ON HOMOGENEOUS PÁLES MEANS

László Losonczi (Debrecen, Hungary)

Dedicated to the 75th birthday of Academicians
Zoltán Daróczy and Imre Kátai

Communicated by Antal Járai

(Received May 31, 2013; accepted September 24, 2013)

Abstract. Let $f, g : I \to \mathbb{R}$ be continuous functions such that g is positive, f/g is strictly monotonic on the open interval I and let μ be a probability measure on the Borel sets of $[0, 1]$. The two-variable mean $M_{f,g;\mu} : I^2 \to I$ defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t)}{\int\limits_{[0,1]} g(tx + (1-t)y)d\mu(t)} \right).$$

was proposed by Zsolt Páles in 2005. It is a common generalization of quasi-arithmetic, Bajraktarević, Lagrangian and Cauchy means. Extending earlier results we solve the homogeneity equation

$$M_{f,g;\mu}(tx, ty) = tM_{f,g,\mu}(x, y) \quad (x, y \in I, t \in \mathbb{R} \text{ such that } tx, ty \in I)$$

if the centralized moments c_k of the measure satisfy the condition

$$5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 \neq 0.$$

Key words and phrases: Functional means, functional equation, homogeneous functions, symmetric means, moments.
2010 Mathematics Subject Classification: 39B22.
This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402.

https://doi.org/10.71352/ac.41.103

1. Introduction

Let I be a real open interval. The classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$ respectively.

A continuous function $M : I^2 \to I$ is called a *two-variable mean* on I if the mean value inequality

$$\min(x, y) \leq M(x, y) \leq \max\{x, y) \qquad (x, y \in I)$$

holds.

Let now $f, g : I \to \mathbb{R}$ be continuous functions on I with $g \in \mathcal{CP}(I)$, $h := f/g \in \mathcal{CM}(I)$ and let further μ be a probability measure on the Borel sets of $[0, 1]$. Applying the mean value theorem of the integral calculus (for the integral of $f = h \cdot g$ in the numerator) one can easily see that the two-variable function $M_{f,g;\mu} : I^2 \to I$ defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int\limits_{[0,1]} f(tx + (1-t)y) d\mu(t)}{\int\limits_{[0,1]} g(tx + (1-t)y) d\mu(t)} \right) \qquad (x, y \in I).$$

is a two-variable mean on I. This mean was proposed by Zsolt Páles [37] and henceforth will be called Páles mean (in some previous papers it was called (two variable) functional mean generated by two functions and a measure). Functional equations and inequalities concerning (special) Páles means were discussed in [27], [28], [31], [32]. With suitable choice of μ the quasi-arithmetic, Bajraktarević, Lagrangian and Cauchy (and also several other) means can be obtained as special cases of Páles means.

If $g = p \in \mathcal{CP}(I)$, $f = p\varphi$ $(\varphi \in \mathcal{CM}(I))$, $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ where δ_t is the Dirac measure concentrated at the point t, then

$$M_{f,g;\mu}(x, y) = B_{\varphi,p}(x, y) := \varphi^{-1}\left(\frac{p(x)\varphi(x) + p(y)\varphi(y)}{p(x) + p(y)}\right) \qquad (x, y \in I)$$

is the Bajraktarević mean (Bajraktarević [3], [4]). Several papers dealt with functional equations and inequalities for these means and their multi-variable versions, see e.g. Aczél-Daróczy [1], Beckenbach [5], Danskin [10], Daróczy [11], Dresher [13], Losonczi [18], [19], [26], Losonczi and Páles [29] and the references there, and Páles [33], [34], [35], [36].

Let now $\varphi, \psi : I \to \mathbb{R}$ be given functions such that $\varphi'/\psi' \in \mathcal{CM}(I)$ and $\psi' \in \mathcal{CP}(I)$. If $f = \varphi', g = \psi'$ and μ= the Lebesgue measure then

$$M_{f,g;\mu}(x,y) = C_{\varphi,\psi}(x,y) := \begin{cases} \left(\frac{\varphi'}{\psi'}\right)^{-1}\left(\frac{\varphi(y)-\varphi(x)}{\psi(y)-\psi(x)}\right) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \quad (x, y \in I)$$

is the Cauchy mean (or difference mean). It was was first defined and studied by Leach and Scholander [14], [15], [16] (they called it *extended* mean). Some homogeneous two variable Cauchy means were discovered earlier by Stolarsky [43], [44]. Inequalities for these special means have been studied extensively, see Alzer [2], Brenner [6], Brenner and Carlson [7], Burk [8], Carlson [9], Dodd [12], Losonczi [21], [23], [24], Losonczi and Páles [30], Lin [17], Páles [33], Pittenger [38], [39], Sándor [40], Seiffert [41], [42], Székely [45].

