

QUASI-SUM REPRESENTATION FOR AN m -PLACE FUNCTION

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*Dedicated to the 75th birthday
of Professors Zoltán Daróczy and Imre Káta*

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Abstract. Aczél, in his book, reported on the associativity equation

$$F(F(x, y), z) = F(x, F(y, z))$$

on a real interval. Conditions leading to the quasi-sum representation

$$F(x, y) = f(f^{-1}(x) + f^{-1}(y))$$

were obtained. We expand on this and obtain results for an m -place function.

1. Introduction

The associativity equation $F(F(x, y), z) = F(x, F(y, z))$ is featured in the book of Aczél [1]. In the case of semigroups the main result is the following

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Theorem 1.1 (see [1], §6.2.2). *Let I be an interval. If for all x, y in I , $F(x, y)$ always lies in I and F is reducible on both sides, (i.e. F is injective in both variables), then*

$$(1.1) \quad F(x, y) = f(f^{-1}(x) + f^{-1}(y))$$

with continuous and strictly monotonic f is the general continuous solution of the associativity equation

$$(1.2) \quad F(F(x, y), z) = F(x, F(y, z)).$$

Craig and Páles ([2]) offered a new proof.

We shall extend the result to m -place functions. The following uniqueness theorem will be invoked.

Theorem 1.2 (see [3]). *Let X and Y be real intervals and $T : X \times Y \rightarrow \mathbb{R}$ be a function continuous in each of its two variables. Consider the functional equation*

$$(1.3) \quad \begin{aligned} \phi(x) + \psi(y) &= \eta(T(x, y)), \quad x \in X, y \in Y, \\ \phi : X &\rightarrow \mathbb{R}, \psi : Y \rightarrow \mathbb{R}, \eta : T(X \times Y) \rightarrow \mathbb{R}. \end{aligned}$$

If (ϕ_0, ψ_0, η_0) is a solution with non-constant continuous ϕ_0 and non-constant ψ_0 , then

$$\phi = \alpha\phi_0 + \beta_1, \quad \psi = \alpha\psi_0 + \beta_2, \quad \eta = \alpha\eta_0 + \beta_1 + \beta_2$$

where $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ are constants, give the general solutions (ϕ, ψ, η) with continuous ϕ .

2. A result for m -place functions

Theorem 2.1. *Let I be an interval. Let $F : I^m \rightarrow I$ be a function having the following properties:*

- (i) $F(x_1, x_2, \dots, x_m)$ is continuous in the variables (x_1, x_2) , injective in x_1 and in x_2 .
- (ii) $F(x_1, x_2, \dots, x_m)$ is symmetric in the variables $x_2, x_3, \dots, x_m$.
- (iii)

$$(2.1) \quad \begin{aligned} F(F(x_1, x_2, \dots, x_m), x_{m+1}, \dots, x_{2m-1}) &= \\ &= F(x_1, F(x_2, \dots, x_m, x_{m+1}), x_{m+2}, \dots, x_{2m-1}). \end{aligned}$$

Then F admits the representation

$$(2.2) \quad f(F(x_1, x_2, \dots, x_m)) = \sum_{i=1}^m f(x_i)$$

for some strictly increasing continuous bijective function $f : I \rightarrow J$ (for some interval J).

Proof. *Step 1.* We claim that, for arbitrarily fixed $x_3, \dots, x_m$, the function

$$H(x_1, x_2) := F(x_1, x_2, \dots, x_m)$$

satisfies the associativity equation

$$H(H(u, v), w) = H(u, H(v, w)).$$

Proof of the claim: If $m = 2$, there is really no $x_3, \dots, x_m$ to fix, assumption (ii) is redundant and $H = F$. The associativity equation and the equation (2.1) coincide.

We now work with $m > 2$.

$$\begin{aligned} H(H(u, v), w) &= F(F(u, v, x_3, \dots, x_m), w, x_3, \dots, x_m) = \\ &= F(u, F(v, x_3, \dots, x_m, w), x_3, \dots, x_m) = && \text{by (2.1)} \\ &= F(u, F(v, w, x_3, \dots, x_m), x_3, \dots, x_m) = && \text{by assumption (ii)} \\ &= H(u, H(v, w)). \end{aligned}$$

Step 2. By Theorem 1.1 there exists a function f such that H has the representation

$$H(u, v) = f^{-1}(f(u) + f(v)).$$

As a consequence, H is symmetric.

