1} = r_{\alpha_2} = 0; \\ 1 & \text{if } r_{\alpha_1} = 0, r_{\alpha_2} = 1; \\ 2 & \text{if } r_{\alpha_1} = r_{\alpha_2} = 1; \end{cases}$$

$$|\tilde{K}_2(\alpha_0 \bar{q}_1, \alpha_1 \bar{q}_1, \alpha_2 \bar{q}_1; q_2)| \leq \begin{cases} 3^{\omega(q_2)} q_2^{\frac{3}{2}} & \text{if } \min(r_{\alpha_1}, r_{\alpha_2}) = 0 \\ & \text{for all } p|q_2; \\ 0 & \text{else.} \end{cases}$$

4. Main results

First we will suppose that $(\ell, q) = 1$ and ℓ is a norm residue mod q .

For $\Re s > 1$ we denote

$$(4.1) \quad F(s; \ell; q) := \sum_{\substack{\alpha \in G \\ S(C)}} \frac{\tau_3(\alpha)}{N(\alpha)^s},$$

where $C : \{\alpha \in G : N(\alpha) \equiv \ell \pmod{q}, N(\alpha) \leq x\}$;

$$(4.2) \quad F^*(s; \ell, q) = F(s; \ell, q) - \frac{A(\ell)}{\ell^s},$$

where

$$A(\ell) = \sum_{\substack{\alpha \\ N(\alpha) = \ell}} \tau_3(\alpha).$$

It is clear, that

$$(4.3) \quad F(s; \ell, q) = \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in G_q^* \\ N(\alpha_1, \alpha_2, \alpha_3) \equiv \ell \pmod{q}}} Z\left(s; \frac{\alpha_1}{q}, 0\right) Z\left(s; \frac{\alpha_2}{q}, 0\right) Z\left(s; \frac{\alpha_3}{q}, 0\right) N(q)^{-3s}.$$

Hence, be a simple generalization of the well-known Perron's formula for the Dirichlet series on an arithmetic progression (see, [8]) we have

$$(4.4) \quad \begin{aligned} T(x; \ell, q) := & \sum_{\substack{N(\alpha) \equiv \ell \pmod{q} \\ N(\alpha) \leq x}} \tau_3(\alpha) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F^*(s; \ell, q) \frac{x^s}{s} ds + \\ & + O\left(\frac{x^c}{Tq(c-1)^3}\right) + O(x^\varepsilon), \end{aligned}$$

where $c > 1, 1 < T \leq x$ are the parameters to be chosen later.

Now we obtain by moving the path of integration to the line $\Re s = -b$, $b > 0$,

$$(4.5) \quad \begin{aligned} T(x; \ell, q) = & \sum_{\substack{N(\alpha) \equiv \ell \pmod{q} \\ N(\alpha) \leq x}} \tau_3(\alpha) = \left(\operatorname{res}_{s=1} + \operatorname{res}_{s=0}\right) \left(F^*(s; \ell, q) \frac{x^s}{s}\right) + \\ & + \frac{1}{2\pi i} \int_{-b-iT}^{-b+iT} F^*(s; \ell, q) \frac{x^s}{s} ds + \frac{1}{2\pi i} \int_{-b+iT}^{c+iT} F^*(s; \ell, q) \frac{x^s}{s} ds - \\ & - \frac{1}{2\pi i} \int_{-b-iT}^{c-iT} F^*(s; \ell, q) \frac{x^s}{s} ds + O\left(\frac{x^c}{Tq(c-1)^3}\right) + O(x^\varepsilon). \end{aligned}$$

For the calculation of integrals in (4.5) we consider the function $F^*(s; \ell, q)$ in the strip $-\varepsilon \leq \Re s \leq c$.

