NOTE ON THE IDENTITY FUNCTION

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Dedicated to Professors Zoltán Daróczy and Imre Kátai
on the occasion of their 75th birthday

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Abstract. We consider the functional equation

$$f\left(n^2+m^2+a+b\right)=g(n^2+a)+g(m^2+b) \quad \text{for all} \quad n,m\in\mathbb{N},$$

where a, b are non-negative integers with $a+b>0$ and f, g are multiplicative functions.

1. Introduction

In the following, let $\mathbb{N}$ and $\mathcal{P}$ be the set of positive integers and prime numbers, respectively. We denote by $\mathcal{M}$ the set of all multiplicative functions f such that $f(1)=1$. Furthermore, we deal with the set $\mathcal{B}$ of non-negative integers which can be represented as a sum of two squares of integers and with $\mathcal{S}$ the set of all squares of positive integers. (m,n) denotes the greatest common divisor of the integers m, n and $(\frac{x}{p})$ denotes the Legendre symbol.

We say that subsets A and B of $\mathbb{N}$ are additive uniqueness sets (AU-sets) for $\mathcal{M}$ if there is exactly one element $f\in\mathcal{M}$ which satisfies

$$f(a+b)=f(a)+f(b) \quad \text{for all} \quad a\in A \quad \text{and} \quad b\in B.$$

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In 1992, C. Spiro [11] showed that $A = B = \mathcal{P}$ are AU-sets for $\mathcal{M}$. In [3] written jointly with J.-M. De Koninck and I. Kátai we proved that $A = \mathcal{S}$ and $B = \mathcal{P}$ are also AU-sets for $\mathcal{M}$. For other results we refer to [1], [2], [4], [5], [7], [8], [9] and [10]. For example, we proved the following two results:

Theorem A. ([9]) *If $a \in \mathbb{N}$ and $f \in \mathcal{M}$ satisfy the conditions $f(4)f(9) \neq 0$ and*

$$f\left(n^2 + m^2 + a\right) = f(n^2 + a) + f(m^2) \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2a) = 1$.

Theorem B. ([7]) *If a non-negative integer a and $f \in \mathcal{M}$ satisfy the conditions $f(2)f(5) \neq 0$ and*

$$f\left(n^2 + m^2 + a + 1\right) = f(n^2 + a) + f(m^2 + 1) \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2) = 1$.

Our purpose of this note is to prove the following

Theorem 1. *Assume that non-negative integers a, b with $a + b > 0$ and $f,\ g \in \mathcal{M}$ satisfy the condition*

$$f\left(n^2 + m^2 + a + b\right) = g(n^2 + a) + g(m^2 + b) \quad \text{for all} \quad n, m \in \mathbb{N}.$$

If either

$$g(i^2 + a) = i^2 + a \quad \text{for} \quad i = 1, 2 \dots, 6$$

or

$$g(j^2 + b) = j^2 + b \quad \text{for} \quad j = 1, 2 \dots, 6,$$

then

$$g(k^2 + a) = k^2 + a,\ \ g(k^2 + b) = k^2 + b \quad \text{for all } k \in \mathbb{N}$$

and

$$f(n) = n \quad \text{for all} \quad n \in \mathbb{N},\ \ (n, 2(a + b)) = 1.$$

For the case $f = g$, we have

Theorem 2. *Assume that non-negative integers a, b with $a + b > 0$ and $f \in \mathcal{M}$ satisfy the condition*

$$f\left(n^2 + m^2 + a + b\right) = f(n^2 + a) + f(m^2 + b) \quad \text{for all} \quad n, m \in \mathbb{N}.$$

If either

$$f(i^2 + a) = i^2 + a \quad \text{for} \quad i = 1, 2 \dots, 6$$

or

$$f(j^2+b)=j^2+b \quad \textit{for} \quad j=1,2\dots,6,$$

then

$$f(k^2+a)=k^2+a, \;\; f(k^2+b)=k^2+b \quad \textit{for all } k\in\mathbb{N}$$

and

$$f(n)=n \quad \textit{for all} \quad n\in\mathbb{N}, \;\; (n,2K)=1,$$

where

$$K=K(a,b):=(a,b)\prod_{\substack{p\mid a+b\\ \left(\frac{a}{p}\right)=\left(\frac{b}{p}\right)=-1}} p.$$

2. Proof of Theorem 1

We shall use the following results:

Lemma 1. *Let a and b be non-negative integers and F,G be arithmetical functions, for which the condition*

(1) $$F(n^2+m^2+a+b)=G(n^2+a)+G(m^2+b)$$

is satisfied for all $n,m\in\mathbb{N}$. For each $j\in\mathbb{N}$ let $S_j:=G(j^2+a)$. Then

(2) $$S_{n+12}=S_{n+9}+S_{n+8}+S_{n+7}-S_{n+5}-S_{n+4}-S_{n+3}+S_n$$

holds for all $n\in\mathbb{N}$ and

(3) $$\begin{cases} S_7 &= 2S_5-S_1\\ S_8 &= 2S_5+S_4-2S_1\\ S_9 &= S_6+2S_5-S_2-S_1\\ S_{10} &= S_6+3S_5-S_3-2S_1\\ S_{11} &= S_6+4S_5-S_3-S_2-2S_1\\ S_{12} &= S_6+4S_5+S_4-S_2-4S_1\end{cases}$$

Proof. The proof is similar to that in Lemma 1 of [9].