If $m = 2$, we have arrived at the representation (2.2) as $F = H$ and we are done.

We now continue with the case $m > 2$.

To make clear that f may depend on the fixed $x_3, \dots, x_m$ we denote it as $f_{x_3, \dots, x_m}$ and write the above representation as

$$(2.3) \quad F(x_1, x_2, \dots, x_m) = f_{x_3, \dots, x_m}^{-1}(f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2)).$$

The symmetry of H translates into

$$F(x_1, x_2, \dots, x_m) \text{ is symmetric in } x_1 \text{ and } x_2.$$

Step 3. The symmetry of $F(x_1, x_2, \dots, x_m)$ in x_1 and x_2 and the symmetry assumption (ii) imply that

$$(2.4) \quad F \text{ is a symmetric function of all of its variables.}$$

Step 4. Consider the function $G : I^{2m-1} \rightarrow I$ induced by

$$G(x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{2m-1}) := F(F(x_1, x_2, \dots, x_m), x_{m+1}, \dots, x_{2m-1}).$$

The symmetry of F , (2.4), and the identity (2.1), along with combinatorial manipulation, imply that

$$(2.5) \quad G \text{ is a symmetric function of all of its variables.}$$

Step 5. Let $c \in I$ be arbitrarily fixed. Then (2.5) yields in particular

$$(2.6) \quad F(F(x_1, x_2, \dots, x_m), c, c, \dots, c) = F(F(x_1, x_2, c, c, \dots, c), c, x_3, \dots, x_m).$$

Note that we do not require the existence of a common c such that (2.6) holds. In fact, one can fix $x_{m+1} = c_2, x_{m+2} = c_3, \dots, x_{2m-1} = c_m$ arbitrarily and the remainder of the proof can be adjusted by just adding extra subscripts to c .

Step 6. The strict monotonicity of F in its first argument allows us to infer from (2.6) that

$$(2.7) \quad F(x_1, x_2, \dots, x_m) = \phi(F(x_1, x_2, c, c, \dots, c), x_3, \dots, x_m)$$

for some function ϕ .

In fact, let ψ denote the inverse function of $y \mapsto F(y, c, c, \dots, c)$ and we may take $\phi = \psi \circ F$.

Step 7. Consider (2.3) which gives rise to

$$F(x_1, x_2, \dots, x_m) = f_{x_3, \dots, x_m}^{-1}(f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2))$$

and

$$F(x_1, x_2, c, \dots, c) = f_{c, c, \dots, c}^{-1}(f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2)).$$

Putting them into (2.7) we get

$$\begin{aligned} f_{x_3, \dots, x_m}^{-1}(f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2)) &= \\ &= \phi(f_{c, c, \dots, c}^{-1}(f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2)), x_3, \dots, x_m). \end{aligned}$$

It can be rewritten:

$$\begin{aligned} f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2) &= \\ &= f_{x_3, \dots, x_m}(\phi(f_{c, c, \dots, c}^{-1}(f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2)), x_3, \dots, x_m)) = \\ &= h_{x_3, \dots, x_m}(f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2)) \end{aligned}$$

where $h_{x_3, \dots, x_m}$ is defined by

$$h_{x_3, \dots, x_m}(t) = f_{x_3, \dots, x_m}(\phi(f_{c, c, \dots, c}^{-1}(t), x_3, \dots, x_m)).$$

The resulting equation

$$\begin{aligned} f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2) &= \\ &= h_{x_3, \dots, x_m}(f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2)) \end{aligned}$$

is in a form to which Theorem 1.2 can be readily applied. It leads to the existence of constants $\alpha(x_3, \dots, x_m) > 0$ and $\beta(x_3, \dots, x_m)$ such that

$$(2.8) \quad f_{x_3, \dots, x_m}(x) = \alpha(x_3, \dots, x_m)f_{c, c, \dots, c}(x) + \beta(x_3, \dots, x_m).$$

