Annales Univ. Sci. Budapest., Sect. Comp. 41 (2013) 119–124

THE EULER FUNCTION OF FIBONACCI AND LUCAS NUMBERS AND FACTORIALS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their $L_9 - 1$'st birthday

Communicated by Bui Minh Phong

(Received December 22, 2012; accepted July 18, 2013)

Abstract. Here, we look at the Fibonacci and Lucas numbers whose Euler function is a factorial, as well as Lucas numbers whose Euler function is a product of power of two and power of three.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let $(L_n)_{n\geq 0}$ be the companion Lucas sequence satisfying the same recurrence with initial conditions, $L_0 = 2$, $L_1 = 1$. In our previous paper [2], we noticed the relation

$$F_1 F_2 F_3 F_4 F_5 F_6 F_7 F_8 F_{10} F_{12} = 11!$$

and proved that it is the largest solution of the Diophantine equation

$$F_{n_1} F_{n_2} \cdots F_{n_k} = m_1! m_2! \cdots m_\ell!$$

Key words and phrases: Fibonacci and Lucas numbers, factorials, diophantine equations.
2010 Mathematics Subject Classification: 11B39, 11D45.
This paper was written during a visit of F. L. to the Applied Mathematics Department in December, 2012. During the preparation of this paper, F. L. was supported in part by Project PAPIIT IN104512 (UNAM), VSP N62909-12-1-4046 (Department of the US Navy, ONR–Global) and a Marcos Moshinsky fellowship.
https://doi.org/10.71352/ac.41.119

in positive integers $n_1 < n_2 < \cdots < n_k$ and $m_1 \le m_2 \le \cdots \le m_\ell$ where by "largest" we mean that the number appearing in the left (or right) hand side of the above equation is largest among all solutions. Here, we note that

$$\phi(F_{21}) = 7! \text{ and } \phi(L_6) = 3!$$

and conjecture that the above solutions are the largest solutions of the equation

$$\phi(F_n) = m!, \text{ respectively, } \phi(L_n) = m!$$

but have no idea how to attack this problem. Instead, we put

$$\mathcal{N} = \{n : \phi(F_n) = m! \text{ for some positive integer } m\}$$

and prove the following properties of the set $\mathcal{N}$. Put $\mathcal{N}(x) = \mathcal{N} \cap [1, x]$. For a positive real number x we write $\log x$ for the natural logarithm of x.

Theorem 1.1. *The following hold:*

(i) $\#\mathcal{N}(x) \ll \dfrac{x \log\log x}{\log x}$, *and so* $\mathcal{N}$ *is of asymptotic density zero.*

(ii) *The only primes in* $\mathcal{N}$ *are* 2 *and* 3.

In [1] it was shown that $F_9 = 34$ and $L_3 = 4$ are the largest Fibonacci and Lucas numbers, respectively, whose Euler function is a power of 2. Here, we show the following result.

Theorem 1.2. *The only solutions in nonnegative integers of the equation* $\phi(L_n) = 2^a 3^b$ *are*

$$(n, a, b) = (0, 0, 0),\ (1, 0, 0),\ (2, 1, 0),\ (3, 1, 0),\ (4, 1, 1),\ (6, 1, 1),\ (9, 2, 2).$$

We do not know how to find all the nonnegative solutions (n, a, b) of the Diophantine equation

$$\phi(F_n) = 2^a 3^b.$$

Also, we noted that $\phi(L_{30}) = 5!7!$, but we do not even know how to prove that the set of positive integers n such that

$$\phi(F_n) = m_1! \cdots m_\ell! \qquad \text{or} \qquad \phi(L_n) = m_1! \cdots m_\ell!$$

for some integers $1 \le m_1 \le \cdots \le m_\ell$ is of asymptotic density zero. We leave such questions for the reader.