It is obviously that

$$(4.6) \quad |F^*(c+it; \ell, q)| \leq \sum_{\substack{N(\alpha) \equiv \ell \pmod{q} \\ N(\alpha) > q}} \frac{\tau_3(\alpha)}{N(\alpha)^c} \leq q^{-1+\varepsilon}, \quad (\varepsilon > 0).$$

On the line $\Re s = -\varepsilon$ we apply the functional equation for $Z(s; \delta, 0)$ and then obtain

(4.7)

$$F^*(s; \ell, q) = \pi^{-3} \left(\frac{\pi}{q} \right)^{6s} \frac{\Gamma^3(1-s)}{\Gamma^3(s)} \times \\ \times \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in G_q \\ N(\alpha_1 \alpha_2 \alpha_3) \equiv \ell \pmod{q}}} \sum_{\omega_1, \omega_2, \omega_3 \in G} e_q \left(\Re \left(\sum_{j=1}^3 \alpha_j \omega_j \right) \right) N(\omega_1 \omega_2 \omega_3)^{-1+s} - \frac{A(\ell)}{\ell^s}.$$

By the absolute convergence of the series at $\omega_1, \omega_2, \omega_3$ we may write

$$(4.8) \quad F^*(s; \ell, q) = \pi^3 \left(\frac{\pi}{q} \right)^{6s} \frac{\Gamma^3(1-s)}{\Gamma^3(s)} \times \\ \times \sum_{\substack{\delta \in G \\ \delta | q}} \sum_{\substack{\omega \in G \\ (\omega, q) = \delta}} N(\omega)^{-1+s} \sum_{\substack{\omega_1, \omega_2, \omega_3 \in G \\ \omega_1 \omega_2 \omega_3 = \omega}} \tilde{K}_2(\omega_1, \omega_2, u_0 \omega_3; q),$$

where $N(u_0) \equiv \ell \pmod{q}$.

Putting $\delta = \delta_1 \delta_2 \delta_3$ and $\omega = \delta \omega' = \delta_1 \omega'_1 \delta_2 \omega'_2 \delta_3 \omega'_3$ and applying Lemma 6 we have for $s = -\varepsilon + it$:

$$(4.9) \quad F^*(s; \ell, q) \ll q^{-6\varepsilon + \frac{3}{2}2^{\nu(q)}} \left| \frac{\Gamma^3(1-s)}{\Gamma^3(s)} \right| \ll q^{\frac{3}{2}-6\varepsilon} 2^{\nu(q)} (|t| + 3)^{3+6\varepsilon}.$$

Hence, by (4.6)-(4.9) and the Phragmen-Lindelöf theorem we infer for $-\varepsilon \leq \Re s \leq c$, $|\Im s| \geq 3$

$$(4.10) \quad F^*(s; \ell, q) \ll q^{\frac{3c-5\sigma-5\varepsilon}{2(c+\varepsilon)}} |t|^{\frac{3(c-\sigma)}{c+\varepsilon}}.$$

Thus trivially we have

$$(4.11) \quad \int_{-b \pm iT}^{c \pm iT} F^*(s; \ell, q) \frac{x^s}{s} ds \ll q^{\frac{3}{2}-6b} 2^{\nu(q)} T^{2+6b} + \frac{x}{T q^{1-b}}.$$

Moreover, by Stirling's formula and a simple transformation, we obtain

$$(4.12) \quad \frac{1}{2\pi i} \int_{-b-iT}^{-b+iT} F^*(s; \ell, q) \frac{x^s}{s} ds = \\ = \sum_{\substack{\delta | q \\ \delta \in G}} \sum_{\substack{\omega \in G \\ (\omega, q) = \delta}} N(\omega)^{-1} \Phi(\omega) \left(\mathfrak{I}(\omega) + O \left(T^{2+6b} \left(\frac{x N(\omega)}{q^6} \right)^{\frac{-b}{6}} \right) \right),$$

