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SIMPLE REMARKS ON SOME DIRICHLET SERIES

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Dedicated to Professors Zoltán Daróczy and Imre Kátai
on their 75th birthday

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Abstract. In this article, we shall reconsider, discuss, extend, some results related to the value-distribution of twisted L-functions, an extension of the universality of the Riemann Zeta function, and the self-similarity for the Riemann Zeta function.

1. Introduction

1.1. Menu

In this article, we shall consider three topics which are (in order): the value-distribution of twisted L-functions, some extension of the universality of the Riemann Zeta function, the self-similarity for the Riemann Zeta function.

The results (resp. the tools) are (same order): a general result on the value-distribution of twisted Dirichlet series (abstract measure-theory), how to extend some ghost universality in a space of continuous functions (general topology), a general self-similarity result for a class of Dirichlet series (Besicovitch almost-periodicity).

Since the main purpose of this contribution is a kind of clarification of the content of some results which can be found in the recent mathematical literature, and as a consequence, a simple extension of them, there will be no aim at the greatest generality.

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1.2. The space $\mathcal{V}(\alpha, \sigma_a)$.

For a given real number $\alpha > 0$, to be an element of the set $\mathcal{V}(\alpha, \sigma_a)$, a function $\mathbb{C} \to \mathbb{C}$ must be a Dirichlet series $\Sigma_{n \geq 0} A_n n^{-s}$ absolutely convergent for $\operatorname{Re} s = \sigma > \sigma_a$, defining a function $f(s)$ meromorphic in a wider half-plane $\operatorname{Re} s = \sigma > \alpha$, $\alpha \leq \sigma_a$, with eventually a finite set $\mathcal{Z}$ of poles and satisfying the following conditions:

(i) $\sup(1/T) \int_{-T}^{+T} |f(\sigma + it)|^2 \, dt = K_\sigma$ is finite for all $\sigma > \alpha$, $\sigma \notin \operatorname{Re} \mathcal{Z}$,

(ii) there exist a positive integer m and positive constants D and U_0 such that

$$|f(\sigma + it)| < D \left| \frac{t}{2} \right|^{m - \frac{1}{2}}$$

for

$$\alpha < \sigma, \quad |t| > U_0.$$

We remark that $\mathcal{Z}$ can be empty, that many of the Dirichlet series considered in number theory are in some $\mathcal{V}(\alpha, \sigma_a)$, and for more on the properties of the spaces $\mathcal{V}(\alpha, \sigma_a)$, see [9] and [10].

2. Value-distribution of twisted elements of $\mathcal{V}$.

2.1. Introduction

We denote by $L(s, F, 1)$ the Dirichlet series

$$L(s, F, 1) = \sum_{m \geq 1} \frac{c(m)}{m^s}$$

which is absolutely convergent for $\operatorname{Re} s > \sigma_a \, (> 0)$.

Now let $q \in \mathbb{N}$, q prime, and let $\chi(m)$ be a Dirichlet character modulo q. Then the twisted L-function $L(s, F, \chi)$ is defined for $\operatorname{Re} s > \sigma_a$ by the Dirichlet series

$$L(s, F, \chi) = \sum_{m \geq 1} \frac{\chi(m) c(m)}{m^s}.$$

For $Q \geq 2$, denote

$$M_Q = \sum_{q \leq Q} \sum_{\substack{\chi = \chi \bmod q \\ \chi \neq \chi_0}} 1$$

where q are prime numbers and χ_0 denotes the principal character $\operatorname{mod} q$.

Denote by

$$\mu_Q = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi \bmod q \\ \chi \ne \chi_0 \\ \cdots}} 1$$

where in place of dots a condition satisfied by a pair $(q, \chi \bmod q)$ is to be written.

We consider now the special case where

$$L(s, F, 1) = \sum_{m \ge 1} \frac{c(m)}{m^s}$$

is the L-function associated to a holomorphic normalized Hecke eigen cusp form of weight κ and $\sigma_a = (\kappa + 1)/2$.