In Losonczi [20] and [22] the equality and homogeneity problems were solved for several variable Cauchy means.

The Bajraktarević and the Cauchy means are symmetric means. If we take $g = p, f = p\varphi$ ($\varphi \in \mathcal{CM}(I)$, $p \in \mathcal{CP}(I)$), $\mu_1 = \frac{1}{\alpha+2}(\delta_0 + \alpha\delta_{\frac{1}{3}} + \delta_1)$, $(\alpha > 0)$ then

$$M_{f,g;\mu_1}(x,y) = \varphi^{-1}\left(\frac{p(x)\varphi(x) + \alpha p\left(\frac{x+2y}{3}\right)\varphi\left(\frac{x+2y}{3}\right) + p(y)\varphi(y)}{p(x) + \alpha p\left(\frac{x+2y}{3}\right) + p(y)}\right) \quad (x, y \in I)$$

is a *non-symmetric mean* which can be considered as a modified Bajraktarević mean.

By $g = p, f = p\varphi$ ($\varphi \in \mathcal{CM}(I)$, $p \in \mathcal{CP}(I)$), $\mu_2 = \frac{1}{2\alpha+2}(\delta_0 + \alpha(\delta_{\frac{1}{3}} + \delta_{\frac{2}{3}}) + \delta_1)$, $(\alpha > 0)$ we get a *symmetric mean*

$$M_{f,g;\mu_2}(x, y = \varphi^{-1}\left(\frac{p(x)\varphi(x) + \alpha p\left(\frac{x+2y}{3}\right)\varphi\left(\frac{x+2y}{3}\right) + \alpha p\left(\frac{2x+y}{3}\right)\varphi\left(\frac{2x+y}{3}\right) + p(y)\varphi(y)}{p(x) + \alpha p\left(\frac{x+2y}{3}\right) + \alpha p\left(\frac{2x+y}{3}\right) + p(y)}\right)$$

$x, y \in I$ which can again be considered as another modified Bajraktarević mean.

In the latter two means the measures depend on the parameter α. Changing this parameter we can influence the moments of the measure. With $(\alpha = 0)$ both means are reduced to the Bajraktarević mean.

The homogeneity problem for Páles means is finding all means $M_{f,g;\mu}$ (or function f, g) which satisfy the homogeneity equation

$$M_{f,g;\mu}(tx, ty) = tM_{f,g;\mu}(x, y) \quad (x, y \in I, t \in \mathbb{R} \text{ such that } tx, ty \in I).$$

In the sequel we assume that I *is an open multiplicatively symmetric* (*i.e.* $x \in I$ *implies* $1/x \in I$) *subinterval of the positive half line* $\mathbb{R}_+$. This clearly implies that $1 \in I$.

In [28] the author determined all homogeneous Bajraktarević and Cauchy means, while in [27] the non-symmetric homogeneous Páles means (assuming that μ has non vanishing third central moment) were found.

The aim of this paper is to find all homogeneous Páles means. The measure μ enters into our calculations through its moments.

The *kth moments of* μ are the integrals

$$m_k := \int_0^1 t^k d\mu(t) \quad (k = 0, 1 \dots).$$

Clearly $m_0 = 1$, and for symmetric means $m_1 = 1/2$.

The *centralized kth moments of* μ

$$c_k := \int_0^1 (t - m_1)^k d\mu(t) \quad (k = 0, 1 \dots)$$

are more suitable for our calculations. It is clear that $c_0 = 1, c_1 = 0$. Further $c_{2k} = 0\,(k = 1, 2, \dots)$ if and only if $\mu = \delta_{m_1}$. For this measure $M_{f,g;\mu}(x, y) = m_1 x + (1 - m_1)y$ $(x, y \in I)$ is a weighted arithmetic mean. *Thus excluding weighted arithmetic means we may assume that* $c_{2k} > 0$ $(k = 1, 2, \dots)$.

It is easy to transform the moments to centralized moments and vice versa.

2. Results

For the sake of shortness introduce the notations $\mathcal{E}(x) := \exp(x)$, $(x \in \mathbb{R})$ and $\mathcal{E}_a(x) := \exp\left(\frac{1}{a}\arctan(x)\right)$, $(x \in \mathbb{R}, a > 0)$. Concerning the homogeneity of Páles means we have