2. The proofs

2.1. The proof of Theorem 1.1

(i) Let x be a large real number and $\gamma = (1+\sqrt{5})/2$ be the golden section. Let $n \in \mathcal{N}(x)$. Since

$$\left(\frac{m}{e}\right)^m < m! = \phi(F_n) < F_n < \gamma^n \le \gamma^x,$$

it follows that for large x we have $m \le x/\log x$. Let us denote $K = \lfloor x/\log x \rfloor$. For $k = 1, \ldots, K$, put

$$\mathcal{N}_k(x) = \{n \le x : \phi(F_n) = k!\}\}.$$

Fix k and let $n_1 < n_2 < \ldots < n_t$ be all elements in $\mathcal{N}_k(x)$. Since

$$1 \le \frac{F_n}{\phi(F_n)} \ll \log\log F_n \ll \log x,$$

we get that

$$\frac{F_{n_t}}{F_{n_1}} = \left(\frac{F_{n_t}}{k!}\right)\left(\frac{k!}{F_{n_1}}\right) = \left(\frac{F_{n_t}}{\phi(F_{n_t})}\right)\left(\frac{\phi(F_{n_1})}{F_{n_1}}\right) \ll \log x.$$

Since $\gamma^{n-2} \le F_n \le \gamma^{n-1}$ holds for all n, we get that $F_{n_t}/F_{n_1} \ge \gamma^{n_t - n_1 - 1}$. Hence,

$$\gamma^{n_t - n_1 - 1} \ll \log x \qquad \text{yielding} \qquad \#\mathcal{N}_k(x) \le n_t - n_1 \ll \log\log x.$$

Since certainly

$$\mathcal{N}(x) = \bigcup_{1 \le k \le K} \mathcal{N}_k(x),$$

it follows that

$$\#\mathcal{N}(x) \le \sum_{k=1}^{K} \#\mathcal{N}_k(x) \ll K \log\log x \ll \frac{x \log\log x}{\log x},$$

which completes the proof of (i).

(ii) Assume that $p > 12$ is in $\mathcal{N}$. Then all prime factors q of F_p satisfy the relation $q \equiv (5|q) \pmod{p}$, where $(a|q)$ is the Legendre symbol of a with

respect to q. If $q \equiv 1, 4 \pmod{p}$, then $p \mid (q-1) \mid \phi(F_p)$. Since $\phi(F_p) = m!$ for some integer m, we get that $m \geq p$. Thus,

$$\gamma^p > F_p \geq \phi(F_p) \geq p! \geq (p/e)^p,$$

an inequality which is false for any $p > 12$. A similar argument proves that F_p is square free. Indeed, if $q^2 \mid F_p$, then $q \mid \phi(F_p)$, therefore $m \geq q$. Since $q \equiv \pm 1 \pmod{p}$, we get that $q \geq 2p - 1 > p$, and we get again that $\phi(F_p) \geq q! > p!$, a contradiction. Thus, F_p is square free and $q \equiv 2, 3 \pmod 5$ for all prime factors q of F_p. Since the above congruence is true for all prime factors q of F_p, we get that $5 \nmid \phi(F_p)$, so that $m \leq 4$. Hence, $\phi(F_p) \leq 4! = 24$. This is false if F_p is a prime, or if F_p has at least one prime factor > 23, or if F_p has at least four distinct prime factors because $(2-1)(3-1)(5-1)(7-1) > 24$. Hence, $F_p < 23^3$, leading to $p \leq 19$. A quick search now completes the proof of (ii).■

Remark 1. The argument used to prove (ii) shows that for each fixed positive integer a, there are only finitely many primes p such that $ap \in \mathcal{N}$. To see why, assume that $p > 12$ and $ap \in \mathcal{N}$. Then every prime factor q of F_{ap} either is a prime factor of F_a, or is a primitive prime factor of F_{dp} for some divisor d of a. In the second case, either $q \equiv 1 \pmod{p}$, and we get

$$\gamma^{ap} > F_{ap} > \phi(F_{ap}) \geq p! \geq (p/e)^p \qquad \text{therefore} \qquad p < e^a\gamma,$$