Simple consequence of the uniform almost-periodicity, for any s such that $\operatorname{Re} s > (\kappa + 1)/2$, and any character χ, the function $t \mapsto |L(s + it, F, \chi)|$ has a limit distribution, i.e. for $T \to +\infty$ and any Borel set A in $\mathbb{R}$, $(A \in \mathcal{B}(\mathbb{R}))$,

$$\frac{1}{T} meas \{t \in \mathbb{R}, 0 \le t \le T; |L(s + it, F, \chi)| \in A\}$$

converge weakly to a limit law, say $m_\chi(A)$.

A. Laurinčikas and A. Kolupayeva in [2] deal with this kind of function, and their result is that

Theorem 1. *Let $\mu_Q(|L(\sigma + it, F, \chi)| \in A)$ defined by*

$$\mu_Q(|L(\sigma + it, F, \chi)| \in A) = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi \bmod q \\ \chi \ne \chi_0 \\ |L(\sigma + it, F, \chi)| \in A}} 1, \qquad A \in \mathcal{B}(\mathbb{R}).$$

The weak limit of $\mu_Q(|L(\sigma + it, F, \chi)| \in A)$, $Q \to +\infty$, exists.

It can be remarked that in [2], this limit is identified only by its characteristic transform.

In another article [3], they study a similar problem for the argument of $L(s + it, F, \chi)$ under the same hypothesis as above.

2.2. A problem

At the end of the introduction in [3], one can find (p. 287) the following statement:

”In this paper, as in [2], we consider the weak convergence of probability measures defined by the twists $L(s,F,\chi)$ in the half-plane of absolute convergence of the series defining $L(s,F,\chi)$. We believe that the results can be extended to the region $\sigma > \kappa/2$; however, in our opinion, this problem is very difficult. The principal difference from the case of Dirichlet L-function is that the series defining $L(s,F,\chi)$, $\chi \neq \chi_0$, does not converge in the above region.”

In such a case, the next result, which extend Theorem 1 and is rather general, and more specially, its method of proof, which does not require any convergence condition, can be of interest. A similar approach can be followed to deal with the argument of Dirichlet series as considered in [3]. This is left to the interested reader.

2.3. A result on the value-distribution of twisted elements of $\mathcal{V}$

Using the same notations as above:

Theorem 2. *Assume that $L(s,F,1)$ and all the $L(s,F,\chi)$, $\chi(m)$ being any Dirichlet character modulo q, q prime, are in $\mathcal{V}(\alpha,\sigma_a)$, and are holomorphic in $\mathrm{Re}\, s > \alpha$.*

Then, given any s such that $\mathrm{Re}\, s = \sigma > \alpha$,

$$\frac{1}{T} meas\left\{t \in \mathbb{R}, 0 \leq t \leq T; |L(s+it,F,\chi)| \in A, A \in \mathcal{B}(\mathbb{R})\right\}$$

converge weakly to a limit law, say $m_\chi(A)$, and the weak limit of

$$\mu_Q\left(|L(\sigma+it,F,\chi)| \in A\right), \quad Q \to +\infty,$$

exists and is the distribution of $t \mapsto |L(s+it,F,1)|$.

Proof.

$$\frac{1}{T} meas\left\{t \in \mathbb{R}, 0 \leq t \leq T; |L(s+it,F,\chi)| \in A, A \in \mathcal{B}(\mathbb{R})\right\}$$

converge weakly to a limit law is an immediate consequence of the fact that if a Dirichlet series

$$h(s) = \sum_{n>0} a_n n^{-s}$$

is in $\mathcal{V}(\alpha,\sigma_a)$, then, given any s such that $\mathrm{Re}\, s > \alpha$, the function $t \mapsto h(s+it)$ is $B^2.A.P$ (Besicovitch almost-periodic for the exponent 2) and it has a limit distribution function.

We shall use now the theory of $B^2.A.P.$ as described in [10] (see the section 11, p.241 and seq.)

$\mathcal{P}$ denoting the set of the primes, consider the compact group

$$\Omega = \prod_{p \in \mathcal{P}} \Omega_p,$$

where Ω_p is the dual group of the discrete group $\mathbb{Z} \log p$, equipped as usual with its (normalized) Haar measure $\mu = \bigotimes_{p \in \mathcal{P}} d\mu_p$. So, any ω of Ω is a sequence $(\omega_p ; \omega_p \in \Omega_p, p \in \mathcal{P})$.