Theorem 2.1. *Suppose that* I *is an open multiplicatively symmetric subinterval of* $\mathbb{R}_+$, $f, g : I \to \mathbb{R}$ *are six times continuously differentiable functions with* $g \in \mathcal{CP}(I)$,$h' = \left(\frac{f}{g}\right)' \neq 0$, *and the moment condition*

$$5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 \neq 0 \tag{2.1}$$

holds. Then the only homogeneous Páles means are the weighted arithmetic *means and the means*

$$\left(\frac{\int\limits_{[0,1]}(tx+(1-t)y)^{c+a}d\mu(t)}{\int\limits_{[0,1]}(tx+(1-t)y)^{c-a}d\mu(t)}\right)^{\frac{1}{2a}}\qquad (x,y\in I),$$

$$\mathcal{E}\left(\frac{\int\limits_{[0,1]}(tx+(1-t)y)^{c}\ln(tx+(1-t)y)d\mu(t)}{\int\limits_{[0,1]}(tx+(1-t)y)^{c}d\mu(t)}\right)\qquad (x,y\in I),$$

$$\mathcal{E}_a\left(\frac{\int\limits_{[0,1]}(tx+(1-t)y)^{c}\sin(a\ln(tx+(1-t)y))d\mu(t)}{\int\limits_{[0,1]}(tx+(1-t)y)^{c}\cos(a\ln(tx+(1-t)y))d\mu(t)}\right)\qquad (x,y\in I),$$

where $a>0, c\in\mathbb{R}$ *are constants. In case of the first two means the maximal interval* I *is* $]0,\infty[$, *in case of the last mean the maximal interval is* $\left]\exp\left(-\frac{\pi}{2a}\right),\exp\left(\frac{\pi}{2a}\right)\right[$.

For the modified Bajraktarević means M_{f,g,μ_1} and M_{f,g,μ_2} we have the following

Theorem 2.2. *Suppose that* I *is an open multiplicatively symmetric subinterval of* $\mathbb{R}_+$, $f,g: I\to\mathbb{R}$ *are three times continuously differentiable functions with* $g\in\mathcal{CP}(I)$,$h'=\left(\frac{f}{g}\right)'\neq 0$. *Then the only homogeneous non-symmetric modified Bajraktarević means* M_{f,g,μ_1} *are the means*

$$\left(\frac{x^{c+a}+\alpha\left(\frac{x+2y}{3}\right)^{c+a}+y^{c+a}}{x^{c-a}+\alpha\left(\frac{x+2y}{3}\right)^{c-a}+y^{c-a}}\right)^{\frac{1}{2a}},$$

$$\mathcal{E}\left(\frac{x^c\ln x+\alpha\left(\frac{x+2y}{3}\right)^{c}\ln\left(\frac{x+2y}{3}\right)+y^c\ln y}{x^c+\alpha\left(\frac{x+2y}{3}\right)^{c}+y^c}\right),$$

$$\mathcal{E}_a\left(\frac{x^c\sin(\ln x^a)+\alpha\left(\frac{x+2y}{3}\right)^{c}\sin\left(\ln\left(\frac{x+2y}{3}\right)^{a}\right)+y^c\sin(\ln y^a)}{x^c\cos(\ln x^a)+\alpha\left(\frac{x+2y}{3}\right)^{c}\cos\left(\ln\left(\frac{x+2y}{3}\right)^{a}\right)+y^c\cos(\ln y^a)}\right),$$

where $x,y\in I$, $a>0, c\in\mathbb{R}$ *are constants. In case of the first two means the maximal interval* I *is* $]0,\infty[$, *in case of the last mean the maximal interval is* $\left]\exp\left(-\frac{\pi}{2a}\right),\exp\left(\frac{\pi}{2a}\right)\right[$.

Theorem 2.3. *Suppose that I is an open multiplicatively symmetric subinterval of $\mathbb{R}_+$, $f,g : I \to \mathbb{R}$ are six times continuously differentiable functions with $g \in \mathcal{CP}(I)$,$h' = \left(\frac{f}{g}\right)' \neq 0$. If $0 < \alpha \neq 81/11$ then the only homogeneous symmetric modified Bajraktarević means M_{f,g,μ_2} are*

$$\left(\frac{x^{c+a} + \alpha\left(\frac{x+2y}{3}\right)^{c+a} + \alpha\left(\frac{2x+y}{3}\right)^{c+a} + y^{c+a}}{x^{c-a} + \alpha\left(\frac{x+2y}{3}\right)^{c-a} + \alpha\left(\frac{2x+y}{3}\right)^{c-a} + y^{c-a}}\right)^{\frac{1}{2a}},$$

$$\mathcal{E}\left(\frac{x^c \ln x + \alpha\left(\frac{x+2y}{3}\right)^c \ln\left(\frac{x+2y}{3}\right) + \alpha\left(\frac{2x+y}{3}\right)^c \ln\left(\frac{2x+y}{3}\right) + y^c \ln y}{x^c + \alpha\left(\frac{x+2y}{3}\right)^c + \alpha\left(\frac{2x+y}{3}\right)^c + y^c}\right),$$

