THE EULER FUNCTION OF FIBONACCI AND LUCAS NUMBERS AND FACTORIALS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their $L_9 - 1$'st birthday

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Abstract. Here, we look at the Fibonacci and Lucas numbers whose Euler function is a factorial, as well as Lucas numbers whose Euler function is a product of power of two and power of three.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let $(L_n)_{n\geq 0}$ be the companion Lucas sequence satisfying the same recurrence with initial conditions, $L_0 = 2$, $L_1 = 1$. In our previous paper [2], we noticed the relation

$$F_1F_2F_3F_4F_5F_6F_7F_8F_{10}F_{12} = 11!$$

and proved that it is the largest solution of the Diophantine equation

$$F_{n_1}F_{n_2}\cdots F_{n_k} = m_1!m_2!\cdots m_\ell!$$

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in positive integers $n_1 < n_2 < \cdots < n_k$ and $m_1 \le m_2 \le \cdots \le m_\ell$ where by "largest" we mean that the number appearing in the left (or right) hand side of the above equation is largest among all solutions. Here, we note that

$$\phi(F_{21}) = 7! \text{ and } \phi(L_6) = 3!$$

and conjecture that the above solutions are the largest solutions of the equation

$$\phi(F_n) = m!, \text{ respectively, } \phi(L_n) = m!$$

but have no idea how to attack this problem. Instead, we put

$$\mathcal{N} = \{n : \phi(F_n) = m! \text{ for some positive integer } m\}$$

and prove the following properties of the set $\mathcal{N}$. Put $\mathcal{N}(x) = \mathcal{N} \cap [1, x]$. For a positive real number x we write $\log x$ for the natural logarithm of x.

Theorem 1.1. *The following hold:*

(i) $\#\mathcal{N}(x) \ll \dfrac{x \log\log x}{\log x}$, *and so $\mathcal{N}$ is of asymptotic density zero.*

(ii) *The only primes in $\mathcal{N}$ are 2 and 3.*

In [1] it was shown that $F_9 = 34$ and $L_3 = 4$ are the largest Fibonacci and Lucas numbers, respectively, whose Euler function is a power of 2. Here, we show the following result.

Theorem 1.2. *The only solutions in nonnegative integers of the equation $\phi(L_n) = 2^a 3^b$ are*

$$(n, a, b) = (0, 0, 0),\ (1, 0, 0),\ (2, 1, 0),\ (3, 1, 0),\ (4, 1, 1),\ (6, 1, 1),\ (9, 2, 2).$$

We do not know how to find all the nonnegative solutions (n, a, b) of the Diophantine equation

$$\phi(F_n) = 2^a 3^b.$$

Also, we noted that $\phi(L_{30}) = 5!7!$, but we do not even know how to prove that the set of positive integers n such that

$$\phi(F_n) = m_1! \cdots m_\ell! \qquad \text{or} \qquad \phi(L_n) = m_1! \cdots m_\ell!$$

for some integers $1 \le m_1 \le \cdots \le m_\ell$ is of asymptotic density zero. We leave such questions for the reader.

2. The proofs

2.1. The proof of Theorem 1.1

(i) Let x be a large real number and $\gamma = (1+\sqrt{5})/2$ be the golden section. Let $n \in \mathcal{N}(x)$. Since

$$\left(\frac{m}{e}\right)^m < m! = \phi(F_n) < F_n < \gamma^n \le \gamma^x,$$

it follows that for large x we have $m \le x/\log x$. Let us denote $K = \lfloor x/\log x \rfloor$. For $k = 1, \ldots, K$, put

$$\mathcal{N}_k(x) = \{n \le x : \phi(F_n) = k!\}\}.$$

Fix k and let $n_1 < n_2 < \ldots < n_t$ be all elements in $\mathcal{N}_k(x)$. Since

$$1 \le \frac{F_n}{\phi(F_n)} \ll \log\log F_n \ll \log x,$$

we get that

$$\frac{F_{n_t}}{F_{n_1}} = \left(\frac{F_{n_t}}{k!}\right)\left(\frac{k!}{F_{n_1}}\right) = \left(\frac{F_{n_t}}{\phi(F_{n_t})}\right)\left(\frac{\phi(F_{n_1})}{F_{n_1}}\right) \ll \log x.$$

Since $\gamma^{n-2} \le F_n \le \gamma^{n-1}$ holds for all n, we get that $F_{n_t}/F_{n_1} \ge \gamma^{n_t-n_1-1}$. Hence,

$$\gamma^{n_t-n_1-1} \ll \log x \qquad \text{yielding} \qquad \#\mathcal{N}_k(x) \le n_t - n_1 \ll \log\log x.$$

Since certainly

$$\mathcal{N}(x) = \bigcup_{1\le k\le K} \mathcal{N}_k(x),$$

it follows that

$$\#\mathcal{N}(x) \le \sum_{k=1}^{K} \#\mathcal{N}_k(x) \ll K\log\log x \ll \frac{x\log\log x}{\log x},$$

which completes the proof of (i).