Define $L(s, \omega, F, \chi)$ as

$$L(s, \omega, F, \chi) = \sum_{m \geq 1} \frac{\chi(m) c(m) \omega(m)}{m^s}$$

where $\omega(m)$ is

$$\omega(m) = \prod_{p^\alpha \| m} \omega_p^\alpha.$$

$L(s, \omega, F, \chi)$ is in $\mathcal{L}^2(\Omega, d\mu)$ since

$$\sum_{m \geq 1} \frac{|c(m)|^2}{m^{2\sigma}} < +\infty.$$

Moreover, for a given prime q, there exists some θ in Ω such that

$$\begin{aligned} \theta_p &= \chi(p) \text{ if } p \neq q \\ &= 1 \text{ if } p = q. \end{aligned}$$

We recall that the asymptotic distribution of $L(s + it, F, \chi)$ exists and is the same as the distribution of the function $L(s, \omega, F, \chi)$.

But for a given prime q, denoting by χ_0 the principal character $\bmod q$ and by χ any Dirichlet character modulo q, we have

$$L(s, \omega, F, \chi) = L(s, \omega\theta, F, \chi_0)$$

and by invariance for translation of the Haar measure, this distribution is the distribution of $L(s, \omega, F, \chi_0)$ which is itself for the same reason the distribution of $L(\sigma, \omega, F, \chi_0)$.

Now, the Fourier series of $\omega \mapsto (L(\sigma, \omega, F, 1) - L(\sigma, \omega, F, \chi_0))$ is

$$\sum_{m \geq 1} c(m) \omega(m) m^{-\sigma} - \sum_{\substack{m \geq 1 \\ (m,q)=1}} c(m) \omega(m) m^{-\sigma}$$

and this implies that

$$\begin{aligned}
&\left|\int_\Omega |L(\sigma,\omega,F,1)|\,d\omega - \int_\Omega |L(\sigma,\omega,F,\chi_0)|\,d\omega\right| \leq \\
\leq\ & \int_\Omega |L(\sigma,\omega,F,1) - L(\sigma,\omega,F,\chi_0)|\,d\omega \leq \\
\leq\ & \left(\int_\Omega |L(\sigma,\omega,F,1) - L(\sigma,\omega,F,\chi_0)|^2\,d\omega\right)^{\frac{1}{2}} \times \left(\int_\Omega d\omega\right)^{\frac{1}{2}} = \\
=\ & \left(\sum_{\substack{m\geq 1\\ q|m}} \frac{|c(m)|^2}{m^{2\sigma}}\right)^{\frac{1}{2}} = \\
=\ & o(1), q \to +\infty,
\end{aligned}$$

since

$$\sum_{\substack{m\geq 1\\ q|m}} \frac{|c(m)|^2}{m^{2\sigma}} \leq \sum_{m\geq q} \frac{|c(m)|^2}{m^{2\sigma}}$$

and

$$\sum_{m\geq 1} \frac{|c(m)|^2}{m^{2\sigma}} < +\infty.$$

This gives us that the distribution of $L(\sigma,\omega,F,\chi)$, which is the asymptotic distribution of $L(s+it,F,\chi)$, converges weakly to the distribution of $L(\sigma,\omega,F,1)$, which is the asymptotic distribution of $L(s+it,F,1)$.

This implies immediately that on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, for the weak convergence,

$$\lim_{Q\to+\infty} \mu_Q\left(|L(s+it,F,\chi)| \in A\right) = \mu_1\left(|L(s+it,F,1)| \in A\right). \qquad \blacksquare$$

3. Extension of universality and general topology.

3.1. Notations

D is defined by

$$D = \left\{ s \in \mathbb{C}; Res \in \left]\frac{1}{2}, 1\right[\right\},$$

S_ζ by

$$S_\zeta = \{g \in H(D) \mid \forall s \in D, g(s) \neq 0 \text{ or } g(s) \equiv 0 \text{ on } D\}\,.$$

$H(D)$ is the set of functions holomorphic on D.