$$\mathcal{E}_a\left(\frac{x^c \sin(\ln x^a) + \alpha\left(\frac{x+2y}{3}\right)^c \sin\left(\ln\left(\frac{x+2y}{3}\right)^a\right) + \alpha\left(\frac{2x+y}{3}\right)^c \sin\left(\ln\left(\frac{2x+y}{3}\right)^a\right) + y^c \sin(\ln y^a)}{x^c \cos(\ln x^a) + \alpha\left(\frac{x+2y}{3}\right)^c \cos\left(\ln\left(\frac{x+2y}{3}\right)^a\right) + \alpha\left(\frac{2x+y}{3}\right)^c \cos\left(\ln\left(\frac{2x+y}{3}\right)^a\right) + y^c \cos(\ln y^a)}\right),$$

where $x, y \in I$, $a > 0, c \in \mathbb{R}$ are constants. In case of the first two means the maximal interval I is $]0,\infty[$, in case of the last mean the maximal interval is $\left]\exp\left(-\frac{\pi}{2a}\right), \exp\left(\frac{\pi}{2a}\right)\right[$.

3. Proofs

Proof of Theorem 2.1. Excluding weighted arithmetic means (which are homogeneous Páles means) we may assume that $c_2c_4c_6 \neq 0$.

The partial derivatives

$$E_{k,l}(x) := \left.\frac{\partial^{k+l} M_{f,g;\mu}(x,y)}{\partial x^k \partial y^l}\right|_{y=x} \qquad (x \in I)$$

of a homogeneous mean $M_{f,g;\mu}$ are homogeneous functions of degree $1-(k+l)$ therefore

$$E_{k,l}(x) = c_{k,l}x^{1-(k+l)} \tag{3.1}$$

with a suitable constants $c_{k,l}$.

It turns out that the functions $E_{k,l}$ are polynomials of the functions

$$G = G(x) := \frac{g'(x)}{g(x)}, \quad H = H(x) := \frac{h''(x)}{h'(x)} \quad (x \in I),$$

and their derivatives (and the coefficients in these polynomials are polynomials of the moments of the measure μ).

Using equations (3.1) we deduce systems of differential equations for the unknown functions G, H (and thus for f, g).

Concerning the measure μ we distinguish three cases:

- Case 1: $c_3 \neq 0$,
- Case 2: $c_3 = 0$ and $c_5 \neq 0$,
- Case 3: $c_3 = c_5 = 0$ and $5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 \neq 0$.

We remark that Case 1 (in slightly modified form) was discussed in [27] but for the sake of completeness we also give the details here.

Case 1: $c_3 \neq 0$.

We start with the equations

(3.2)
$$\begin{aligned} E_{1,1}(x) &= \frac{c_{1,1}}{x} = -c_2(H+2G) \\ E_{2,1}(x) &= \frac{c_{2,1}}{x^2} = -c_3(3G^2-3GH-H'-3G'-H^2)+c_2(1-3m_1)(H'+2G'). \end{aligned}$$

Introducing a new constant $c := -\frac{c_{1,1}}{2c_2}$ and substituting $G(x) = \frac{c}{x} - \frac{1}{2}H(x)$, $G'(x) = -\frac{c}{x^2} - \frac{1}{2}H'(x)$, from the first equation into the second we get the Riccati equation

$$H' - \frac{H^2}{2} = \frac{d_1}{x^2} \tag{3.3}$$

where $d_1 := 6(c^2 - c) + \frac{2c_{2,1}+4cc_2(1-3m_1)}{c_3}$ is a constant. By

$$z(x) = e^{-\frac{1}{2}\int H(x)\,dx} (> 0), \qquad H(x) = -2\frac{d}{dx}(\ln z(x)) \tag{3.4}$$

(3.3) transforms to the Euler's equation

$$z'' + \frac{d_1}{x^2}z = 0. \tag{3.5}$$

The nonzero solutions of (3.5) are

$$\begin{array}{lll} z &= R\sqrt{x}\cos(\ln x^a + D), & \text{if } 0 > 1 - 4d_1 = -(2a)^2, \\ z &= R\sqrt{x}\cosh(\ln x^a + D), & \text{if } 0 < 1 - 4d_1 = (2a)^2, \\ z &= R\sqrt{x}\sinh(\ln x^a + D), & \text{if } 0 < 1 - 4d_1 = (2a)^2, \\ z &= R\sqrt{x}\exp(\pm\ln x^a), & \text{if } 0 < 1 - 4d_1 = (2a)^2, \\ z &= R\sqrt{x}, & \text{if } 0 = 1 - 4d_1, \\ z &= R\sqrt{x}(\ln x + D), & \text{if } 0 = 1 - 4d_1, \end{array} \tag{3.6}$$

where $R \neq 0, a \neq 0, D$ are constants (which should be chosen such that $z(x) > 0$).