or $q \equiv 2, 3 \pmod 5$. If this last scenario happens for all prime factors q of F_{ap} which are not prime factors of F_a, we then deduce that $\nu_5(m!) = \nu_5(\phi(F_a))$, where $\nu_5(m)$ is the exponent of 5 in the factorization of m. Since certainly $\nu_5(m!) \geq \lfloor m/5 \rfloor$, we get that $\lfloor m/5 \rfloor \leq \nu_5(\phi(F_a))$, so that $m \leq 5\nu_5(\phi(F_a)) + 4$. This in turn puts an upper bound on ap. For example, for $a \in \{2, 3, 4\}$, we get that either $p < e^4\gamma$, therefore $p \leq 19$, or $m \leq 4\nu_5(\phi(F_a)) + 4 = 4$, so $\phi(F_{ap}) \leq 4!$, which again gives that $p \leq 19$, and a quick search reveals that the only such values of ap in $\mathcal{N}$ are 4 and 21.

Remark 2. The conclusions of the above theorem (with the same bounds and primes membership in $\mathcal{N}$) as well as the above Remark 1 still hold if we replace the Fibonacci numbers by Lucas numbers. One just uses the inequalities $\gamma^{n-1} \leq L_n \leq \gamma^{n+1}$ valid for all $n \geq 1$.

2.2. The proof of Theorem 1.2

Assume that $n = 2^\alpha m$ for some odd positive integer m. We start by showing that $\alpha \leq 2$. Assume that $\alpha \geq 4$. Since

$$L_{2^\alpha} = L_{2^{\alpha-1}}^2 - 2,$$

it follows that $L_{2^\alpha} \equiv 3 \pmod 4$. In particular, there exists a prime factor q of L_{2^α} such that $q \equiv 3 \pmod 4$. Reducing the relation $L_{2^\alpha}^2 - 5F_{2^\alpha}^2 = 4$ modulo q, we get that $(-5|q) = 1$. Since $q \equiv 3 \pmod 4$, we deduce that $(-1|q) = -1$, therefore $(5|q) = -1$. It follows that $q \equiv -1 \pmod{2^\alpha}$. Write $q = 2^a 3^b + 1$. Then since $q \equiv 3 \pmod 4$, we get that $a = 1$. Thus, $2^\alpha \mid (q+1)$, or $2^{\alpha-1} \mid 3^b+1$, and this is impossible for $\alpha \geq 4$ because $\nu_2(3^b+1) = 1,\ 2$ according to whether b is even or odd. This shows that $\alpha \leq 3$. The case $\alpha = 3$ is not possible since it would lead to $L_8 \mid L_n$, hence $23 \mid \phi(L_8) \mid \phi(L_n)$, a contradiction. We now look at the prime factors of m. Since $107 \mid \phi(L_{27}),\ 41 \mid \phi(L_{18})$ and $11 \mid \phi(L_{36})$, it follows that $3^3 \nmid m$. In fact, if $\alpha \in \{1, 2\}$, then $3^2 \nmid m$.

Now assume that $p > 3$ is a prime factor of m. Then $L_{2^\alpha p}$ has the same property that its Euler function is divisible only by primes which are at most 3. Let $q > 2$ be any prime factor of $L_{2^\alpha p}$ which is not a prime factor of L_{2^α}. If $\alpha = 0$, then reducing the formula $L_p^2 - 5F_p^2 = -4$ modulo q, we get that $(5|q) = 1$. This shows that $q \equiv 1 \pmod p$, therefore $p \mid \phi(L_p)$, which is a contradiction because $p > 3$. This shows that the only acceptable solutions when $\alpha = 0$ are $n = 3,\ 9$. Assume now that $\alpha \geq 1$. Reducing the formula $L_{2^\alpha p}^2 - 5F_{2^\alpha p}^2 = 4$ modulo q we get $(-5|q) = 1$. If $q \equiv 1 \pmod 4$, then we get $q \equiv 1 \pmod p$, leading to $p \mid \phi(L_{2^\alpha p})$, which is a contradiction for $p > 3$. So, we get that $n \in \{2, 4, 6, 12\}$ and the solution $n = 12$ is not convenient. So, we need to treat the case when $q \equiv -1 \pmod 4$ for all prime factors q of $L_{2^\alpha p}/L_{2^\alpha}$, which leads to the conclusion that $q = 2 \cdot 3^{b_q} + 1$. Moreover, $q \equiv -1 \pmod p$, therefore $2 \cdot 3^{b_q} + 1 = a_q p - 1$ for some even integer a_q. Further, it is clear that $L_{2^\alpha p}/L_{2^\alpha}$ is square free. Thus, we get that

