A NOTE ON REGULAR MORPHISMS

Truong Cong Quynh (DaNang City, Vietnam)
M. Tamer Koşan (Gebze-Kocaeli, Turkey)
Phan The Hai (Hue City, Vietnam)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

Communicated by Bui Minh Phong

(Received May 28, 2013; accepted September 28, 2013)

Abstract. In this paper, we give some other characterizations of regular $Hom(M,N)$. A property regular homomorphism via lifting is studied. Moreover, we also study regular submodules conditions and this is one of way to check regularity of $Hom(M,N)$. On the other hand, we consider regular $Hom(M,N)$ via weakly M-torsionless, idempotent submodules conditions. Finally, we give some results of Δ-regular, ∇-regular homomorphisms and some well-known results are obtained.

1. Introduction

The concept of the regularity of $[M,N]$ was introduced by Kasch and Mader in [4] to extend the notion of regularity ring to $[M,N]$. Recall that $\alpha \in [M,N]$ is called *regular* if $\alpha = \alpha\beta\alpha$ for some $\beta \in [N,M]$. The module $[M,N]$ is said to be *regular* if each $\alpha \in [M,N]$ is regular. M is called a *direct projective* module if whenever a factor module M/K is isomorphic to a summand of M then K

Key words and phrases: Regular morphism, singular and co-singular ideal.
2010 Mathematics Subject Classification: 16D40, 16E50, 16N20.
https://doi.org/10.71352/ac.41.249

is a summand of M (see [7]). According to Nicholson-Zhou [8], N is *direct M-projective* if $M/K \cong P \leq^{\oplus} N$ implies that $K \leq^{\oplus} M$. In [8, Theorem 4], it is shown that $[M, N]$ is regular if and only if $\alpha(M)$ is a direct summand of N for every $\alpha \in [M, N]$ and N is direct M-projective. In Section 2, we show that for every $\alpha \in [M, N]$, α is regular if and only if $\alpha(M)$ is a direct summand of N and, for R-homomorphisms $f : M \to \alpha(M)$ and $g : \alpha(M) \to \alpha(M)$, there exists an R-homomorphism $h : \alpha(M) \to M$ with $fh = g$. We also show that if M is N-injective, then $[M, N]$ is regular if and only if $[M, \alpha(M)]$ is regular for every $\alpha \in [M, N]$ if and only if, for every $\alpha \in [M, N]$, and for every R-homomorphism $f : M \to \alpha(M)$ and $g : \alpha(M) \to \alpha(M)$, there exists an R-homomorphism $h : \alpha(M) \to M$ with $fh = g$.

An important line of research in this module classes is to investigate relationships of regularity to substructures such as Jacobson radical $J[M, N]$ of $[M, N]$, to the singular $\Delta[M, N]$ and cosingular $\nabla[M, N]$ ideals of $[M, N]$, and to the notion of lying over or under a direct summand. Beidar and Kasch [2] defined and studied the singular ideal $\Delta[M, N]$ and the co-singular ideal $\nabla[M, N]$ such as:

$$\begin{aligned} \Delta[M, N] &= \{f \in [M, N] : Ker(f) \leq^{e} M\} \\ \nabla[M, N] &= \{f \in [M, N] : Im(f) \ll N\}. \end{aligned}$$

The other substructure, Jacobson radical $J[M, N]$ of $[M, N]$ was introduced and studied by Kasch-Mader [4] and Nicholson-Zhou [8]. If $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ are left R-modules, then (using the canonical injections and projections) $[M, N]$ has a natural matrix representation as.

$$[M, N] = \begin{pmatrix} [M_1, N_1] & [M_1, N_2] & \cdots & [M_1, N_t] \\ [M_2, N_1] & [M_2, N_2] & \cdots & [M_2, N_t] \\ \cdots & \cdots & \cdots & \\ [M_s, N_1] & [M_s, N_2] & \cdots & [M_s, N_t] \end{pmatrix} = ([M_i, N_j])$$

where the elements of M and N are written as rows, and the matrix $([M_i, N_j])$ acts by right matrix multiplication. In [8, Theorem 10], it is shown that if $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ are modules, then $J[M, N] = (J[M_i, N_j])$. In [10], the authors proved that $\Delta[M, N] = (\Delta[M_i, N_j])$ and $\nabla[M, N] = (\nabla[M_i, N_j])$.