From (3.4)

$$(\ln |h'|)' = \frac{h''}{h'} = H = -2\frac{z'}{z} = \left(\ln z^{-2}\right)',$$

hence, by integrating, we obtain

$$h' = K_1 z^{-2}, \qquad h = K_1 \int z^{-2} + L. \tag{3.7}$$

Integrating the equation $G(x) = \frac{c}{x} - \frac{1}{2}H(x)$ we get that $h'(x)g(x)^2 = \beta x^{2c}$ where β is an arbitrary constant. From this and (3.7) we obtain that

$$g = K_2\, x^c\, z, \quad f = g\, h \tag{3.8}$$

where $K_1 \neq 0, K_2 \neq 0, L$ are arbitrary constants.

Performing the integrations with the solutions z given by (3.6) and introducing some new constants if necessary we get the following solutions for h, g and f

(3.9)

h	g	f
$K \tan(\ln x^a + D) + L$	$Mx^c \cos(\ln x^a + D)$	$Mx^c\,[K \sin(\ln x^a + D) + L \cos(\ln x^a + D)]$
$K \tanh(\ln x^a + D) + L$	$Mx^c \cosh(\ln x^a + D)$	$Mx^c\,[K \sinh(\ln x^a + D) + L \cosh(\ln x^a + D)]$
$K \coth(\ln x^a + D) + L$	$Mx^c \sinh(\ln x^a + D)$	$Mx^c\,[K \cosh(\ln x^a + D) + L \sinh(\ln x^a + D)]$
$Kx^{\mp 2a} + L$	$Mx^c x^{\pm 2a}$	$Mx^c\left[Kx^{\mp 2a} + Lx^{\pm 2a}\right]$
$K(\ln x + D)^{-1} + L$	$Mx^c\sqrt{x}(\ln x + D)$	$Mx^c\sqrt{x}\,[K + L(\ln x + D)]$
$K \ln x + L$	$Mx^c\sqrt{x}$	$Mx^c\sqrt{x}\,[K \ln x + L]$

where $a, c, D, K, L, M \in \mathbb{R}$ are arbitrary constants apart from the restrictions $aKM \neq 0$.

It is easy to check that

$$M_{f_1,g_1,\mu}(x,y) = M_{f_2,g_2,\mu}(x,y) \quad (x,y \in I)$$

if

$$\begin{aligned} f_2(x) &= pf_1(x) + qg_1(x) \quad (x \in I) \\ g_2(x) &= rf_1(x) + sg_1(x) \quad (x \in I) \end{aligned} \tag{3.10}$$

where $p, q, r, s \in \mathbb{R}$ are constants for which $ps - qr \neq 0$. Let us call the pairs (f_1, g_1) and (f_2, g_2) *equivalent* (with respect to the mean $M_{f,g,\mu}$) if (3.10) holds. Equivalent pairs generate the same mean.

By the addition formulae of $\sin, \cos, \sinh, \cosh$ the pairs (f, g) listed below are equivalent to the pairs given in (3.9)

(3.11)

g	f	condition
$x^c \cos(\ln x^a)$	$x^c \sin(\ln x^a)$	$\ln x^a \in]\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}[$ if $x \in I$
$x^c \cosh(\ln x^a)$	$x^c \sinh(\ln x^a)$	none
$x^c \sinh(\ln x^a)$	$x^c \cosh(\ln x^a)$	$\ln x^a > 0$ or $\ln x^a < 0$ if $x \in I$
x^{c+a}	x^{c-a}	none
x^c	$x^c \ln x$	$\ln x > 0$ or $\ln x < 0$ if $x \in I$

where $a \neq 0, c, d \neq c$ are arbitrary constants and k is an integer (for the uniform appearance of the constants in (3.11) we replaced $2a$ by a in the fourth line of (3.9) and we replaced $c + \frac{1}{2}$ by c in lines five and six).

The first pair (f, g) of (3.11) gives the third mean of Theorem 2.1, and the maximal interval I as stated.

The second, third, fourth pairs (f, g) of (3.11) build up the same mean, as they are linear combinations of $x^c \exp a \ln x = x^{c+a}$ and $x^c \exp(-a \ln x) = = x^{c-a}$. From them we get the first mean of Theorem 2.1, the maximal interval being $I =]0, \infty[$.

Finally the last pair (f, g) of (3.11) builds up the second mean of Theorem 2.1, with the maximal interval again $I =]0, \infty[$.

We may assume that the constant a is positive as changing a to $-a$ leaves our means unchanged.

It is easy to check that the three means listed in Theorem 2.1 are homogeneous indeed.

Case 2: $c_3 = 0, c_5 \neq 0$.