where

$$\begin{aligned}
 \Phi(\omega) &= \sum_{\omega_1 \omega_2 \omega_3 = \omega} \tilde{K}_2(\omega_1, \omega_2, u_0 \omega_3; q), \quad (N(u_0) \equiv \ell \pmod{q}) \\
 \mathfrak{I}(\omega) &= \frac{y^{-b}}{\sqrt{2\pi}} \times \\
 (4.13) \quad &\times \frac{1}{2\pi i} \int_{T_0}^T t^{2+6b} \left(y^{-\frac{it}{6}} e^{it(\log t - 1)} + y^{\frac{it}{6}} e^{-it(\log t - 1)} \right) g(t) dt + \\
 &+ O(T_0^{3+6b} y^{-b}),
 \end{aligned}$$

where $y = \frac{\pi^6}{q^6}(xN(\omega))$, $T_0 > 1$ and will select later.

In the last integral put $t = y^{\frac{1}{6}} t_1$, $f(t) = t(\log \frac{t}{y^{\frac{1}{6}}} - 1) = y^{\frac{1}{6}} t_1(\log t_1 - 1)$.

Since, $f'(t_1) = 0$ only if $t_1 = 1$, and $f''(1) = y^{\frac{1}{6}} > 0$, we conclude, that the integral $\mathfrak{I}(\omega)$ has only one stationary point $t_1 = 1$ if $1 \in [y^{-\frac{1}{6}} T_0, y^{\frac{1}{6}} T]$.

Hence, for $N(\omega) > \frac{T^6 q^6}{\pi^6 x}$, an integration by parts yields estimate

$$(4.14) \quad \mathfrak{I}(\omega) \ll y^{-b} \min \left(T^{\frac{5}{2}+6b}, T^{2+6b} \log \left(\frac{N(\omega)x}{T^6 q^6} \right)^{-1} \right) \log T.$$

For $N(\omega) \leq \frac{T^6 q^6}{\pi^6 x}$, by the method of stationary phase (see, [7]: Theorem 1.4, p.162) we deduce

$$\begin{aligned}
 \mathfrak{I}(\omega) &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\pi} \sin \left(\frac{\pi}{4} - y^{\frac{1}{6}} \right) \left(y^{\frac{5}{12}} + a_1 y^{\frac{3}{12}} + a_2 y^{\frac{1}{12}} + O \left(y^{-\frac{1}{2}} \right) \right) + \\
 (4.15) \quad &+ O(T^{2+6b}) + O(T_0^{3+6b})
 \end{aligned}$$

with the absolute constants a_1, a_2 in symbol "O".

So, to sum up, we have obtained from (4.4), (4.5), (4.11)-(4.15) the following relation

$$\begin{aligned}
 T(x; \ell, q) &= \left(\operatorname{res}_{s=1} \operatorname{res}_{s=0} \right) \left(F^*(s; \ell, q) \frac{x^s}{s} \right) + \sum_{\substack{\delta \in G \\ \delta | q}} \sum_{\substack{(\omega, q) = \delta \\ N(\omega) \leq X}} \left(\frac{\Phi(\omega) N(\omega)^{-1}}{\pi \sqrt{2\pi}} \cdot \right. \\
 (4.16) \quad &\cdot \sin \left(\frac{\pi}{4} - \frac{\pi \sqrt[6]{xN(\omega)}}{q} \right) \left(\frac{\pi^{\frac{5}{2}}}{q^{\frac{5}{2}}} (xN(\omega))^{\frac{5}{12}} \right) \Bigg) + \\
 &+ O \left(\left(\frac{xN(\omega)}{q^6} \right)^{\frac{3}{12}} \right) + O \left(q^{\frac{5}{2}} T^{\frac{5}{2}} x^\varepsilon \log T \right) + O \left(q^{\frac{5}{2}+\varepsilon} T_0^{3+6b} \log T \right),
 \end{aligned}$$

where $X = \frac{T^6 q^6}{\pi^6 x}$.

After all this preliminary work, it is straight-forward to prove the main result of this paper.