In this paper, we continue further research regular homomorphisms. In [13, Theorem 2.2], Zelmanowitz proved that a module M is regular if and only if mR is projective and a direct summand of M. Approach as above we can see easy that if M is projective then $\alpha \in [M, N]$ is regular if and only if $\alpha(M)$ is projective and a direct summand of N. We can be generalized this result with weakly projective condition and prove that $\alpha \in [M, N]$ is regular if and only if $\alpha(M)$ is "weakly" M-projective and a direct summand of N (Theorem 2.3).

On the other hand, with regular condition submodules of M, we also have a result of regular $[M,N]$; we show that $[M,N]$ is regular if and only if $[M,\alpha(M)]$ is regular for every $\alpha \in [M,N]$; where M is N-injective (Theorem 2.4).

The authors Chen-Nicholson proved that a module M is regular if and only if M is torsionless and for every $m \in M$, $m[M,R_R] = eR$ for some $e^2 = e \in R$, see [3, Theorem 2.1]. We will extend this result for $[M,N]$ and show that $[M,N]$ is regular if and only if N is weakly M-torsionless and, for any $x \in [M,N]$, $[N,M]x = E_M e$ for some $e^2 = e \in E_M$ (Theorem 2.6).

In addition, we show that some characterizations of $[M,N]$ with property: When are $H[M,N] = H$ for every non-empty subset H of $[M,N]$?. We prove that $[M,N]$ is regular if and only if for every non-empty subset H of $[M,N]$ with $H[N,M]H \subseteq H$ implies $H[M,N]H = H$ if and only if for every non-empty subset H of $[M,N]$ with $E_N H \cap HE_M \subseteq H$ implies $H[N,M]H = H$ (Proposition 2.8).

Following [6], a submodule X of N is called a *semisupplement* of Y in N if $N = X + Y$ and $X \cap Y \ll N$. X is called a *semicomplement* of Y in M if $X \cap Y = 0$ and $X + Y$ is essential in M. The authors Lee-Zhou give some characterizations Δ-regular and ∇-regular of endomorphism ring via semisupplement and semicomplement submodules. In this paper, we also have some similar results for regular $[M,N]$ with the condition on semisupplement and semicomplement submodules (Proposition 2.10 and Proposition 2.11).

In this paper, R will present an associative ring with identity and all modules over R are unitary right modules. We write M_R to indicate that M is a right R-module. Throughout this paper, homomorphisms of modules are written on the left of their arguments. Let M and N be modules. For convenience of the readers, we follow the notations used in [8] or [14], let $E_M := End_R(M)$ and $[M,N] := Hom_R(M,N)$. Then $[M,N]$ is an (E_N, E_M)-bimodule. We also denote $J(R)$ and $Rad(M)$ for the Jacobson radical of R and module M, respectively. For a submodule N of M, we use $N \leq M$ $(N < M)$ and $N \leq^{\oplus} M$ to mean that N is a submodule of M (respectively, proper submodule), N is a direct summand of M, and we write $N \leq^e M$ and $N \ll M$ to indicate that N is an essential, respectively small of M. For a subset X of R, let $r(X)$ denote the right annihilator of X in R.

2. Some results of regular morphisms

We call that a right R-module A is called *semi M-projective* if, for any submodule B of M, every epimorphism $\pi : M \to B$ and every R-homomorphism

$\alpha : A \to B$, there exists an R-homomorphism $\beta : A \to M$ such that $\pi\alpha = \beta$. Following Wisbauer ([12]), M is semi-projective if M is semi M-projective.

Lemma 2.1. *Let M be a right R-module. The following conditions are equivalent:*

(1) *For every $s \in E_M$, $Ker(s)$ is a direct summand of M.*

(2) *$s(M)$ is semi M-projective for every $s \in E_M$.*

Proof. (1) $\Rightarrow$ (2). We first claim that M is semi-projective. Let $f : M \to A$ be an epimorphism and $g : M \to A$ be an R-homomorphism with $A \leq M$. Let $\iota : A \to M$ be the inclusion. By the hypothesis $Ker(\iota f) = e(M)$ for some $e^2 = e \in S$. But $Ker(\iota f) = Ker(f)$ and so $Ker(f) \leq^{\oplus} M$. It follows that f is an epimorphism splits. There exists $h : A \to M$ such that $fh = id_A$. We have $f(hg) = (fh)g = g$. Thus M is semi-projective.