Calculating the $E_{1,1}, E_{2,2}$ (using Maple V) in terms of G, H and their derivatives and introducing new constants $c := -\frac{c_{1,1}}{2c_2}$ (as before) $d := -\frac{c_{2,2}}{2c_4}$ we get the system of differential equations

$$
\begin{aligned}
H + 2G &= \frac{2c}{x} \\
H^3 + 4G^3 + 4H^2G + 6HG^2 & \\
+6HG' + 4H'G + 3HH' + 12GG' + 4G'' + H'' & \\
-3\frac{c_2^2}{c_4}(H^3 + 4G^3 + 4H^2G + 6HG^2 + 2HG' + 4GG') & \\
+\frac{5c_2}{2c_4}(2G'' + H'') &= -\frac{2d}{x^3}.
\end{aligned} \tag{3.12}
$$

Substituting G, G', G'' from the first equation into the second and introducing the new unknown function

$$S(x) := \frac{1}{2}\left(H'(x) - \frac{H(x)^2}{2}\right) \tag{3.13}$$

we get the following first order linear differential equation

$$S'(x) + \frac{2cuS(x)}{x} = \frac{k}{x^3} \tag{3.14}$$

where $k = k(c, d, c_2, u)$

$$\begin{aligned} u: &= 1 - \frac{3c_2^2}{c_4}, \\ k: &= d + 2uc^3 - (4+2u)c^2 + \left(4 + \frac{5(1-u)}{3c_2}\right) \end{aligned}$$

We may assume here that $c \neq 0$, otherwise from (3.14) we get $S(x) = \dfrac{\text{constant}}{x^2}$ and the Riccati equation (3.13) (for the unknown function H) transforms to an Euler's equation and we can get the solution as in Case 1.

The general solution of (3.14) is

$$S(x) = \begin{cases} k(c, d, c_2, u)x^{-8cu}\dfrac{x^{8cu-2} - 1}{8cu - 2} + Cx^{-8cu} & \text{if } cu \neq 1/4, \\ k(c, d, c_2, 1/(4c))x^{-2}\ln x + Cx^{-2} & \text{if } cu = 1/4, \end{cases}$$

where C is an arbitrary constant. The next step is to determine H from the (Riccati) equation (3.13)

$$H' - \frac{H^2}{2} = 2S.$$

With $z(x) = e^{-\frac{1}{2}\int H(x)\,dx}(> 0)$, $H(x) = -2\left(\ln z(x)\right)'$ this transforms to the linear homogeneous differential equation

$$z'' + S\,z = 0. \tag{3.15}$$

Unfortunately the Riccati equation cannot be be solved by quadratures for all values of the parameters. At this point the equation obtained from the *fifth derivative* helps. We perform the following steps (all Maple calculations):

- calculation of $E_{3,2}$ in terms of G, H and their derivatives,
- substitution of $G(x) = \dfrac{c}{x} - \dfrac{1}{2}H(x)$, $G'(x) = -\dfrac{c}{x^2} - \dfrac{1}{2}H'(x)$, etc. into $E_{3,2}$ it becomes a function of H, H', H'' etc.,

- substitution of $H' = 2S + \dfrac{H^2}{2}$, $H'' = 2S' + 2HS + \dfrac{H^3}{2}$, etc. into the previous form of $E_{3,2}$ it becomes a function of S, S', S'' etc., and H disappears!
- substitution of $S'(x) = -\dfrac{2cuS(x)}{x} + \dfrac{k}{x^3}$ and from this S'', S''' etc. into the previous form of $E_{3,2}$ it becomes a function of S only.

Equation (3.1) that is $E_{3,2}(x) = \dfrac{e}{x^4}$ (after multiplication by x^4) goes over into the second degree algebraic equation

$$A(x^2S(x))^2 + B(x^2S(x)) + C = 0 \tag{3.16}$$

where the coefficients A, B, C depend on the parameters $c, d, e, m_1, c_2, c_4, c_5$. In particular $A = c_5c_4^2 \neq 0$, therefore

$$S(x) = \frac{\text{constant}}{x^2},$$

thus (3.15) is an Euler's equation again from which we get the unknown functions and the homogeneous means the same way as in Case 1.

Case 3: $c_3 = c_5 = 0$ and $5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 \neq 0$.

We do the same as in the previous Case, but instead of the fifth derivative we use the *sixth derivative* $E_{3,3}(x) = \dfrac{e}{x^5}$. This again goes over into the second degree algebraic equation

$$A^*(x^2S(x))^2 + B^*(x^2S(x)) + C^* = 0 \tag{3.17}$$

but now the coefficient A^* has the form

$$A^* = -6\frac{c}{c_4}\left(5c_2c_4^2 + c_4c_6 - 6c_2^2c_6\right) \neq 0.$$

therefore

$$S(x) = \frac{\text{constant}}{x^2}.$$

(3.15) is an Euler's equation again from which we get the unknown functions and the homogeneous means the same way as in Case 1. ■

