Annales Univ. Sci. Budapest., Sect. Comp. 41 (2013) 125–136

ON ADDITIVE FUNCTIONS WHICH DIFFERENTIATE ELEMENTARY FUNCTIONS IN SOME SENSE

Gyula Maksa (Debrecen, Hungary)

Dedicated to the 75th birthday of Professors Zoltán Daróczy and Imre Kátai

Communicated by Antal Járai

(Received March 16, 2013; accepted May 03, 2013)

Abstract. In this note, we prove that an additive function which differentiates any of a function from a rather rich list of so-called elementary functions, in some sense, is real derivation. Furthermore, we discuss the partially exceptional case of power functions and open problems will also be formulated.

1. Introduction

Derivations play an important role in several areas of mathematics. In our setting, we deal only with real derivations (in the following, we call these functions, simply, derivations) and we define them as the solutions of the following

Key words and phrases: Additive functions, real derivations, algebraic numbers, transcendental numbers, Gelfond-Schneider theorem.

2010 Mathematics Subject Classification: Primary: 39B22, Secondary: 39B72.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 81402 and by the TÁMOP 4.2.2.C-11/1/KONV-2012-0010 project implemented through the New Hungary Development Plan co-financed by the European Social Fund and the European Regional Development Fund.

https://doi.org/10.71352/ac.41.125

system of functional equations, that is, a derivation is a function $d : \mathbb{R} \to \mathbb{R}$ (the set of all real numbers) for which the functional equations

$$d(x+y) = d(x) + d(y) \qquad (x, y \in \mathbb{R}) \tag{1.1}$$

and

$$d(xy) = xd(y) + yd(x) \qquad (x, y \in \mathbb{R}) \tag{1.2}$$

hold simultaneously for all $x, y \in \mathbb{R}$. The solutions of (1.1) are called additive functions and, somewhat surprisingly, it is true that there is non-zero additive function d that fulfils (1.2), too. (See e.g. Kuczma [8].) We mention here just one application of this fact. (For the details see Aczél–Daróczy [1].) In 1970, Daróczy and Kátai [4] proved that every nonnegative information function f coincides with the Shannon information function S defined by

$$S(x) = -x \log_2 x - (1-x) \log_2 (1-x) \quad (x \in [0,1],\ 0 \log_2 0 = 0)$$

at the rational points of the closed interval $[0,1]$, and if, in addition, f is bounded above then $f = S$ on $[0,1]$. This result, which is very important from the point of view of the characterizations of the Shannon entropy, supported the conjecture that every nonnegative information function coincides with S on the entire interval $[0,1]$. However, seven years later, it was proved in Daróczy–Maksa [5], that $f(x) \geq S(x)$ for all nonnegative information functions f and for all $x \in [0,1]$, but there exists nonnegative information function f_0 which is different from S, namely, if d is a non-zero derivation then f_0 defined by

$$f_0(x) = \begin{cases} S(x) + \frac{d(x)^2}{x(1-x)}, & \text{if} \quad 0 < x < 1, \\ 0, & \text{if} \quad x \in \{0,1\} \end{cases}$$

is such a function.

In this paper, we deal with the following problem. Suppose that the function $d : \mathbb{R} \to \mathbb{R}$ satisfies equation (1.1). What has to be supposed additionally on d to obtain that it is a derivation. There are several known answers to this question in the literature (see among others Kurepa [9], [10], Nishiyama–Horinouchi [11], Boros–Erdei [2], Boros–Gselmann [3], Gselmann [7]). In this note, we intend to contribute to these results.

2. Preliminary results

In this section, firstly we collect some known results related to our problem.

Theorem 2.1. *Let $I \subset \mathbb{R}$ be an interval of positive length and $\varphi : I \to \mathbb{R}$ be a differentiable function. Suppose that $d : \mathbb{R} \to \mathbb{R}$ satisfies* (1.1) *and*

$$d(\varphi(x)) = \varphi'(x)d(x) \qquad (x \in I). \tag{2.1}$$

Then d is a derivation in any of the following cases.