First we infer from (1) that

$$G(n^2+a)+G(m^2+b)=G(m^2+a)+G(n^2+b)$$

for all $n, m \in \mathbb{N}$, and so

$$G(n^2+b)-G(n^2+a)=G(1+b)-G(1+a) \quad \text{for all} \quad n\in\mathbb{N}.$$

Let

$$D := G(1+b)-G(1+a).$$

Then, we infer from (1) that

$$(4) \qquad F(n^2+m^2+a+b)=G(n^2+a)+G(m^2+a)+D \quad (\forall n,m\in\mathbb{N}).$$

In the following, for each $j\in\mathbb{N}$ let $S_j := G(j^2+a)$. It follows from (4) that if positive integers $k,\ l,\ u$ and v satisfying the condition

$$k^2+l^2=u^2+v^2,$$

then

$$F(k^2+l^2+a+b)=G(k^2+a)+G(l^2+a)+D=$$
$$=F(u^2+v^2+a+b)=G(u^2+a)+G(v^2+a)+D,$$

which shows that

$$(5) \qquad k^2+l^2=u^2+v^2 \quad \text{implies} \quad S_k+S_l=S_u+S_v.$$

Since

$$(2n+1)^2+(n-2)^2=(2n-1)^2+(n+2)^2$$

and

$$(2n+1)^2+(n-7)^2=(2n-5)^2+(n+5)^2$$

hold for all $n\in\mathbb{N}$, we get from (5) that

$$(6) \qquad S_{2n+1}+S_{n-2}=S_{2n-1}+S_{n+2}$$

and

$$S_{2n+1}+S_{n-7}=S_{2n-5}+S_{n+5}.$$

These imply that

$$S_{n+5}-S_{n+2}+S_{n-2}-S_{n-7}=S_{2n-1}-S_{2n-5}=$$
$$S_{n+1}-S_{n-3}+S_{2n-3}-S_{2n-5}=S_{n+1}-S_{n-3}+S_n-S_{n-4},$$

which proves (2).

Now we prove (3). Indeed, by using (6), we have

$$S_7=S_{2.3+1}=2S_5-S_1,$$

$$S_9 = S_{2.4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2.5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Finally, by using (5) and the facts

$$8^2 + 1^2 = 7^2 + 4^2, \quad 10^2 + 5^2 = 11^2 + 2^2 \quad \text{and} \quad 12^2 + 1^2 = 9^2 + 8^2,$$

we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1,$$

which completes the proof (3). Lemma 1 is proved. ∎

Lemma 2. (K-H. Indlekofer and N. M. Timofeev [6].) *Let C be non-zero integer and $A,\ B \in \mathbb{N}$ such that $(A,\ B) = 1$, $(AB,\ 2C) = 1$. Then there exists a positive constant $\theta = \theta(A, B, C)$ such that*

$$|\{n \le x : \ A(n + C) = B(m + C),\ (A,\ n + C) = 1,\ n,\ m \in \mathcal{B}\}| > \theta \frac{x}{\log x}$$

holds for all $x \ge x_0(A, B, C)$. Hence $\mathcal{B}$ is the set of non-negative integers which can be represented as a sum of two squares of integers.

Proof of Theorem 1. Assume that non-negative integers a, b with $a + b > 0$ and $f,\ g \in \mathcal{M}$ satisfy the condition

$$f(n^2 + m^2 + a + b) = g(n^2 + a) + g(m^2 + b)$$

for all $n,\ m \in \mathbb{N}$.

Case I: $g(i^2 + a) = i^2 + a$ for $i = 1, 2 \cdots, 6$.

Apply Lemma 1 with $f = F$ and $g = G$, it is clear to check from (3) that $S_i := g(i^2 + a) = i^2 + a$ is also true for all $1 \le\ i \le 12$. Assume that $S_n = n^2 + a$ for all $n \le N,\ N \ge 12$. Then we infer from (2) that

$$\begin{aligned} S_N = {} & [(N-3)^2 + a] + [(N-4)^2 + a] + [(N-5)^2 + a] - \\ & - [(N-7)^2 + a] - [(N-8)^2 + a] - [(N-9)^2 + a] + [(N-12)^2 + a] = \\ & = N^2 + a. \end{aligned}$$

Thus, we have proved that

$$S_n = g(n^2 + a) = n^2 + a \quad \text{for all} \quad n \in \mathbb{N}. \tag{7}$$

Next, we shall prove that

$$S_n = g(n^2 + b) = n^2 + b \quad \text{for all} \quad n \in \mathbb{N}. \tag{8}$$