Proof of Theorem 2.2. For the measure $\mu_1 = \frac{1}{\alpha+2}(\delta_0 + \alpha\delta_{\frac{1}{3}} + \delta_1)$, $(\alpha > 0)$ we have $m_1 = \frac{\alpha+3}{3(\alpha+2)}$ and

$$3^3(\alpha+2)^4c_3 = (-\alpha-3)^3 - \alpha + (2\alpha+3)^3 = 7\alpha^3 + 27\alpha^2 + 26\alpha = \alpha(\alpha+2)(7\alpha+13)$$

therefore $c_3 > 0$ for $\alpha > 0$, thus Case 1. of the proof of Theorem 2.1 is applicable. ■

Proof of Theorem 2.3. For the measure $\mu_2 = \frac{1}{2\alpha+2}(\delta_0 + \alpha(\delta_{\frac{1}{3}} + \delta_{\frac{2}{3}}) + \delta_1)$, $(\alpha > 0)$ we have $m_1 = \frac{1}{2}$ and

$$c_k = \frac{(3^k + \alpha)((-1)^k + 1)}{2 \cdot 6^k(\alpha + 1)}$$

therefore $c_3 = 0$. However

$$5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 = \frac{256\alpha(81 - 11\alpha)}{6^{10}(\alpha + 1)^3}$$

therefore if $0 < \alpha \neq \frac{81}{11}$ then Case 3. is applicable. ■

4. Closing remarks

Using the inequality $c_4^2 \leq c_2c_6$ (which follows from Hölder's inequality) we can estimate the essential factor of A^* as

$$K := 5c_2c_4^2 + c_4c_6 - 6c_2^2c_6 \leq 5c_2^2c_6 + c_4c_6 - 6c_2^2c_6 = c_6(c_4 - c_2^2),$$

and here the last factor is nonnegative: $c_4 - c_2^2 \geq 0$ (again by Hölder's inequality). Thus the constant K can be positive, zero or negative. Indeed for the measure μ_2 we have that (with $\alpha > 0$)

$$K \begin{matrix} < \\ = \\ > \end{matrix} 0 \quad \text{if and only if} \quad \frac{81}{11} \begin{matrix} < \\ = \\ > \end{matrix} \alpha.$$

For Bajraktarević and for Cauchy means $K = 0, A^* = 0$. For these means in (3.17) either $B^* \neq 0$ or the parameters u, k in (3.14) are such that the equation (3.15) (obtained by transforming the Riccati equation) can be solved by integration. Using this method *all homogeneous Bajraktarević and Cauchy means have been determined* by the author in [28]. Perhaps a similar argument (solving the equation (3.15) by integration) may lead to getting rid of the moment condition (2.1).

References

[1] **Aczél, J. and Z. Daróczy,** Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind, *Publ. Math. Debrecen,* **10** (1963), 171–190.

[2] **Alzer, H.,** Bestmögliche Abschätzungen für spezielle Mittelwerte, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.,* **23(1)** (1993), 331–346.

[3] **Bajraktarević, M.,** Sur une équation fonctionnelle aux valeurs moyennes, *Glasnik Mat.-Fiz. i Astr.*, **13** (1958), 243–248.

[4] **Bajraktarević, M.,** Über die Vergleichbarkeit der mit Gewichtsfunktionen gebildeten Mittelwerte, *Studia Sci. Math. Hung.,* **4** (1969), 3–8.

[5] **Beckenbach, E.F.,** A class of mean value functions, *Amer. Math. Monthly,* **57** (1950), 1–6.

[6] **Brenner, J.,** A unified treatment and extension of some means of classical analysis I. Comparison theorems, *J. Combin. Inform System Sci.,* **3** (1978), 175–199.

[7] **Brenner, J. and B. C. Carlson,** Homogeneous mean values: Weights and asymptotics, *J. Math. Anal. Appl.,* **13** (1987), 265–280.

[8] **Burk, F.,** By all means, *Amer. Math. Monthly,* **92** (1985), 50.

[9] **Carlson, B.C.,** The logarithmic mean, *Amer. Math. Monthly,* **79** (1972), 615–618.

[10] **Danskin, J.M.,** Dresher's inequality, *Amer. Math. Monthly,* **59** (1952), 687–688.

[11] **Daróczy, Z.,** A general inequality for means, *Aequationes Math.,* **7** (1972), 16–21.

[12] **Dodd, E.L.,** Some generalization of the logarithmic mean and of similar means of two variates which become indeterminate when the two variates are equal, *Ann. Math. Statist.,* **12** (1971), 422–428.

[13] **Dresher, M.,** Moment spaces and inequalities, *Duke Math. J.,* **20** (1953), 261–271.

[14] **Leach E. and M. Sholander,** Extended mean values, *Amer. Math. Monthly,* **85** (1978), 84–90.