Now, consider the set U of functions F defined on $H(D)$ with values in $H(D)$ such that

F is continuous on $H(D)$,

$\forall\, G$ an open set of $H(D)$, $F^{-1}(G) \cap S_\zeta \neq \emptyset$.

3.2. Problem and result

In [4], among other theorems, the following result is proved:

Theorem 3. *Let K be any compact subset of the strip $Res \in \left]\frac{1}{2}, 1\right[$ with connected complement, F in U, and $f(s)$ be any function analytic in $\overset{o}{K}$, the interior of K, and continuous on K. Then for every $\varepsilon > 0$,*

$$\liminf_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s\in K} |F(\zeta(s+it)) - f(s)| \leq \varepsilon \right\} > 0.$$

This result is an extension, in some sense, of a universality theorem for the Riemann Zeta function, which says that

Theorem 4. *Let K be any compact subset of the strip $Res \in \left]\frac{1}{2}, 1\right[$ with connected complement, and $f(s)$ be any non-vanishing function analytic in $\overset{o}{K}$, the interior of K, and continuous on K. Then for every $\varepsilon > 0$,*

$$\liminf_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s\in K} |\zeta(s+it) - f(s)| \leq \varepsilon \right\} > 0.$$

Now, as mentioned by the author, "the principal ingredient for the proof of universality for $F(\zeta(s))$ is a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions".

Other results of this kind can be found in the literature (see for instance [5], [6], [7], [8] for variations on the Hurwitz zeta function, the Lerch zeta function, discrete universality, etc.) but in all these cases, the method of proof is basically the same, relying on some limit theorem in the sense of weak convergence of probability measures in a space of analytic functions.

But in fact, an elementary argument of a topological nature is sufficient to prove this kind of result. To see that, first, we shall reconsider the case of the Riemann zeta function, and more precisely, the hypothesis of Theorem 3 above relative to the set U, considered by the author of [4] as difficult to check. In such case, it is necessary to consider carefully this hypothesis.

So, what is U? Answer: U is a set of continuous functions $H(D) \to H(D)$ such that $F(S_\zeta)$ is dense in the whole of $H(D)$.

To see that, assume that there is some function $g(s)$ in $H(D)$, a metric space with a distance d, such that for some $r > 0$, there is no $\gamma(s)$ in S_ζ such that $d(g(s), F(\gamma(s)) \leq r$.

So, the open ball $B(g(s), r)$ defined as

$$B(g(s), r) = \{l(s) \in H(D) \mid d(g(s), l(s)) < r\}$$

is such that $B(g(s), r) \cap F(S_\zeta) = \emptyset$. And as a consequence, $F^{-1}(B(g(s), r)) \cap$ $\cap S_\zeta = \emptyset$, which contradicts the definition of U.

As we shall see now, such an hypothesis is sufficient to extend universality even if this original universality does not exist.

3.3. Extension of a ghost universality to $\mathcal{C}(D, \mathbb{C})$.

Let $Z(s)$ be in $\mathcal{C}(D, \mathbb{C})$, the space of continuous complex-valued functions defined on D equipped with the uniform topology, and denote by S_Z the set of the $h(s)$ in $\mathcal{C}(D, \mathbb{C})$ such that for any K compact of D and any $\varepsilon > 0$,

$$\liminf_{T \to +\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s \in K} |Z(s + it) - h(s)| \leq \varepsilon \right\} > 0$$

Now, consider the set V of functions F defined on $\mathcal{C}(D, \mathbb{C})$ with values in $\mathcal{C}(D, \mathbb{C})$ such that

F is continuous on $\mathcal{C}(D, \mathbb{C})$,

$\forall$ G an open set of $\mathcal{C}(D, \mathbb{C})$, $F^{-1}(G) \cap S_Z \neq \emptyset$.

(Remark that this is the same hypothesis as what was stated above for the set U.)

We have the following universality result:

Theorem 5. *Let K be any compact subset of the strip $Res \in \left]\frac{1}{2}, 1\right[$, F in V, and $h(s)$ be any function belonging to $\mathcal{C}(D, \mathbb{C})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to +\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s \in K} |F(Z(s + it)) - h(s)| \leq \varepsilon \right\} > 0.$$

Proof. As remarked above, V is a set of continuous functions $\mathcal{C}(D, \mathbb{C}) \to$ $\to \mathcal{C}(D, \mathbb{C})$ such that if F is in V, $F(S_Z)$ is dense in the whole of $\mathcal{C}(D, \mathbb{C})$.