Since $g(n^2 + b) = g(n^2 + a) + D,\ D = g(b + 1) - g(a + 1)$, we get from (7) that

$$g(n^2 + b) = (n^2 + a) + [g(b + 1) - g(a + 1)] = n^2 + [g(b + 1) - 1] = n^2 + L \tag{9}$$

for all $n \in \mathbb{N}$, where $L := g(b + 1) - 1$. From the relation

$$[n^2 + b][(n + 1)^2 + b] = (n^2 + n + b)^2 + b,$$

we infer from the multiplicativity of g that

$$g[n^2 + b]g[(n + 1)^2 + b] = g[(n^2 + n + b)^2 + b] \quad \text{if} \quad (2n + 1, 4b + 1) = 1.$$

This with (9) shows that

$$[n^2 + L][(n + 1)^2 + L] = [(n^2 + n + b)^2 + L] \quad \text{if} \quad (2n + 1, 4b + 1) = 1,$$

which gives

$$2n(n + 1)L + L^2 = 2n(n + 1)b + b^2 \quad \text{if} \quad (2n + 1, 4b + 1) = 1.$$

Since there are infinitely many $n \in \mathbb{N}$ such that $(2n + 1, 4b + 1) = 1$, the last relation shows that $L = b$. Therefore (9) completes the proof of (8).

Let $C := a + b$. We get from our assumptions and (7)–(8) that

$$f(\alpha + C) = \alpha + C \quad \text{for all} \quad \alpha \in \mathcal{B}, \tag{10}$$

where $\mathcal{B}$ denotes the set of non-negative integers which can be represented as a sum of two squares of integers.

By using Lemma 2, for each $n \in \mathbb{N},\ (n,\ 2C) = 1$ there are $\alpha,\ \beta \in \mathcal{B}$ such that

$$n(\alpha + C) = \beta + C, \quad (n,\ \alpha + C) = 1,$$

which with (10) implies

$$f(n)(\alpha + C) = f(n)f(\alpha + C) = f[n(\alpha + C)] = f(\beta + C) = \beta + C = n(\alpha + C).$$

Therefore

$$f(n) = n \quad \text{holds for all} \quad n \in \mathbb{N}, \quad (n,\ 2C) = 1. \tag{11}$$

Case II: $g(j^2 + b) = j^2 + b$ for $j = 1, 2 \cdots, 6$.

The proof is similar to Case I.

Theorem 1 is proved. ■

3. Proof of Theorem 2

Assume that non-negative integers a, b with $a+b>0$ and $f \in \mathcal{M}$ satisfy all conditions of Theorem 2. We infer from Theorem 1 that

$$f(k^2+a)=k^2+a,\ \ f(k^2+b)=k^2+b \ \text{ for all } k\in\mathbb{N} \tag{12}$$

and

$$f(n)=n \ \text{ for all } \ n\in\mathbb{N},\ (n,2(a+b))=1. \tag{13}$$

It is clear that Theorem 2 will follow if we can prove the following:

$$f(p^\ell)=p^\ell \ \text{ for } \ p\in\mathcal{P},\ p\nmid 2K=(a,b)\prod_{\substack{p\mid a+b\\ \left(\frac{a}{p}\right)=\left(\frac{b}{p}\right)=-1}} p. \tag{14}$$

Assume first that $p\in\mathcal{P},\ p>2,\ p|C,\ p\nmid(a,b),\ \ell\in\mathbb{N}$ and $(\frac{a}{p})=1$. We consider the equation

$$x^2+b=p^\ell y. \tag{15}$$

Since $(\frac{-b}{p})=(\frac{a}{p})=1$, therefore $(ab,p)=1$ and there are $x_\ell, y_\ell\in\mathbb{N}$ such that

$$x_\ell^2+b=y_\ell p^\ell \ \text{ and } \ \left(p^\ell-x_\ell\right)^2+b=\left(p^\ell-2x_\ell+y_\ell\right)p^\ell.$$

It is obvious that one of y_ℓ and $p^\ell-2x_\ell+y_\ell$ is coprime to p. Assume that $x_\ell, y_\ell\in\mathbb{N}$ satisfy (15) and $(y_\ell,p)=1$. Let $x=p^\ell t+x_\ell$ and $y=p^\ell t^2+2x_\ell t+y_\ell$. Then (x,y) is also a solution of (15).

Hence an application of the Chinese Remainder Theorem shows that there is $t_0\in\mathbb{N}$ for which

$$\left(p^\ell t_0^2+2x_\ell t_0+y_\ell, 2(k+1)\right)=1.$$

Thus we have proved that

$$(x_0,y_0)=(p^\ell t_0+x_\ell, p^\ell t_0^2+2x_\ell t_0+y_\ell)$$

is a solution of (15) with the condition $(y_0,2(k+1))=1$.

Finally, we infer from (12) and (13) that

$$p^\ell y_0=x_0^2+b=f\left(x_0^2+b\right)=f\left(p^\ell y_0\right)=f\left(p^\ell\right)f(y_0)=f\left(p^\ell\right)y_0,$$

which proves (14) for the case $(\frac{a}{p})=1$. Similarly, we prove (14) for the case $(\frac{b}{p})=1$.

Theorem 2 is proved. ■

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