(ii) Assume that $p > 12$ is in $\mathcal{N}$. Then all prime factors q of F_p satisfy the relation $q \equiv (5|q) \pmod{p}$, where $(a|q)$ is the Legendre symbol of a with respect to q. If $q \equiv 1, 4 \pmod p$, then $p \mid (q-1) \mid \phi(F_p)$. Since $\phi(F_p) = m!$ for some integer m, we get that $m \geq p$. Thus,

$$\gamma^p > F_p \geq \phi(F_p) \geq p! \geq (p/e)^p,$$

an inequality which is false for any $p > 12$. A similar argument proves that F_p is square free. Indeed, if $q^2 \mid F_p$, then $q \mid \phi(F_p)$, therefore $m \geq q$. Since $q \equiv \pm 1 \pmod p$, we get that $q \geq 2p - 1 > p$, and we get again that $\phi(F_p) \geq q! > p!$, a contradiction. Thus, F_p is square free and $q \equiv 2, 3 \pmod 5$ for all prime factors q of F_p. Since the above congruence is true for all prime factors q of F_p, we get that $5 \nmid \phi(F_p)$, so that $m \leq 4$. Hence, $\phi(F_p) \leq 4! = 24$. This is false if F_p is a prime, or if F_p has at least one prime factor > 23, or if F_p has at least four distinct prime factors because $(2-1)(3-1)(5-1)(7-1) > 24$. Hence, $F_p < 23^3$, leading to $p \leq 19$. A quick search now completes the proof of (ii). ■

Remark 1. The argument used to prove (ii) shows that for each fixed positive integer a, there are only finitely many primes p such that $ap \in \mathcal{N}$. To see why, assume that $p > 12$ and $ap \in \mathcal{N}$. Then every prime factor q of F_{ap} either is a prime factor of F_a, or is a primitive prime factor of F_{dp} for some divisor d of a. In the second case, either $q \equiv 1 \pmod p$, and we get

$$\gamma^{ap} > F_{ap} > \phi(F_{ap}) \geq p! \geq (p/e)^p \qquad \text{therefore} \qquad p < e^a \gamma,$$

or $q \equiv 2, 3 \pmod 5$. If this last scenario happens for all prime factors q of F_{ap} which are not prime factors of F_a, we then deduce that $\nu_5(m!) = \nu_5(\phi(F_a))$, where $\nu_5(m)$ is the exponent of 5 in the factorization of m. Since certainly $\nu_5(m!) \geq \lfloor m/5 \rfloor$, we get that $\lfloor m/5 \rfloor \leq \nu_5(\phi(F_a))$, so that $m \leq 5\nu_5(\phi(F_a)) + 4$. This in turn puts an upper bound on ap. For example, for $a \in \{2, 3, 4\}$, we get that either $p < e^4 \gamma$, therefore $p \leq 19$, or $m \leq 4\nu_5(\phi(F_a)) + 4 = 4$, so $\phi(F_{ap}) \leq 4!$, which again gives that $p \leq 19$, and a quick search reveals that the only such values of ap in $\mathcal{N}$ are 4 and 21.

Remark 2. The conclusions of the above theorem (with the same bounds and primes membership in $\mathcal{N}$) as well as the above Remark 1 still hold if we replace the Fibonacci numbers by Lucas numbers. One just uses the inequalities $\gamma^{n-1} \leq L_n \leq \gamma^{n+1}$ valid for all $n \geq 1$.

2.2. The proof of Theorem 1.2

Assume that $n = 2^\alpha m$ for some odd positive integer m. We start by showing that $\alpha \leq 2$. Assume that $\alpha \geq 4$. Since

$$L_{2^\alpha} = L_{2^{\alpha-1}}^2 - 2,$$

it follows that $L_{2^\alpha} \equiv 3 \pmod 4$. In particular, there exists a prime factor q of L_{2^α} such that $q \equiv 3 \pmod 4$. Reducing the relation $L_{2^\alpha}^2 - 5F_{2^\alpha}^2 = 4$ modulo q, we get that $(-5|q) = 1$. Since $q \equiv 3 \pmod 4$, we deduce that $(-1|q) = -1$, therefore $(5|q) = -1$. It follows that $q \equiv -1 \pmod{2^\alpha}$. Write $q = 2^a 3^b + 1$. Then since $q \equiv 3 \pmod 4$, we get that $a = 1$. Thus, $2^\alpha \mid (q+1)$, or $2^{\alpha-1} \mid 3^b+1$, and this is impossible for $\alpha \geq 4$ because $\nu_2(3^b+1) = 1,\ 2$ according to whether b is even or odd. This shows that $\alpha \leq 3$. The case $\alpha = 3$ is not possible since it would lead to $L_8 \mid L_n$, hence $23 \mid \phi(L_8) \mid \phi(L_n)$, a contradiction. We now look at the prime factors of m. Since $107 \mid \phi(L_{27})$, $41 \mid \phi(L_{18})$ and $11 \mid \phi(L_{36})$, it follows that $3^3 \nmid m$. In fact, if $\alpha \in \{1, 2\}$, then $3^2 \nmid m$.