That is, $f_{x_3, \dots, x_m}$ and $f_{c, c, \dots, c}$ are affine functions of each other. Rewrite (2.3) first as

$$f_{x_3, \dots, x_m}(F(x_1, x_2, \dots, x_m)) = f_{x_3, \dots, x_m}(x_1) + f_{x_3, \dots, x_m}(x_2)$$

and then replace $f_{x_3, \dots, x_m}$ using (2.8) we get

$$\begin{aligned} \alpha(x_3, \dots, x_m)f_{c, c, \dots, c}(F(x_1, x_2, \dots, x_m)) + \beta(x_3, \dots, x_m) &= \\ &= \alpha(x_3, \dots, x_m)f_{c, c, \dots, c}(x_1) + \beta(x_3, \dots, x_m) + \\ &+ \alpha(x_3, \dots, x_m)f_{c, c, \dots, c}(x_2) + \beta(x_3, \dots, x_m). \end{aligned}$$

Eliminating a common term and dividing through by $\alpha(x_3, \dots, x_m)$ we get

$$(2.9) \quad \begin{aligned} f_{c, c, \dots, c}(F(x_1, x_2, \dots, x_m)) &= \\ &= f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2) + \gamma(x_3, \dots, x_m) \end{aligned}$$

where $\gamma := \beta/\alpha$.

Step 8. The symmetry of F on the left side of (2.9) yields the symmetry of γ and the symmetry of $f_{c, c, \dots, c}(x_1) + f_{c, c, \dots, c}(x_2) + \gamma(x_3, \dots, x_m)$ in all variables. The latter further implies that

$$(2.10) \quad \gamma(x_3, \dots, x_m) = f_{c, c, \dots, c}(x_3) + f_{c, c, \dots, c}(x_4) + \dots + f_{c, c, \dots, c}(x_m) + k$$

for some constant k . This is done by implicit simple induction as indicated below.

(a) The symmetry of $f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_2) + \gamma(x_3, \dots, x_m)$ gives in particular

$$\begin{aligned} f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_2) + \gamma(x_3, x_4, \dots, x_m) = \\ = f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_3) + \gamma(x_2, x_4, \dots, x_m). \end{aligned}$$

Simplifying and rearranging terms we get

$$\gamma(x_2, x_4, \dots, x_m) - f_{c,c,\dots,c}(x_2) = \gamma(x_3, x_4, \dots, x_m) - f_{c,c,\dots,c}(x_3).$$

Because the left side does not depend on x_3 and the right side does not depend on x_2 we get

$$\gamma(x_3, x_4, \dots, x_m) - f_{c,c,\dots,c}(x_3) =: \gamma_4(x_4, \dots, x_m).$$

Hence

$$(2.11) \quad \gamma(x_3, x_4, \dots, x_m) = f_{c,c,\dots,c}(x_3) + \gamma_4(x_4, \dots, x_m)$$

where, as mentioned earlier, γ is symmetric.

(b) Repeating the scheme of step (a) we get from the symmetry of γ and (2.11) the symmetry of γ_4 and that

$$\gamma_4(x_4, \dots, x_m) = f_{c,c,\dots,c}(x_4) + \gamma_5(x_5, \dots, x_m)$$

for some function γ_5 .

(c) Repeating the above inductively till we reach

$$\gamma_m(x_m) = f_{c,c,\dots,c}(x_m) + k.$$

This proves (2.10) following a sequence of substitutions.

Step 9. Putting (2.10) into (2.9) we obtain

$$\begin{aligned} f_{c,c,\dots,c}(F(x_1, x_2, \dots, x_m)) \\ = k + \sum_{i=1}^m f_{c,c,\dots,c}(x_i). \end{aligned} \quad (2.12)$$

The constant k can be absorbed into the function $f_{c,c,\dots,c}$. In fact, letting

$$f(x) := f_{c,c,\dots,c}(x) + \frac{k}{m-1}$$

(2.12) becomes

$$f(F(x_1, x_2, \dots, x_m)) = \sum_{i=1}^m f(x_i).$$

This proves the representation (2.2) in the case $m > 2$. ■

References

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