Now, denoting by $\|.\|$ the norm $\sup\limits_{s\in K}|.|$, for $h(s)\in\mathcal{C}(D,\mathbb{C})$, $\gamma(s)\in S_Z$,

$$\|F(Z(s+it))-h(s)\|\leq\|F(Z(s+it))-F(\gamma(s))\|+\|F(\gamma(s))-h(s)\|\,.$$

Let $\varepsilon>0$. Since $F(S_Z)$ is dense in the whole of $\mathcal{C}(D,\mathbb{C})$, we select a $\gamma(s)$ in S_Z such that $\|F(\gamma(s))-h(s)\|\leq\varepsilon/2$.

Since F is continuous on $\mathcal{C}(D,\mathbb{C})$, for any given $a(s)$ in $\mathcal{C}(D,\mathbb{C})$, there exists some $\eta>0$ such that if $b(s)$ is in $\mathcal{C}(D,\mathbb{C})$ and satisfies $\|a(s)-b(s)\|\leq\eta$, then the inequality $\|F\left(a(s)\right)-F(b(s))\|\leq\varepsilon/2$ holds.

By hypothesis,

$$\liminf_{T\to+\infty}\frac{1}{T}meas\left\{t\in\mathbb{R},0\leq t\leq T;\|Z(s+it)-\gamma(s)\|\leq\eta\ \right\}>0.$$

Since for a given t

$$\|Z(s+it)-\gamma(s)\|\leq\eta \text{ implies that } \|F(Z(s+it))-F(\gamma(s))\|\leq\varepsilon/2,$$

we get that on the set E_η defined as

$$E_\eta=\left\{t\in\mathbb{R},0\leq t\leq T;\|Z(s+it)-\gamma(s)\|\leq\eta\ \right\},$$

we have

$$\begin{aligned}\|F(Z(s+it))-h(s)\| &\leq \|F(Z(s+it))-F(\gamma(s))\|+\|F(\gamma(s))-h(s)\|\leq\\ &\leq \frac{1}{2}\varepsilon+\|F(\gamma(s))-h(s)\|\,,\end{aligned}$$

and by choice of $\gamma(s)$, we obtain that

$$\begin{aligned}\|F(Z(s+it))-h(s)\| &\leq \|F(Z(s+it))-F(\gamma(s))\|+\|F(\gamma(s))-h(s)\|\leq\\ &\leq \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon\leq\\ &\leq \varepsilon.\end{aligned}$$

Hence we get that on the set E_η, we have $\|F(Z(s+it))-h(s)\|\leq\varepsilon$, and since

$$\lim\inf_{T\to+\infty}\frac{1}{T}meas\left\{0\leq t\leq T,t\in E_\eta\right\}>0,$$

we get that for any given $\varepsilon>0$, any $h(s)$ in $\mathcal{C}(D,\mathbb{C})$, we have

$$\liminf_{T\to+\infty}\frac{1}{T}meas\left\{t\in\mathbb{R},0\leq t\leq T;\sup_{s\in K}|F(Z(s+it))-h(s)|\leq\varepsilon\right\}>0.$$

Similarly, if by hypothesis,

$$\liminf_{N\to+\infty}\frac{1}{N}\sharp\left\{n\in\mathbb{N},1\le n\le N;\|Z(s+inv)-\gamma(s)\|\le\eta\ \right\}>0.$$

since for a given t

$$\|Z(s+it)-\gamma(s)\|\le\eta \text{ implies that } \|F(Z(s+it))-F(\gamma(s))\|\le\varepsilon/2,$$

we get that on the set E_η^* defined as

$$E_\eta^*=\left\{n\in\mathbb{N},1\le n\le N;;\|Z(s+inv)-\gamma(s)\|\le\eta\ \right\},$$

we have

$$\begin{aligned}\|F(Z(s+inv))-h(s)\| &\le \|F(Z(s+inv))-F(\gamma(s))\|+\|F(\gamma(s))-h(s)\|\le\\ &\le \frac{1}{2}\varepsilon+\|F(\gamma(s))-h(s)\|,\end{aligned}$$

and by choice of $\gamma(s)$, we obtain that

$$\begin{aligned}\|F(Z(s+inv))-h(s)\| &\le \|F(Z(s+inv))-F(\gamma(s))\|+\|F(\gamma(s))-h(s)\|\le\\ &\le \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon\le\\ &\le \varepsilon.\end{aligned}$$