(i) *Kurepa* [9]*: $\varphi(x) = x^2$, $x \in \mathbb{R}$,*

(ii) *Kurepa* [10]*: $\varphi(x) = x^{-1}$, $0 \neq x \in \mathbb{R}$,*

(iii) *Nishiyama–Horinouchi* [11] *and Boros–Gselmann* [3]*: $\varphi(x) = x^a$, $0 < x \in \mathbb{R}$, where $a \in \mathbb{Q} \setminus \{0, 1\}$ is fixed ($\mathbb{Q}$ denotes the set of all rational numbers),*

(iv) *Boros–Erdei* [2]*: $\varphi(x) = \sqrt{1 - x^2}$, $-1 < x < 1$.*

Furthermore, any derivation d satisfies (2.1) *in all of the above cases.*

In what follows, by algebraic and transcendental number we always mean algebraic and transcendental real number over $\mathbb{Q}$, respectively. Furthermore we say that *the additive function d differentiates the differentiable function* $\varphi : I \to \mathbb{R}$, if (2.1) holds for all $x \in I$. Our aim is to extend the above list of differentiable functions φ so that equations (1.1) and (2.1) should imply that d is a derivation. To do this we use some basic properties of derivations summarizing in the following lemma (see e.g. [8], pp. 346–352).

Lemma 2.1. *The following statemants are true.*

(i) *If d is a derivation then $d(\alpha) = 0$ for all algebraic $\alpha \in \mathbb{R}$.*

(ii) *If $t \in \mathbb{R}$ is transcendental and $s \in \mathbb{R}$ then there exists a derivation d such that $d(t) = s$.*

(iii) *If d is a derivation and it is bounded above or below on a set of positive Lebesgue measure then d is identically zero.*

An other tool of our investigations will be the following lemma which is an immediate consequence of Lemma 7. in [7] with $n = 2, m = 1$, and $\kappa = 2$.

Lemma 2.2. *If $d : \mathbb{R} \to \mathbb{R}$ satisfies* (1.1) *and there exist a measurable set $A \subset \mathbb{R}$ of positive Lebesgue measure and $K \in \mathbb{R}$ such that*

$$|d(x^2) - 2xd(x)| \leq K \qquad (x \in A)$$

then the function $x \mapsto d(x) - d(1)x$, $x \in \mathbb{R}$ is a derivation. If, in addition, $d(1) = 0$ or $K = 0$ then d is a derivation itself.

3. Main results

A part of our results are contained in the following

Theorem 3.1. *Suppose that the additive function $d : \mathbb{R} \to \mathbb{R}$ differentiates any of the following functions (i.e. (1.1) and (2.1) hold with φ)*

(i) $\varphi(x) = a^x,\ x \in \mathbb{R},\ (0 < a \neq 1)$,

(ii) $\varphi(x) = \cos x,\ 0 < x < \pi$,

(iii) $\varphi(x) = \sin x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$,

(iv) $\varphi(x) = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$,

(v) $\varphi(x) = \cot x,\ 0 < x < \pi$,

(vi) $\varphi(x) = \cosh x,\ 0 < x \in \mathbb{R}$,

(vii) $\varphi(x) = \sinh x,\ x \in \mathbb{R}$,

(viii) $\varphi(x) = \tanh x,\ x \in \mathbb{R}$,

(ix) $\varphi(x) = \coth x,\ 0 < x \in \mathbb{R}$.

Then d is a derivation.

Proof. *Case* (i). In this case, $d(a^x) = a^x \ln a\, d(x)$ for all $x \in \mathbb{R}$. Therefore

$$d(a^{2x}) = a^{2x} \ln a\, d(2x) = 2a^x a^x \ln a\, d(x) = 2a^x d(a^x) \qquad (x \in \mathbb{R}).$$

Thus $d(x^2) = 2xd(x)$ holds for all $0 < x \in \mathbb{R}$, and so Lemma 2.2. implies that d is a derivation.

Case (ii). In this case, $d(\cos x) = -\sin x\, d(x)$ for all $0 < x < \pi$. Therefore $d(\cos 2x) = -\sin 2x\, d(2x)$ for all $0 < x < \frac{\pi}{2}$. Thus, by well-known trigonometric identities, we have that

$$d(2\cos^2 x - 1) = -4 \sin x \cos x\, d(x) = 4 \cos x d(\cos x) \qquad \left(0 < x < \frac{\pi}{2}\right)$$

whence

$$d(2x^2 - 1) = 4xd(x) \qquad (0 < x < 1) \tag{3.1}$$

follows. With the substitution $x = \frac{1}{2}$, (3.1) implies that $d(1) = 0$ (since d is additive). Thus, again by (3.1), we get that $d(x^2) = 2xd(x)$ holds for all $0 < x < 1$, and so Lemma 2.2. implies again that d is a derivation.