For all $s \in S$, $Ker(s) \leq^{\oplus} M$ and so $s(M) \simeq e(M)$ for some $e^2 = e \in S$. We consider the following diagram:

$$\begin{tikzcd} & M \arrow[d, "p"'] & \\ & e(M) \arrow[d, "g"'] & \\ M \arrow[r, "f"] & A \arrow[r] & 0 \end{tikzcd}$$

with $A \leq M$. Let $\iota : e(M) \to M$ be the inclusion and $p : M \to e(M)$ be the projection. Since M is semi-projective, there exists $h : M \to M$ such that $fh = gp$. This implies that $f(h\iota) = g$. Thus $e(M)$ is semi M-projective and so $s(M)$ is too.

(2) $\Rightarrow$ (1). For each $s \in S$, we have

$$\begin{tikzcd} & s(M) \arrow[d, "id_{s(M)}"'] & \\ M \arrow[r, "s"] & s(M) \arrow[r] & 0 \end{tikzcd}$$

Since $s(M)$ is semi M-projective, there exists $h : s(M) \to M$ such that $sh = = id_{s(M)}$. Therefore s is an epimorphism splits and so $Ker(s) \leq^{\oplus} M$. ■

Corollary 2.2. *Let M be a right R-module. The following conditions are equivalent:*

(1) *E_M is regular.*

(2) *$\alpha(M)$ is a direct summand of N and $\alpha(M)$ is semi M-projective for all $\alpha \in E_M$.*

Theorem 2.3. *Let M and N be modules and $\alpha \in [M, N]$. The following conditions are equivalent for $\alpha \in [M, N]$:*

(1) *α is regular.*

(2) *$\alpha(M)$ is a direct summand of N and, for R-homomorphisms $f : M \to \to \alpha(M)$ and $g : \alpha(M) \to \alpha(M)$, there exists an R-homomorphism $h : \alpha(M) \to M$ with $fh = g$.*

$$\begin{tikzcd}
 & \alpha(M) \arrow[dl, dashed] \arrow[d, "g"] & \\
M \arrow[r, "f"'] & \alpha(M) \arrow[r] & 0
\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). By regularity of α, we can obtain that $\alpha(M)$ is a direct summand of N and $Ker(\alpha)$ is a direct summand of M. Hence there exists $u : \alpha(M) \to M$ such that $fu = id_{\alpha(M)}$ because of the diagram. Let $h = ug$. Then $fh = f(ug) = g$.
(2) $\Rightarrow$ (1) is obvious. ■

The following theorem extends Nicholson-Zhou [8, Theorem 4].

Theorem 2.4. *Assume that M is N-injective. The following conditions are equivalent:*

(1) *$[M, N]$ is regular.*

(2) *$[M, \alpha(M)]$ is regular for every $\alpha \in [M, N]$.*

(3) *For every $\alpha \in [M, N]$, and for every R-homomorphism $f : M \to \alpha(M)$ and $g : \alpha(M) \to \alpha(M)$, there exists an R-homomorphism $h : \alpha(M) \to M$ with $fh = g$.*

$$\begin{tikzcd}
 & \alpha(M) \arrow[dl, dashed] \arrow[d, "g"] & \\
M \arrow[r, "f"'] & \alpha(M) \arrow[r] & 0
\end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in [M, N]$ and $f \in [M, \alpha(M)]$. By (1), there exists $g \in [N, M]$ such that $\iota f = (\iota f)g(\iota f)$ with the inclusion $\iota : \alpha(M) \to N$. Let $\beta = g|_{\alpha(M)} : \alpha(M) \to M$. Then $\alpha = \alpha\beta\alpha$.