Hence we get that on the set E_η^*, we have $\|F(Z(s+inv))-h(s)\|\le\varepsilon$, and since

$$\liminf_{N\to+\infty}\frac{1}{N}\sharp\left\{n\in E_h^*,1\le n\le N\right\}>0,$$

we get that for any given $\varepsilon>0$, any $h(s)$ in $\mathcal{C}(D,\mathbb{C})$, we have

$$\liminf_{T\to+\infty}\frac{1}{N}\sharp\left\{n\in\mathbb{N},1\le n\le N;\sup_{s\in K}|F(Z(s+inv))-h(s)|\le\varepsilon\right\}>0.$$

We remark that the proof is purely topological and relies essentially on the hypothesis on the set V which provides all the tools needed to get the result. ■

4. Self-similarity and almost-periodicity

4.1. Introduction

The generalized strong recurrence property for the Riemann Zeta function ζ with respect to d means that for any compact K of $\mathcal{V}(1/2,1)$ with connected complement and any $\varepsilon > 0$,

$$\liminf_{T \to +\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s \in K} |Z(s+it) - Z(s+idt)| \leq \varepsilon \right\} > 0.$$

To simplify, we shall remain in the case with only one parameter d. Dealing with a finite set of parameters is similar.

B. Bagchi showed [1] that the Riemann Hypothesis is equivalent to the strong recurrence property, alias the generalized strong recurrence property with respect to $d = 0$.

The generalized strong recurrence for ζ with respect to almost all real numbers was verified by T. Nakamura [11]. Presented in the context of the universality of ζ, - as related to the strong recurrence property, and, of course, in relation with the Riemann hypothesis -, this property is in fact general for elements of $\mathcal{V}(1/2, 1)$.

4.2. A general result for elements of $\mathcal{V}(1/2, 1)$

We have the following result:

Theorem 6. *let $Z(s)$ be an analytic element of $\mathcal{V}(1/2, 1)$. Then, apart for an atmost countable set of real numbers S, the following statement holds:*

given $\varepsilon > 0$, d a real number $\neq 0$ and not in S, K any compact set of the strip $\operatorname{Re} s \in]1/2, 1[$, the relation

$$\liminf_{T \to +\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T; \sup_{s \in K} |Z(s+it) - Z(s+idt)| \leq \varepsilon \right\} > 0$$

holds.

Proof. $\mathcal{C}(K, \mathbb{C})$ denoting the space of continuous complex-valued functions defined on K equipped with the uniform topology, we recall that for any $a(s)$ in $\mathcal{C}(K, \mathbb{C})$, the function

$$t \mapsto \sup_{s \in K} |Z(s+it) - a(s)|$$

is $B^2.A.P.$ [9], and so,it has a distribution law $v_a(.)$ and at its continuity points r, we can define the measure m of the ball $B(a, r)$ centered in a of radius r by

$$m(B(a, r)) = v_a(r)$$

and extend it as a Radon measure, still denoted m, to the whole of $\mathcal{C}(K, \mathbb{C})$. Since $\mathcal{C}(K, \mathbb{C})$ is a polish space, it is a Lindelöf space and as a simple consequence, the support of m is of full measure and so, it is not empty.

So, let $\varepsilon > 0$, $a(s)$ be in the support of m. The function

$$t \mapsto \sup_{s\in K} |Z(s+it) - a(s)|$$

is $B^2.A.P.$ with a distribution law $v_a(.)$ and at its continuity points r, the function

$$\begin{aligned} I_{a,r}(t) &= 1 \text{ if } \sup_{s\in K} |Z(s+it) - a(s)| < r \\ &= 0 \text{ if } \sup_{s\in K} |Z(s+it) - a(s)| \geq r \end{aligned}$$

is $B^2.A.P.$ and its spectrum is in the set $\log \mathbb{Q}^*$, where $\mathbb{Q}^*$ denotes the set of the positive rationals.