Theorem 1. *Let ℓ, q be the positive integers, $1 \leq \ell < q$, $(\ell, q) = 1$. Then for $x \rightarrow \infty$ we have*

$$\begin{aligned}
 \sum_{\substack{N(\omega) \equiv \ell \pmod{q} \\ N(\omega) \leq x}} \tau_3(\alpha) &= \frac{x}{q^2} \mathfrak{I}(\ell, q) \prod_{\mathfrak{p}|q} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^2 P_2(\log x) + \\
 (4.17) \quad &+ \frac{x}{q^2} \mathfrak{I}(\ell, q) \prod_{\mathfrak{p}|q} \left(1 - \frac{1}{N(\mathfrak{p})}\right) P_1(\log x) + \\
 &+ \frac{12x}{q^2} \mathfrak{I}(q, \ell) + O\left(x^{\frac{5}{7}+\varepsilon}\right),
 \end{aligned}$$

where $\mathfrak{I}(q, \ell)$ is determined by Lemma 1 and $P_j(u)$ are the polynomials of j^{th} -degree with the computable coefficients, moreover these coefficients and the constant in the error term do not depend on x, ℓ, q .

Proof. Let χ be an arbitrary character and let χ_0 be a principal character, both from the group of characters $\widehat{G}_q$ modulo q . The Hecke Z-function with a character χ is defined by the series

$$Z(s, \chi) = \sum \frac{\chi(\omega)}{N(\omega)^s}, \quad (\Re s > 1).$$

For $\chi = \chi_0$

$$Z(s, \chi_0) = \prod_{\mathfrak{p} \in G}^* \left(1 - \frac{\chi_0(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} = \prod_{\mathfrak{p}|q} (1 - N^{-s}(\mathfrak{p})) Z(s).$$

Since, $Z(s) = \zeta(s)L(s, \chi_4)$, we have

$$\begin{aligned}
 Z(s, \chi_0) &= \frac{\pi \overline{\varphi}(q)}{4q^2} \cdot \frac{1}{s-1} + \frac{\pi \overline{\varphi}(q)}{4q^2} \sum_{\mathfrak{p}|q}^* \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1} + \\
 &+ \frac{\overline{\varphi}(q)}{q^2} \left(\frac{\pi}{4} E + L'(1, \chi_4) \right) + a'_1(s-1) + \dots = \\
 &= \frac{\pi \overline{\varphi}(q)}{4q^2} \cdot \frac{1}{s-1} + b_{0,q}(\chi_0) + a'_1(s-1) + \dots,
 \end{aligned}$$

where E is the Euler's constant, $L(s, \chi_4)$ is L -function of Dirichlet with the

non-principal character $\chi \pmod{4}$,

$$\begin{aligned}\bar{\varphi}(q) &= q^2 \prod_{\mathfrak{p}|q} \left(1 - \frac{1}{N(\mathfrak{p})}\right), \\ b_{0,q}(\chi_0) &= \frac{\pi \bar{\varphi}(q)}{4q^2} \left(E + \frac{L'(1, \chi_4)}{L(1, \chi_4)} + \sum_{\mathfrak{p}|q}^* \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1} \right).\end{aligned}$$

For an arbitrary character $\chi \in \widehat{G}_q$ we have

$$(4.18) \quad Z(s, \chi) = \frac{\varepsilon(\chi)}{s-1} + b_{0,q}(\chi) + b_1(\chi)(s-1) + \cdots,$$

where

$$\varepsilon(\chi) = \begin{cases} \frac{\pi \bar{\varphi}(q)}{4q^2} & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Next,

$$\begin{aligned}q^{-2s} Z\left(s, \frac{\alpha_1}{q}, 0\right) &= (\bar{\varphi}(q))^{-1} \sum_{\chi \in G_q} \bar{\chi}(\alpha_1) \sum_{\alpha} \frac{\chi(\alpha)}{N(\alpha)^s} = \\ &= \frac{4}{\bar{\varphi}(q)} \sum_{\bar{\chi} \in \widehat{G}_q} \bar{\chi}(\alpha_1) Z(s, \chi).\end{aligned}$$