Now assume that $p > 3$ is a prime factor of m. Then $L_{2^\alpha p}$ has the same property that its Euler function is divisible only by primes which are at most 3. Let $q > 2$ be any prime factor of $L_{2^\alpha p}$ which is not a prime factor of L_{2^α}. If $\alpha = 0$, then reducing the formula $L_p^2 - 5F_p^2 = -4$ modulo q, we get that $(5|q) = 1$. This shows that $q \equiv 1 \pmod p$, therefore $p \mid \phi(L_p)$, which is a contradiction because $p > 3$. This shows that the only acceptable solutions when $\alpha = 0$ are $n = 3,\ 9$. Assume now that $\alpha \geq 1$. Reducing the formula $L_{2^\alpha p}^2 - 5F_{2^\alpha p}^2 = 4$ modulo q we get $(-5|q) = 1$. If $q \equiv 1 \pmod 4$, then we get $q \equiv 1 \pmod p$, leading to $p \mid \phi(L_{2^\alpha p})$, which is a contradiction for $p > 3$. So, we get that $n \in \{2, 4, 6, 12\}$ and the solution $n = 12$ is not convenient. So, we need to treat the case when $q \equiv -1 \pmod 4$ for all prime factors q of $L_{2^\alpha p}/L_{2^\alpha}$, which leads to the conclusion that $q = 2 \cdot 3^{b_q} + 1$. Moreover, $q \equiv -1 \pmod p$, therefore $2 \cdot 3^{b_q} + 1 = a_q p - 1$ for some even integer a_q. Further, it is clear that $L_{2^\alpha p}/L_{2^\alpha}$ is square free. Thus, we get that

$$L_{2^\alpha p} = L_{2^\alpha} q_1 q_2 \cdots q_\ell,$$

where $q_i = 2 \cdot 3^{b_{q_i}} + 1$ for $i = 1, \ldots, \ell$. We may assume that $1 \leq b_{q_1} < \cdots < b_{q_\ell}$. We thus get that

$$3^{b_1} \mid L_{2^\alpha p} - L_{2^\alpha} = 5F_{2^{\alpha-1}(p-1)} F_{2^{\alpha-1}(p+1)}.$$

Now F_m is a multiple of 3 if and only if $4 \mid m$. Moreover, in this case, $\nu_3(F_m) = \nu_3(m) + 1$. Since exactly one of $p-1$ and $p+1$ is a multiple of 3, and exactly one of these two numbers is a multiple of 4, it follows that

$$\begin{aligned} \min\{\nu_3(F_{2^{\alpha-1}(p-1)}, F_{2^{\alpha-1}(p+1)}\} &\leq 1, \\ \max\{\nu_3(F_{2^{\alpha-1}(p-1)}, F_{2^{\alpha-1}(p+1)}\} &\leq 1 + \max\{\nu_3(p-1), \nu_3(p+1)\}. \end{aligned}$$

In particular, we deduce that if $b_{q_1} \geq 2$, then $3^{b_{q_1}-2} \mid (p-1)/2$ or $3^{b_{q_1}-2} \mid (p+1)/2$. On the one hand, writing

$$p = \frac{2 \cdot 3^{b_{q_1}} + 2}{a_{q_1}}, \quad \text{we get that} \quad 3^{b_{q_1}-2} \mid a_{q_1} + 1, \quad \text{or} \quad 3^{b_{q_1}-2} \mid a_{q_1} - 1.$$

Since $(p+1)/2 \geq 3^{b_{q_1}-2}$, we get that

$$\frac{3^{b_{q_1}}+1}{a_1} = \frac{p}{2} > 3^{b_{q_1}-2} - 1.$$

On the one hand, if $a_{q_1} \geq 10$, then $3^{b_{q_1}} + 1 > 10(3^{b_{q_1}-2} - 1)$, or $11 \geq 3^{b_{q_1}-2}$, or $b_{q_1} \leq 4$. On the other hand, if $a_{q_1} \leq 8$, then $3^{b_{q_1}-2}$ divides one of $a_{q_1} - 1$ or $a_{q_1} + 1$, a number which is at most 9, so again $b_{q_1} \leq 4$. Thus, $b_{q_1} \in \{1, 2, 3, 4\}$, so the only possibilities are $q_1 \in \{7, 19, 163\}$. The case $q_1 = 7$ leads to $\alpha = 2$, then $p = 7$, which is false because 7^2 cannot divide $L_{2^\alpha p}$. The case $q_1 = 19$ leads to $p \mid q_1 - 1$, which is false because $p > 3$. The case $q_1 = 163$ leads to $p \mid 164$, so $p = 41$. However, in this case since $q = 163$ divides $L_{2^a p}$, we get that $\alpha = 1$. In this case, $31 \mid \phi(L_{82})$, and we get a contradiction. So, we indeed conclude that n cannot be divisible by any prime $p > 3$, which completes the proof of the theorem. ■

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