Case (iii). In this case, $d(\sin x) = \cos x\, d(x)$ for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Therefore $d(\sin(x - \frac{\pi}{2})) = \cos(x - \frac{\pi}{2})\, d(x - \frac{\pi}{2})$ for all $0 < x < \pi$, that is,

$$d(\cos x) = -\sin x\, d(x) + d\left(\frac{\pi}{2}\right) \sin x \qquad (0 < x < \pi)$$

and so, as in the proof of (ii), we obtain that

$$d(2\cos^2 x - 1) = 4\cos x \; d(\cos x) - d(\pi)\sin x \cos x \qquad \left(0 < x < \frac{\pi}{2}\right).$$

Hence

$$d(2x^2 - 1) = 4xd(x) - d(\pi)x\sqrt{1-x^2} \qquad (0 < x < 1) \tag{3.2}$$

follows. The substitutions $x = \frac{3}{5}$ and $x = \frac{4}{5}$ show that $43d(1) - 12d(\pi) = 0$ and $57d(1) - 12d(\pi) = 0$, respectively. Therefore $d(1) = d(\pi) = 0$ and, by (3.2), Lemma 2.2. implies that d is a derivation.

Case (iv). In this case,

$$d(\tan x) = \frac{1}{\cos^2 x} d(x) \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right). \tag{3.3}$$

Therefore

$$d\left(\tan\left(x + \frac{\pi}{2}\right)\right) = \frac{1}{\cos^2(x + \frac{\pi}{2})} d\left(x + \frac{\pi}{2}\right) \qquad (-\pi < x < 0),$$

that is,

$$d(\cot x) = -\frac{1}{\sin^2 x} d(x) - d\left(\frac{\pi}{2}\right)\frac{1}{\sin^2 x} \qquad (-\pi < x < 0).$$

This and (3.3) imply that

$$\sin^2 x \; d(\cot x) = -d(x) - d\left(\frac{\pi}{2}\right) = -\cos^2 x \; d(\tan x) - d\left(\frac{\pi}{2}\right) \quad \left(-\frac{\pi}{2} < x < 0\right),$$

that is,

$$-d\left(\frac{\pi}{2}\right) = d(\tan x)\cos^2 x + d(\cot x)\sin^2 x \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{3.4}$$

The substitution $x = \frac{\pi}{4}$ shows that $-d(\frac{\pi}{2}) = d(1)$. Applying trigonometric identities and taking into consideration that the image of the interval $]0, \frac{\pi}{2}[$ under the tangent function is the set of all positive real numbers, (3.4) implies that

$$d(1) = d(x)\frac{1}{1+x^2} + d\left(\frac{1}{x}\right)\frac{x^2}{1+x^2} \qquad (x > 0),$$

whence

$$d\left(\frac{1}{x}\right) = d(1)\frac{1+x^2}{x^2} - \frac{1}{x^2}d(x) \qquad (x > 0) \tag{3.5}$$

follows. By using the additivity of d, an idea from Kurepa [10], and (3.5), for all $x > 0$, we get that

$$d(1)\frac{1+x^2(x+1)^2}{x^2(x+1)^2} - \frac{1}{x^2(x+1)^2}d(x(x+1)) = d\left(\frac{1}{x(x+1)}\right) =$$

$$= d\left(\frac{1}{x}\right) - d\left(\frac{1}{x+1}\right) =$$

$$= d(1)\frac{1+x^2}{x^2} - \frac{1}{x^2}d(x) - d(1)\frac{x^2+2x+2}{(x+1)^2} + \frac{1}{(x+1)^2}(d(x)+d(1)).$$

After an easy calculation and some rearrangements, we obtain that

$$d(x^2) - 2xd(x) = d(1)(x^4 + 2x^3 - 2x) \qquad (x > 0).$$

With $x = 1$, we have that $d(1) = 0$ and so Lemma 2.2. can be applied again to get that d is a derivation.