Similarly, the function

$$\begin{aligned} J_{a,r}(t) &= 1 \text{ if } \sup_{s\in K} |Z(s+idt) - a(s)| < r \\ &= 0 \text{ if } \sup_{s\in K} |Z(s+idt) - a(s)| \geq r \end{aligned}$$

is $B^2.A.P.$ and its spectrum is in the set $d.\log \mathbb{Q}^*$.

Now, assume that x is in $(d \log \mathbb{Q}^*) \cap \log \mathbb{Q}^*$, $x \neq 0$. x can be written as $x = d \log q_1 = \log q_2$, q_1, q_2 in $\mathbb{Q}^*$, and so, $d = \log q_2 / \log q_1$. This gives us that the set S of the d such that $d \log \mathbb{Q}^* \cap \log \mathbb{Q}^* \neq \{0\}$ is atmost countable.

Remark that, since if for some t,

$$\sup_{s\in K} |Z(s+it) - a(s)| < r \quad \text{and} \quad \sup_{s\in K} |Z(s+idt) - a(s)| < r,$$

for this t , we shall have $\sup_{s\in K} |Z(s+it) - Z(s+idt)| < 2r$, and so, the set

$$A(T) = \left\{0 \leq t \leq T; \sup_{s\in K} |Z(s+it) - a(s)| < r, \sup_{s\in K} |Z(s+idt) - a(s)| < r\right\}$$

is contained in

$$B(T) = \left\{0 \leq t \leq T; \sup_{s\in K} |Z(s+it) - Z(s+idt)| < 2r\right\}$$

Now, if d is not in S, we get that

$$\liminf_{T\to+\infty} \frac{1}{T} \int_0^T I_{a,r}(t) J_{a,r}(t) dt =$$

$$= \liminf_{T\to+\infty} \frac{1}{T} meas \left\{0 \leq t \leq T; \sup_{s\in K} |Z(s+it) - a(s)| < r, \right.$$

$$\left. \sup_{s\in K} |Z(s+idt) - a(s)| < r \right\} =$$

$$= \lim_{T\to+\infty} \frac{1}{T} meas \left\{ 0 \le t \le T; \sup_{s\in K} |Z(s+it) - a(s)| < r, \right.$$
$$\left. \sup_{s\in K} |Z(s+idt) - a(s)| < r \right\}$$

and this is equal to the product of

$$\lim_{T\to+\infty} \frac{1}{T} meas \left\{ 0 \le t \le T; \sup_{s\in K} |Z(s+it) - a(s)| < r \right\}$$

by

$$\lim_{T\to+\infty} \frac{1}{T} meas \left\{ 0 \le t \le T; \sup_{s\in K} |Z(s+idt) - a(s)| < r \right\}$$

since $(d \log \mathbb{Q}^*) \cap \log \mathbb{Q}^* = \{0\}$, and this is a positive quantity since $a(s)$ is in the support of m. So, we get that

$$0 < \lim_{T\to+\infty} \frac{1}{T} meas \{ t \in A(T) \}$$

and since

$$\liminf_{T\to+\infty} \frac{1}{T} meas \{ t \in A(T) \} = \lim_{T\to+\infty} \frac{1}{T} meas \{ t \in A(T) \}$$

and $A(T) \subseteq B(T)$, we deduce that

$$0 < \liminf_{T\to+\infty} \frac{1}{T} meas \{ t \in A(T) \} \le \liminf_{T\to+\infty} \frac{1}{T} meas \{ t \in B(T) \}$$

and so we have

$$0 < \liminf_{T\to+\infty} \frac{1}{T} meas \left\{ 0 \le t \le T; \sup_{s\in K} |Z(s+it) - Z(s+idt)| < 2r \right\}.$$

This holds for any $r < \varepsilon/2$, which gives that

$$0 < \liminf_{T\to+\infty} \frac{1}{T} meas \left\{ 0 \le t \le T; \sup_{s\in K} |Z(s+it) - Z(s+idt)| < \varepsilon \right\}.$$

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