(2) $\Rightarrow$ (3). Let $f : M \to \alpha(M)$ be an epimorphism. Since $[M, \alpha(M)]$ is regular, we can obtain that $Ker(f)$ is a direct summand of M. Hence there exists $k : \alpha(M) \to M$ such that $fk = 1_{\alpha(M)}$. Let $h = kg$ and so $fh = g$.

(3) $\Rightarrow$ (1). Let $\alpha \in [M, N]$. There exists $\beta \in [\alpha(M), M]$ such that $\alpha = \alpha\beta\alpha$. Since M is N-injective, there exists $\gamma \in [N, M]$ such that $\gamma|_{\alpha(M)} = \beta$. Thus $\alpha = \alpha\gamma\alpha$. ■

Corollary 2.5. *Assume that R is a right self-injective ring. The following conditions are equivalent for a right R-module M:*

(1) *M is regular.*

(2) *Every principal submodule of M is regular.*

(3) *Every principal submodule of M is projective.*

An R-module N is *M-torsionless* if it can be embedded into a direct product of copies of M.

We call an R-module N *weakly M-torsionless*, if $r_{[M,N]}([N, M]) = 0$. It is easy to see that M-torsionless modules are weakly M-torsionless. Moreover, N is R_R-torsionless if and only if N is weakly R_R-torsionless.

Theorem 2.6. *Let M and N be R-modules.*

(1) *$[M, N]$ is regular if and only if N is weakly M-torsionless and, for any $x \in [M, N]$, $[N, M]x = E_M e$ for some $e^2 = e \in E_M$.*

(2) *$[M, N]$ is regular if and only if, for any $x, y \in [M, N]$,*

   (a) *$[N, M]x = E_M e$ for some $e^2 = e \in E_M$.*

   (b) *$y = (y_1 f_1 + \cdots + y_n f_n)y$ for some $y_1, ..., y_n \in [M, N]$ and for some $f_1, ..., f_n \in [N, M]$.*

Proof. (1) Assume that $[M, N]$ is regular. We first show that N is weakly M-torsionless. Let $f \in r_{[M,N]}([N, M])$ and $gf = 0$ for all $g \in [N, M]$. Since $[M, N]$ is regular, we can obtain that $f = fgf = 0$.

Let $x \in [M, N]$. Since $[M, N]$ is regular, there exists $y \in [N, M]$ such that $x = xyx$. Hence $[N, M]x = [N, M]xyx$. Let $e = yx$. Then $e^2 = e \in E_M$. Now $[N, M]x = [N, M]xe \subset E_M e$. The other inclusion is similar.

Assume that N is weakly M-torsionless and, for any $x \in [M,N]$, $[N,M]x = E_M e$ for some $e^2 = e \in E_M$. Then $e = yx$ for some $y \in [N,M]$. We must show that $x = xe$. For all $u \in [N,M]$, we can obtain that $u(x - xe) = ux - uxe = ux - (ux)e = 0$ since $ux \in [N,M]x = E_M e$. Then $x - xe \in r_{[M,N]}([N,M]) = 0$ and so $x = xyx$.

(2) Assume that $[M,N]$ is regular. By (1), we can obtain that $[N,M]x = E_M e$ for some $e^2 = e \in E_M$, i.e., (a) holds. For every $y \in [M,N]$, there exists $z \in [N,M]$ such that $y = yzy$ and so (b) holds.

For converse, let $y \in [M,N]$. Because of (1), we must show that N is weakly M-torsionless, i.e. $r_{[M,N]}([N,M]) = \{f \in [M,N] : gf = 0; \forall g \in [N,M]\}$. Since $y \in r_{[M,N]}([N,M])$, we can obtain that $f_i y = 0$ for some $f_1 \dots f_n \in [N,M]$. Therefore $y = (y_1 f_1 + \cdots + y_n f_n)y = 0$. ■

We have the following corollary.

Corollary 2.7. *The following conditions are equivalent for R-modules M and N:*

(1) $[M,N]$ *is regular.*

(2) *For any* $x_1, x_2, ..., x_n, y \in [M,N]$,

  (a) $\sum_{i=1}^{k} [N,M]x_i = E_M e$ *for some* $e^2 = e \in E_M$

  (b) $y = (y_1 f_1 + \cdots + y_n f_n)y$ *for some* $y_1, ..., y_n \in [M,N]$ *and for some* $f_1, ..., f_n \in [N,M]$.