Case (v). In this case,

$$d(\cot x) = -\frac{1}{\sin^2 x}d(x) \qquad (0 < x < \pi). \tag{3.6}$$

Thus

$$d\left(\cot\left(x + \frac{\pi}{2}\right)\right) = -\frac{1}{\sin^2(x+\frac{\pi}{2})}d\left(x + \frac{\pi}{2}\right) \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right),$$

therefore

$$d(\tan x) = \frac{1}{\cos^2 x}d(x) + d\left(\frac{\pi}{2}\right)\frac{1}{\cos^2 x} \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right).$$

On the other hand, with the substitution $x = \frac{\pi}{2}$, (3.6) implies that $d(\frac{\pi}{2}) = 0$, hence the equation above implies (3.3), and our statement follows from the *Case* (iv).

Case (vi). In this case, $d(\cosh x) = \sinh x\ d(x)$ for all $0 < x \in \mathbb{R}$. Therefore $d(\cosh 2x) = \sinh 2x\ d(2x)$ for all $0 < x \in \mathbb{R}$. Thus, by well-known identities, we have that

$$d(2\cosh^2 x - 1) = 4\sinh x \cosh x\, d(x) = 4\cosh x\ d(\cosh x) \qquad (0 < x \in \mathbb{R})$$

whence

$$2d(x^2) - d(1) = 4xd(x) \qquad (1 < x) \tag{3.7}$$

follows. With the substitution $x = 2$, (3.7) implies that $d(1) = 0$. Thus, again by (3.7), we get that $d(x^2) = 2xd(x)$ holds for all $1 < x$, and so Lemma 2.2. yields that d is a derivation.

Case (vii). In this case,

$$d(\sinh x) = \cosh x \; d(x) \qquad (x \in \mathbb{R}). \tag{3.8}$$

Let $x, y \in \mathbb{R}$ and write $x + y$ in (3.8) instead of x. Using the identities of the hyperbolic sine and the hyperbolic cosine functions, the additivity of d, and (3.8) again, we get that

$$d(\sinh x \cosh y) + d(\sinh y \cosh x) =$$

$$= \cosh x \; d(x) \cosh y + \cosh x \;\; \cosh y \; d(y) + d(x+y) \sinh x \;\; \sinh y =$$

$$= d(\sinh x) \cosh y + d(\sinh y) \cosh x + d(x+y) \sinh x \;\; \sinh y.$$

Repeating this computation with $-y$ instead of y and adding the equation so obtained to the above one, side by side, we have that

$$d(\sinh x \cosh y) = d(\sinh x) \cosh y + d(y) \sinh x \;\; \sinh y \qquad (x, y \in \mathbb{R}). \tag{3.9}$$

With the substitution $x = arsinh \; 1 = \ln(1 + \sqrt{2})$, this implies that

$$d(\cosh y) = d(1) \cosh y + d(y) \sinh y \qquad (y \in \mathbb{R}).$$

Thus, it follows from (3.9) that

$$d(\sinh x \cosh y) = d(\sinh x) \cosh y + d(\cosh y) \sinh x - d(1) \sinh x \cosh y$$

holds for all $x, y \in \mathbb{R}$. Taking into consideration the ranges of the functions involved in the equation above, we obtain that

$$d(xy) = d(x)y + d(y)x - d(1)xy \qquad (x \in \mathbb{R}, 1 \leq y \in \mathbb{R}). \tag{3.10}$$

Now we show that $d(1) = 0$. Indeed, (3.8) implies that, for all $x \in \mathbb{R}$, $d(\sinh 2x) = \cosh 2x \; d(2x)$, hence

$$d(2 \sinh x \;\cosh x) = 2(2\cosh^2 x - 1)\, d(x) = 2\frac{2\cosh^2 x - 1}{\cosh x} d(\sinh x)$$

which implies that

$$d(x\sqrt{1+x^2}) = \frac{2x^2+1}{\sqrt{1+x^2}} d(x) \qquad (x \in \mathbb{R}).$$

Finally, with the substitution $x = \frac{3}{4}$, we obtain that $d(1) = 0$. Thus, it follows from (3.10) that $d(x^2) = 2xd(x)$ holds for all $1 < x$, and so Lemma 2.2. yields again that d is a derivation.