$$L_{2^\alpha p} = L_{2^\alpha} q_1 q_2 \cdots q_\ell,$$

where $q_i = 2 \cdot 3^{b_{q_i}} + 1$ for $i = 1, \ldots, \ell$. We may assume that $1 \leq b_{q_1} < \cdots < b_{q_\ell}$. We thus get that

$$3^{b_1} \mid L_{2^\alpha p} - L_{2^\alpha} = 5F_{2^{\alpha-1}(p-1)} F_{2^{\alpha-1}(p+1)}.$$

Now F_m is a multiple of 3 if and only if $4 \mid m$. Moreover, in this case, $\nu_3(F_m) = \nu_3(m) + 1$. Since exactly one of $p-1$ and $p+1$ is a multiple of 3, and exactly one of these two numbers is a multiple of 4, it follows that

$$\begin{aligned} \min\{\nu_3(F_{2^{\alpha-1}(p-1)}, F_{2^{\alpha-1}(p+1)}\} &\leq 1, \\ \max\{\nu_3(F_{2^{\alpha-1}(p-1)}, F_{2^{\alpha-1}(p+1)}\} &\leq 1 + \max\{\nu_3(p-1), \nu_3(p+1)\}. \end{aligned}$$

In particular, we deduce that if $b_{q_1} \geq 2$, then $3^{b_{q_1}-2} \mid (p-1)/2$ or $3^{b_{q_1}-2} \mid (p+1)/2$. On the one hand, writing

$$p = \frac{2 \cdot 3^{b_{q_1}} + 2}{a_{q_1}}, \quad \text{we get that} \quad 3^{b_{q_1}-2} \mid a_{q_1} + 1, \quad \text{or} \quad 3^{b_{q_1}-2} \mid a_{q_1} - 1.$$

Since $(p+1)/2 \geq 3^{b_{q_1}-2}$, we get that

$$\frac{3^{b_{q_1}}+1}{a_1} = \frac{p}{2} > 3^{b_{q_1}-2} - 1.$$

On the one hand, if $a_{q_1} \geq 10$, then $3^{b_{q_1}} + 1 > 10(3^{b_{q_1}-2} - 1)$, or $11 \geq 3^{b_{q_1}-2}$, or $b_{q_1} \leq 4$. On the other hand, if $a_{q_1} \leq 8$, then $3^{b_{q_1}-2}$ divides one of $a_{q_1} - 1$ or $a_{q_1} + 1$, a number which is at most 9, so again $b_{q_1} \leq 4$. Thus, $b_{q_1} \in \{1, 2, 3, 4\}$, so the only possibilities are $q_1 \in \{7, 19, 163\}$. The case $q_1 = 7$ leads to $\alpha = 2$, then $p = 7$, which is false because 7^2 cannot divide $L_{2^\alpha p}$. The case $q_1 = 19$ leads to $p \mid q_1 - 1$, which is false because $p > 3$. The case $q_1 = 163$ leads to $p \mid 164$, so $p = 41$. However, in this case since $q = 163$ divides $L_{2^a p}$, we get that $\alpha = 1$. In this case, $31 \mid \phi(L_{82})$, and we get a contradiction. So, we indeed conclude that n cannot be divisible by any prime $p > 3$, which completes the proof of the theorem. ■

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