(3) N *is weakly* M*-torsionless and, for every elements* $x_1, x_2, ..., x_n \in [M,N]$, $\sum_{i=1}^{n} [N,M]x_i = E_M e$ *for some* $e^2 = e \in E_M$.

Proof. (1) ⇒ (2). Assume that $[M,N]$ is regular. By Theorem 2.6(2), for each $y \in [M,N]$, we can obtain that $y = (y_1 f_1 + \cdots + y_n f_n)y$ for some $y_1, ..., y_n \in [M,N]$ and for some $f_1, ..., f_n \in [N,M]$.

Now assume that $x_1, x_2, ..., x_n \in [M,N]$. We show that $\sum_{i=1}^{n} [N,M]x_i = E_M e$ for some $e^2 = e \in E_M$. The case $n = 1$ is clear from Theorem 2.6. If $n > 1$, then $[N,M]x_n = E_M f$ for some $f^2 = f \in E_M$. By the hypothesis on induction,

$$\sum_{i=1}^{n-1} [N,M]x_i(1-f) = E_M g$$

for some $g^2 = g \in E_M$. It is easy to see that $gf = 0$, $e = f + g - fg$ is an idempotent, $fe = f = ef$ and $ge = g = eg$. Since $[N,M]x_i f \subset E_M f = [N,M]x_n$ for each $i = 1, 2, ..., n-1$, we can obtain that

$$\begin{aligned} E_M e &= E_M f + E_M g = \\ &= [N,M]x_n + (\textstyle\sum_{i=1}^{n-1} [N,M]x_i)(1-f) = \\ &= [N,M]x_n + \textstyle\sum_{i=1}^{n-1} [N,M]x_i. \end{aligned}$$

(2) $\Rightarrow$ (3). is clear.

(3) $\Rightarrow$ (1). By Theorem 2.6. ■

Regularity subsets are shown in the following result.

Proposition 2.8. *The following conditions are equivalent for R-modules M and N:*

(1) $[M,N]$ *is regular.*

(2) *For every non-empty subset H of $[M,N]$ with $H[N,M]H \subseteq H$ implies $H[M,N]H = H$.*

(3) *For every non-empty subset H of $[M,N]$ with $E_N H \cap HE_M \subseteq H$ implies $H[N,M]H = H$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Note that $H[N,M]H \subseteq E_N H \cap HE_M$. Thus (3) is clear.

(3) $\Rightarrow$ (1). For every $f \in [M,N]$, let $H = E_N f \cap fE_M$. We have that $E_N H \cap HE_M \subseteq H$ and obtain that $H[N,M]H = H$ by (3). Thus $f \in E_N f \cap \cap fE_M = H[N,M]H \subseteq f[N,M]f$, that means $f = fgf$ for some $g \in [N,M]$. ■

Lemma 2.9. *Assume that $u \in [M,N]$ and $v \in [N,M]$. Then:*

(1) $(u - uvu)(M) = u(M) \cap (1 - uv)(N)$ *and* $N = u(M) + (1 - uv)(N)$.

(2) $Ker(u - uvu) = Ker(u) \oplus Ker(1 - vu)$.

Proof. (1). It is easy to see that $u(M)+(1-uv)(N) = M$ and $(u-uvu)(M) \leq \leq u(M) \cap (1-uv)(N)$. On the other hand, for all $m \in u(M) \cap (1-uv)(N)$, we write $m = u(x) = (1 - uv)(n)$ with $x \in M$ and $n \in N$. Then $n = u(x + v(n))$ and hence $x = (1 - uv)(n) = (u - uvu)(x + v(n)) \in (u - uvu)(M)$.

(2). We have $Ker(u - uvu) \leq Ker(u) + Ker(1 - uv)$ and $Ker(u) \cap \cap Ker(1 - uv) = 0$. On the other hand, for all $m \in Ker(u - uvu)$, $m = vu(m)+(1-vu)(m)$ with $vu(m) \in Ker(1-vu)$ and $(1-vu)(m) \in Ker(u)$; so $Ker(u - uvu) = Ker(u) + Ker(1 - vu)$. ■

Let M and N be modules and let I be an E_M-E_N-submodule of $[M,N]$. $f \in [M,N]$ is called *I-regular* if there exists a $g \in [N,M]$ such that $fgf - f \in I$.