Case (viii). In this case,

$$d(\tanh x) = \frac{1}{\cosh^2 x}\, d(x) \qquad (x \in \mathbb{R}),$$

that is,

$$d\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{4}{(e^x + e^{-x})^2}\, d(x) \qquad (x \in \mathbb{R}).$$

Substitute here x by $\ln\sqrt{x}$, $(x > 0)$ to get that

$$d\left(\frac{x-1}{x+1}\right) = \frac{2x}{(x+1)^2}\, d(\ln x) \qquad (0 < x \in \mathbb{R}. \tag{3.11}$$

Define the function p by

$$p(x) = xd(\ln x) \qquad (0 < x \in \mathbb{R}. \tag{3.12}$$

Then, because of the additivity of d, we have that

$$p(xy) = xp(y) + yp(x) \qquad (0 < x,\ 0 < y). \tag{3.13}$$

Particularly,

$$p(x^2) = 2xp(x) \qquad \text{and} \qquad p\left(\frac{1}{x}\right) = -\frac{1}{x^2}p(x) \qquad (0 < x,\ 0 < y). \tag{3.14}$$

On the other hand, for all $x > 0$, we obtain that

$$d(1) = d\left(\frac{x+1}{x+1}\right) = d\left(\frac{x}{x+1}\right) + d\left(\frac{1}{x+1}\right).$$

Therefore it follows from (3.11) and (3.12) that

$$p(x) = \frac{(x+1)^2}{2}\left(d(1) - 2d\left(\frac{1}{x+1}\right)\right) \qquad (0 < x \in \mathbb{R}). \tag{3.15}$$

This and (3.14) imply that

$$\frac{(x^2+1)^2}{2}\left(d(1) - 2d\left(\frac{1}{x^2+1}\right)\right) = x(x+1)^2\left(d(1) - 2d\left(\frac{1}{x+1}\right)\right)$$

holds for all $x > 0$. Let here $x = 2$. Then an easy calculation shows that $d(1) = 0$. Hence, it follows from (3.15) that

$$p(x) = -(x+1)^2 d\left(\frac{1}{x+1}\right) \qquad (0 < x \in \mathbb{R}. \tag{3.16}$$

In what follows, we show that

$$p(x) = p(x+1) \qquad (0 < x \in \mathbb{R}). \tag{3.17}$$

Indeed, by (3.16) and the additivity of d, for all $x > 0$, we have that

$$\begin{aligned} p(x^2+3x+1) &= -(x+1)^2(x+2)^2 d\left(\tfrac{1}{x+1} - \tfrac{1}{x+2}\right) \\ &= (x+2)^2 p(x) - (x+1)^2 p(x+1). \end{aligned}$$

Replacing here x by $\frac{1}{x}$, we get that

$$p\left(\tfrac{x^2+3x+1}{x^2}\right) = \tfrac{(2x+1)^2}{x^2} p\left(\tfrac{1}{x}\right) - \tfrac{(x+1)^2}{x^2} p\left(\tfrac{x+1}{x}\right).$$

holds for all $x > 0$. Taking into consideration (3.13) and (3.14), after an elementary calculation, it turns out from the above equation that

$$p(x^2+3x+1) = \frac{3x^2+5x+1}{x} p(x) - \frac{(x+1)^2}{x^2} p(x+1) \qquad (x > 0).$$

Comparison of the values of $p(x^2+3x+1)$ finally leads to the equality $p(x) = p(x+1)$, $(0 < x \neq 1.)$ Since $d(1) = 0$, (3.16) implies that $p(x) = p(x+1)$, $(0 < x)$. To finish the proof of this case, let $1 < x \in \mathbb{R}$. Then (3.17) and (3.16) imply that

$$p(x) = p(x-1) = -x^2 d\left(\frac{1}{x}\right). \qquad (x > 1)$$

Hence, by (3.14), we obtain that $-\frac{1}{x^2}p(x) = p\left(\frac{1}{x}\right) = -\frac{1}{x^2}d(x)$, that is, $p(x) = d(x)$ for all $0 < x < 1$. Therefore, again by (3.14), $d(x^2) = p(x^2) = 2xp(x) = 2xd(x)$ for all $0 < x < 1$, and it follows from Lemma 2.2. that d is a derivation.