Thus, by (4.18)

$$q^{-2s} Z\left(s, \frac{\alpha_1}{q}, 0\right) = \frac{\pi}{q^2} \cdot \frac{1}{s-1} + \frac{4}{\bar{\varphi}(q)} \sum_{\chi \in \widehat{G}_q} b_{0,q}(\chi) \bar{\chi}(\alpha_1) + \cdots.$$

So,

$$\begin{aligned}& \operatorname{res}_{s=1} \left(F^*(s) \frac{x^s}{s} \right) = \\ &= \operatorname{res}_{s=1} \left\{ \sum_{N(\alpha_1 \alpha_2 \alpha_3) \equiv \ell \pmod{q}} q^{-6s} \frac{x^s}{s} \prod_{j=1}^3 Z\left(s, \frac{\alpha_j}{q}, 0\right) \right\} = \\ &= \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in G_q^* \\ N(\alpha_1 \alpha_2 \alpha_3) \equiv \ell \pmod{q}}} \left\{ \frac{\pi^3}{q^6} x P_2 \log x + \frac{\pi^2 x}{q^4 \bar{\varphi}(q)} x P_1 \log x \sum_{\chi \in G_q} b_{0,q}(\chi) \left(\sum_{j=1}^3 \bar{\chi}(\alpha_j) \right) + \right. \\ & \quad \left. + \frac{4\pi}{q^2 \bar{\varphi}^2(q)} x \sum_{\chi_1, \chi_2 \in G_q} b_{0,q}(\chi_1) b_{0,q}(\chi_2) \left(\sum_{\substack{i=1,2, \\ i \neq j}} \bar{\chi}(\alpha_i) \bar{\chi}(\alpha_j) \right) \right\}.\end{aligned}$$

Taking into account that a summation over $\alpha_1, \alpha_2, \alpha_3$ shows the terms with $\chi_i \neq \chi_0$ gives 0, we obtain

$$(4.19) \quad \operatorname{res}_{s=1} \left(F^*(s) \frac{x^s}{s} \right) = \frac{\pi^3 \bar{\varphi}(q)^2 \mathfrak{J}(q, \ell)}{q^6} x P_2 \log x + \frac{3\pi^2 \mathfrak{J}(q, \ell)}{q^4} x P_1 \log x + \\ + \frac{12\pi \mathfrak{J}(q, \ell)}{q^2 \bar{\varphi}(q)} x,$$

where $\mathfrak{J}(q, \ell)$ is determined by Lemma 1, $\bar{\varphi}(q)$ be totient Euler function in $\mathbb{Z}[i]$.

Moreover,

$$(4.20) \quad \operatorname{res}_{s=0} \left(F^*(s) \frac{x^s}{s} \right) = (\zeta(0) L(0, \chi_4))^3 - 1 = C_0 \text{ is constant.}$$

Now, we see from (16), (18), (19) that the assertion of the Theorem is proved by choosing $T_0 = T^{\frac{1+6b}{2+6b}}$, $b = \frac{1}{\log x} < \varepsilon$, $T = x^{\frac{2}{7}} q^{-1}$. ■

The asymptotic formula (4.17) is non-trivial if $q \leq x^{\frac{2}{7}+\varepsilon}$. Similar asymptotic formula for the divisor function $\tau_2(\alpha)$ under $N(\alpha) \equiv \ell \pmod{q}$ have been obtained in [10] a non-trivial for $q \leq x^{\frac{5}{16}+\varepsilon}$. We claim that with a growth of k the divisor function $\tau_k(\alpha)$ in an arithmetic progression $N(\alpha) \equiv \ell \pmod{q}$ has an asymptotic formula that is a non-trivial in still smaller range of a change q .

The case $(\ell, q) > 1$ can be leaded easily to considered $(\ell, q) = 1$.

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