Following [6], a submodule X of N is called *semisupplement* of Y in N if $N = X + Y$ and $X \cap Y \ll N$.

Proposition 2.10. *The following conditions are equivalent for $u \in [M, N]$:*

(1) *u is ∇-regular.*

(2) *There exist $v \in [N, M]$ and a semisupplement X of $u(M)$ in N such that the following diagram is commutative:*

$$\begin{tikzcd} M \arrow[r, "u"] & N \arrow[d, "\pi_X"] \\ N \arrow[u, "v"] \arrow[r, "\pi_X"] & N/X. \end{tikzcd}$$

(3) *There exists a semisupplement X of $u(M)$ in N such that $(1-uv)(N) \leq X$ for some $v \in [N, M]$.*

Proof. (2) $\Leftrightarrow$ (3) is obvious.

(1) $\Rightarrow$ (2) Assume that there exists $v \in [N, M]$ such that $u - uvu \in \nabla$. Let $H = (u - uvu)(M) \leq N$. Then by Lemma 2.9, $X = (1 - uv)(N)$ is a semisupplement of $u(M)$ in N. Since $H \leq X$, $uvu(m) + X = u(m) + X \in N/X$ for all $m \in M$. It follows that $(\pi_X uv)(u(m)) = \pi_X(u(m))$. We have that $N = u(M) + X$ and obtain that $\pi_X uv = \pi_X$.

(3) $\Rightarrow$ (1) By (3), there exists a semisupplement X of $u(M)$ in N such that $(1 - uv)(N) \leq X$ for some $v \in [N, M]$. Then $(u - uvu)(M) = u(M) \cap$ $\cap (1 - uv)(N) \leq u(M) \cap X \ll N$ and so $(u - uvu)(M) \ll N$. It follows that $u - uvu \in \nabla$. ■

Again according to [6], a submodule X of N is called *semicomplement* of Y in M if $X \oplus Y \leq^e M$.

Proposition 2.11. *The following conditions are equivalent for $u \in [M, N]$:*

(1) *u is Δ-regular.*

(2) *There exists $v \in [N, M]$ and a semicomplement X of $Ker(u)$ in M such that the following diagram is commutative:*

$$\begin{tikzcd} u(X) \arrow[r, "i_1"] \arrow[d, "(u|_X)^{-1}"] & N \arrow[d, "v"] \\ X \arrow[r, "i_2"] & M. \end{tikzcd}$$

(3) *There exists a semicomplement X of $Ker(u)$ in M such that $X \leq$ $\leq Ker(1 - vu)$ for some $v \in [N, M]$.*

Proof. (2) $\Leftrightarrow$ (3) is obvious.

(1) $\Rightarrow$ (2). Assume that there exists $v \in [N, M]$ such that $u - uvu \in \Delta$. Let $H = (u - uvu)(M) \leq N$. Then by Lemma 2.9, $X = Ker(1 - vu)$ is a semicomplement of $Ker(u)$ in M. For all $x \in X$, we have $vu(x) = x$. It follows that following diagram is commutative in (2).

(3) $\Rightarrow$ (1). By (3), there exists a semicomplement X of $Ker(u))$ in M such that $X \leq Ker(1 - vu)$ for some $v \in [N, M]$. Then $Ker(u) \oplus X \leq$ $\leq Ker(u) \oplus Ker(1 - vu) = Ker(u - uvu)$ and so $Ker(u - uvu) \leq^e M$. It follows that $u - uvu \in \Delta$. ■

We call an R-module N *semisupplemented* if every submodule of N has a semisupplement.