Case (ix). In this case,

$$d(\coth x) = -\frac{1}{\sinh^2 x}\, d(x) \qquad (0 < x \in \mathbb{R}).$$

Well-known identities imply that

$$d\left(\frac{1}{\tanh 2x}\right) = -\frac{1}{2\sinh^2 x \cosh^2 x} d(x) = \frac{1}{2\cosh^2 x}\, d\left(\frac{1}{\tanh x}\right),$$

that is,

$$d\left(\frac{1+\tanh^2 x}{2\tanh x}\right) = \frac{1-\tanh^2 x}{2} d\left(\frac{1}{\tanh x}\right)$$

holds for all $0 < x$. Therefore $d\left(\frac{1+x^2}{x}\right) = (1-x^2)d\left(\frac{1}{x}\right)$, if $0 < x < 1$, i.e.,

$$d(x) = -x^2 d\left(\frac{1}{x}\right) \qquad (0 < x < 1). \tag{3.18}$$

The substitution $x = \frac{1}{2}$ and the additivity of d show that $d(1) = 0$ and if we replace x in (3.18) by $\frac{1}{x}$, $(x > 1)$ we obtain that

$$d(x) = -x^2 d\left(\frac{1}{x}\right) \qquad (x > 0).$$

In what follows, we use this equality, the additivity of d, and an idea from Kurepa [10] to get, for all $x > 0$, that

$$-\frac{1}{x^2(x+1)^2} d(x(x+1)) = d\left(\frac{1}{x(x+1)}\right) =$$

$$= d\left(\frac{1}{x}\right) - d\left(\frac{1}{x+1}\right) = -\frac{1}{x^2}d(x) + \frac{1}{(x+1)^2}d(x+1) = -\frac{2x+1}{x^2(x+1)^2}d(x),$$

whence $d(x^2) = 2xd(x)$, $(x > 0)$ follows, and Lemma 2.2. completes the proof also in this case. ∎

With the help of the following simple observation, we can extend the list involved in the theorem we have just proved.

Lemma 3.1. *Let $d : \mathbb{R} \to \mathbb{R}$ be a function, $\varphi : I \to \mathbb{R}$ be a differentiable function with nowhere zero derivative and $\psi = \varphi^{-1}$. Suppose that* (1.1) *and* (2.1)*, that is,*

$$\begin{aligned} d(x+y) &= d(x) + d(y) \qquad (x, y \in \mathbb{R}), \text{ and} \\ d(\varphi(x)) &= \varphi'(x)d(x) \qquad (x \in I) \end{aligned}$$

imply that d is a derivation. Then (1.1) *and*

$$d(\psi(x)) = \psi'(x)d(x) \qquad (x \in \varphi(I)). \tag{3.19}$$

imply the same.

Proof. Let $t \in I$ and $x = \varphi(t)$ in (3.19). Then we get that

$$d(t) = \psi'(\varphi(t))d(\varphi(t)) = \frac{1}{\varphi'(t)}d(\varphi(t))$$

whence (2.1) follows with t instead of x. ∎

As a consequence of the previous lemma and Theorem 3.1., we also can conclude the following

Corollary 3.1. *Let $0 < a \neq 1$ be a fixed real number and suppose that the additive function $d : \mathbb{R} \to \mathbb{R}$ differentiates any of the following functions (i.e.* (1.1) *and* (2.1) *hold with φ)*

(i) $\varphi(x) = \log_a x,\ 0 < x \in \mathbb{R}$,

(ii) $\varphi(x) = \arccos x,\ -1 < x < 1$,

(iii) $\varphi(x) = \arcsin x,\ -1 < x < 1$,

(iv) $\varphi(x) = \arctan x,\ x \in \mathbb{R}$,

(v) $\varphi(x) = \operatorname{arccot} x,\ x \in \mathbb{R}$,

(vi) $\varphi(x) = \operatorname{arcosh} x,\ 1 < x \in \mathbb{R}$,

(vii) $\varphi(x) = \operatorname{arsinh} x,\ x \in \mathbb{R}$,

(viii) $\varphi(x) = \operatorname{artanh} x,\ -1 < x < 1$,

(ix) $\varphi(x) = \operatorname{arcoth} x,\ 1 < x \in \mathbb{R}$.