[15] **Leach, E. and M. Sholander,** Extended mean values II, *J. Math. Anal. Appl.,* **92** (1983), 207–223.

[16] **Leach, E. and M. Sholander,** Multi-variable extended mean values, *J. Math. Anal. Appl.,* **104** (1984), 390–407.

[17] **Lin, T.P.,** The power mean and the logarithmic mean, *Amer. Math. Monthly,* **81** (1974), 879–883.

[18] **Losonczi, L.,** Inequalities for integral mean values, *J. Math. Anal. Appl.,* **61** (1977), 586–606.

[19] **Losonczi, L.,** Equality of two variable weighted means: reduction to differential equations, *Aequationes Math.,* **58** (1999), 223–241.

[20] **Losonczi, L.,** Equality of Cauchy mean values, *Publ. Math. Debrecen,* **57** (2000), 217–230.

[21] **Losonczi, L.,** On the Comparison of Cauchy Mean Values, *J. of Inequalities and Appl.,* **7** (2002), 11–24.

[22] **Losonczi, L.,** Homogeneous Cauchy means, in: Z. Daróczy and Zs. Páles (Ed.) *Functional Equations – Results, Advances* Kluwer Academic Publ., Dordrecht-Boston-London (2002), 209–218.

[23] **Losonczi, L.,** Inequalities for Cauchy mean values, *Math. Ineq. Appl.,* **5** (2002), 349–359.

[24] **Losonczi, L.,** Comparison and subhomogeneity of integral means, *Math. Ineq. Appl.,* **5** (2002), 609–618.

[25] **Losonczi, L.,** Equality of two variable Cauchy mean values, *Aequationes Math.,* **65** (2003), 61–81.

[26] **Losonczi, L.,** Equality of two variable means revisited, *Aequationes Math.,* **71** (2006), 228–245.

[27] **Losonczi, L.,** Homogeneous non-symmetric means of two variables, *Demonstratio Math.,* **40** (2007), 169–180.

[28] **Losonczi, L.,** Homogeneous symmetric means of two variables, *Aequationes Math.,* **74** (2007), 262–281.

[29] **Losonczi, L. and Zs. Páles,** Minkowski's inequality for two variable Gini means, *Acta Sci.* (*Szeged*)*,* **62** (1996), 413–425.

[30] **Losonczi, L. and Zs. Páles,** Minkowski's inequality for two variable difference means, *Proc. Amer. Math. Soc.,* **126** (1998), 779–789.

[31] **Losonczi, L. and Zs. Páles,** Equality of two-variable functional means generated by different measures, *Aequationes Math.,* **81** (2011), 31–53.

[32] **Losonczi, L. and Zs. Páles,** Minkowski-type inequalities for means generated by two functions and a measure, *Publ. Math. Debrecen,* **70** (2011), 743–753.

[33] **Páles, Zs.,** Inequalities for differences of powers, *J. Math. Anal. Appl.,* **131** (1988), 271–281.

[34] **Páles, Zs.,** On the characterization of quasi-arithmetic means with weight function, *Aequationes Math.,* **32** (1987), 171–194.

[35] **Páles, Zs.,** On homogeneous quasideviation means, *Aequationes Math.,* **36** (1988), 132–152.

[36] **Páles, Zs.,** Comparison of two variable homogeneous means, in: *General Inequalities 6.* Proc. 6th Internat. Conf. Math. Res. Inst. Oberwolfach, Birkhäuser Verlag Basel, 1992, pp. 59–69.

[37] **Páles, Zs.,** Oral communication.

[38] **Pittenger, A.O.,** The symmetric, logarithmic, and power means, *Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz.,* **No. 681** (1980), 19–23.

[39] **Pittenger, A.O.,** The logarithmic mean in n variables, *Amer. Math. Monthly,* **92** (1985), 99–104.

[40] **Sándor, J.,** On certain inequalities for means, *J. Math. Anal. Appl.,* **189** (1995), 602–606.

[41] **Seiffert, H.,** Ungleichungen für einen bestimmten Mittelwert, *Nieuw Arch. Wisk.,* **13** (1995), 195–198.

[42] **Seiffert, H.,** Ungleichungen für elementare Mittelwerte, *Arch. Math. (Basel),* **63** (1995), 129–131.

[43] **Stolarsky, K.B.,** Generalizations of the logarithmic mean, *Math. Mag.,* **48** (1975), 87–92.

[44] **Stolarsky, K.B.,** The power and generalized logarithmic means, *Amer. Math. Monthly,* **87** (1980), 545–548.

[45] **Székely, G.,** A classification of means, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.,* **18** (1975), 129–133.

L. Losonczi
Faculty of Economics and Business Administration
Debrecen University
Debrecen
Hungary
losonczi08@gmail.com