Theorem 2.12. *Let M be a finitely generated, self-projective R-module and $N \in Gen(M)$. If $[M, N]_{E_M}$ is semisupplemented, then $[M, N]/\nabla[M, N]$ is semisimple.*

Proof. Let $\bar{A} = A/\nabla[M, N]$ be a submodule of $[M, N]/\nabla[M, N]$. Since $[M, N]$ is semisupplemented, there exists $B \leq [M, N]$ such that $[M, N] = A + B$ and $A \cap B \ll [M, N]$. For any $f \in A \cap B$, it is easy to see that $fE_M \leq A \cap B$ and $fE_M \ll [M, N]$ because $A \cap B \ll [M, N]$. Now we show that $f \in \nabla[M, N]$. Let K be a submodule of N with $M = Imf + K$. By [12, 18.4],

$$[M, N] = [M, f(M)] + [M, K].$$

It follows that $fE_M + [M, K] = [M, N]$. Since $fE_M \ll [M, N]$, we can obtain that $[M, K] = [M, N]$. Now $[M, K] = [M, N]$ gives that $N = [M, N]M =$ $= [M, K]M \leq K$ because of $N \in Gen(M)$. Therefore $N = K$, i.e., $Imf \ll N$. It follows that $f \in \nabla[M, N]$. ■

Recall that;

(D2) For any submodule A of M for which M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

(GD2) For any submodule A of M for which M/A is isomorphic to M, then A is a direct summand of M.

Lemma 2.13. *Let M and N be R-modules. If N satisfies* GD2*, then*

$$\nabla[M, N] \subseteq J[M, N].$$

Proof. See [9, Lemma 3.1]. ■

Corollary 2.14. *Let M be a finitely generated, self-projective R-module and $N \in Gen(M)$ satisfies* GD2*. If $[M, N]$ is semisupplemented, then*

$$[M, N]/J[M, N]$$

is semisimple.

Proof. It is clear from Theorem 2.12 and Lemma 2.13. ■

References

[1] **Anderson, F.W. and K.R. Fuller,** *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.

[2] **Beidar, K.I. and F. Kasch,** Good conditions for the total, *Trends Mathematics. Boston, MA: Birkhäuser*, 2001, 43–65.

[3] **Chen, H. and W.K. Nicholson,** On regular modules, *Bull. Aust. Math. Soc.*, **84(2)** (2011), 280–287.

[4] **Kasch, F. and A. Mader,** *Regularity and Substructures of Hom*, Frontiers in Mathematics, 2009.

[5] **Lee, T.K. and Y. Zhou,** Substructure of Hom. *J. Algebra Appl.*, **10(1)** (2011), 119–127.

[6] **Lee, T.K. and Y. Zhou,** On (strong) lifting of idempotents and semiregular endomorphism rings, *Colloq. Math.*, **125** (2011), 99–113.

[7] **Mohammed, S.H. and B.J. Müller,** *Continous and Discrete Modules*, London Math. Soc. LN 147: Cambridge Univ. Press, 1990.

[8] **Nicholson, W.K. and Y. Zhou,** Semiregular morphism, *Comm. Algebra*, **34** (2006), 219–233.

[9] **Quynh, T.C., M.T. Koşan and L.V. Thuyet,** On (semi)regular morphisms, *Commun. Algebra*, **41(8)** (2013), 2933–2947.

[10] **Quynh, T.C. and M.T. Koşan,** On the substructures ∇ and Δ, to appear in the *Rocky Mountain Journal of Mathematics.*

[11] **Vanaja, N.,** The torsion theory cogenerated by M-small modules, *Comm. Algebra*, **30** (2002), 1449–1460.

[12] **Wisbauer, R.,** *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[13] **Zelmanowitz, J.,** Regular modules, *Trans. Amer. Math. Soc.* **163** (1972), 341–355.

[14] **Zhou, Y.,** On (semi)regularity and the total of rings and modules, *J. Algebra* **322** (2009), 562–578.

T.C. Quynh
Department of Mathematics
Danang University
459 Ton Duc Thang
DaNang City
Vietnam
tcquynh@dce.udn.vn, tcquynh@live.com

M.T. Koşan
Department of Mathematics
Gebze Institute of Technology
Çayirova Campus
41400, Gebze-Kocaeli
Turkey mtkosan@gyte.edu.tr

P.T. Hai
Department of Mathematics
College of Pedagogy
Hue University
34 Le Loi, Hue City
Vietnam haikien2004@yahoo.com.vn