Then d is a derivation.

4. Remarks and problems

Remark 4.1. The power functions $x \mapsto x^a$ are missing from the list of Theorem 3.1. and also from the list of Corollary 3.1. On the other hand, by the results of Nishiyama-Horinouchi[11] and Boros-Gselmann[3], we know that, if $a \in \mathbb{Q} \setminus \{0, 1\}$, then an additive function $d : \mathbb{R} \to \mathbb{R}$ is a derivation if, and only if,

$$d(x^a) = a x^{a-1} d(x) \qquad (0 < x), \tag{4.1}$$

that is, d differentiates the power function $x \mapsto x^a$, $(x > 0)$. We are not able to prove similar statement for irrational exponent a. It may have some interest however that for every irrational algebraic exponent a there is a derivation d which does not differentiate the power function $x \mapsto x^a$, $(x > 0)$. Indeed, if a is an irrational algebraic number then, by the Gelfond-Schneider theorem (Gelfond[6], Schneider[12]) 2^a is transcendental. Taking into consideration Lemma 2.1. (ii) and (i), there exists a derivation d such that $d(2^a) = 1$. Since $d(2) = 0$, equation (4.1) is not satisfied by d at the point $x = 2$.

Remark 4.2. We are not able to prove the reversal of our statements in any case listed in Theorem 3.1. or Corollary 3.1. Therefore we formulate the following open problems.

Problem 4.1. Prove or disprove that the derivations differentiate the function φ listed in Theorem 3.1 or Corollary 3.1., or at least

Problem 4.2. Prove or disprove that there exists non-zero derivation d such that $d(e^x) = e^x d(x)$ holds for all $x \in \mathbb{R}$!

References

[1] **Aczél, J. and Z. Daróczy,** *On Measures of Information and their Characterizations*, Academic Press, New York, San Francisco, and London, 1975.

[2] **Boros, Z. and P. Erdei,** A conditional equation for additive functions, *Aequationes Math.*, **70** (2005), 309–313.

[3] **Boros, Z. and E. Gselmann,** Hyers–Ulam stability of derivations and linear functions, *Aequationes Math.*, **80** (2010), 13–25.

[4] **Daróczy, Z. and I. Kátai,** Additive zahlentheoretische Funktionen und das Mass der Information, *Ann. Univ. Sci. Budapest., Eötvös Sect. Math.*, **13** (1970), 83–88.

[5] **Daróczy, Z. and Gy. Maksa,** Nonnegative information functions, in: Analytic function methods in probability theory (Proc. Colloq. Methods of Complex Anal. in the Theory of Probab. and Statist., Kossuth L. Univ. Debrecen, 1977, **21**, Colloq. Math. Soc. János Bolyai, 21, North-Holland, Amsterdam-New York, 1979, 67–78.

[6] **Gelfond, A.O.,** Sur le septièm problème de Hilbert, *Doklady Akad. Nauk SSSR*, **2** (1934), 1–6.

[7] **Gselmann, E.,** Derivations and linear functions along rational functions, *Monatsh. Math.*, **169(3-4)** (2013), 355–370.

[8] **Kuczma, M.,** *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985.

[9] **Kurepa, S.,** The Cauchy functional equation and scalar product in vector spaces, *Glasnik Mat.-Fiz. Astronom.*, (2) **19** (1964), 23–26.

[10] **Kurepa, S.,** Remarks on the Cauchy functional equation, *Publ. Inst. Math.* (*Beograd*), **5(19)** (1965), 85–88.

[11] **Nishiyama, A. and S. Horinouchi,** On a system of functional equations, *Aequationes Math.*, **1** (1968), 1–5.

[12] **Schneider, Th.,** Transzenderzuntersuchungen periodischer Funktionen: I Transzendenz von Potenzen; II Transzendenzeigenschaften ellipticher Funktionen, *Journal reine Angew. Math.*, **172** (1934), 65–74.

Gy. Maksa
Institute of Mathematics
University of Debrecen
Egyetem tér 1.
Debrecen
Hungary
`maksa@science.unideb.hu`