UNBOUNDED SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. Some sufficient conditions are given under which a solution of a stochastic differential equation is unbounded as $t \to \infty$.

1. Introduction

The asymptotic behavior of solutions of one-dimensional autonomous stochastic differential equations

$$d\zeta(t) = g\left(\zeta(t)\right) dt + \sigma\left(\zeta(t)\right) dw(t), \qquad t \geq 0, \tag{1}$$

is considered in [1], [2], and [3]–[6] as $t \to \infty$. Here w is a standard Wiener process, g and σ are positive continuous functions defined on the set $\mathbf{R} =$

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$= (-\infty, \infty)$ and such that a unique and continuous solution ζ of equation (1) exists.

The same problem was later considered in [7]–[9] for a more general stochastic differential equation

$$d\eta(t) = g(\eta(t))\,\varphi(t)dt + \sigma(\eta(t))\,\theta(t)dw(t), \qquad t \ge 0, \tag{2}$$

where g and σ are continuous positive functions, φ and θ are continuous functions. Some sufficient conditions are obtained in [7]–[9] under which the exact order of growth of a solution η is determined almost surely (a.s.) by a solution μ of the corresponding ordinary differential equation

$$d\mu(t) = g(\mu(t))\,\varphi(t)dt, \qquad t \ge 0.$$

Moreover, the asymptotic equivalence of two solutions of stochastic differential equations with time-depended coefficients and that of the solutions of the corresponding ordinary differential equations are considered in [7]–[9]. One of the basic assumptions in [7]–[9] is that

$$\lim_{t\to\infty} \eta(t) = \infty \qquad \text{a.s.} \tag{3}$$

Unboundedness of solutions of stochastic differential equations is one of the important topics in studies of the asymptotic behavior of stochastic differential equations solutions. General results for the unboundedness of solutions for an autonomous stochastic differential equation can be found, for example, in [1].

In this paper, we provide some sufficient conditions for the unboundedness of a solution of a stochastic differential equation with time-depended coefficient in the general case and those for the case that considered in [7]–[9].

2. Assumptions and the main results

2.1. Unbounded solutions of a stochastic differential equation with time dependent drift and diffusion coefficients

Consider the following stochastic differential equation

$$d\xi(t) = a(t, \xi(t))\,dt + \sigma(t, \xi(t))\,dw(t), \qquad t \ge 0; \tag{4}$$

$$\xi(0) \equiv \xi_0,$$

where w is a standard Wiener process, ξ_0 is a nonrandom positive constant, ξ is a solution of equation (4), a and σ are continuous functions defined for $t \in [0, +\infty)$ and $x \in (-\infty; +\infty)$. We denote by $\mathbb{C}$ ($\mathbb{C}_+$) the class of all continuous (and positive) functions and by $\mathbb{C}^1$ ($\mathbb{C}^1_+$) the class of all continuously differentiable (and positive) functions.

Theorem 1. *Let $a \in \mathbb{C}$ and $\sigma \in \mathbb{C}_+$ be such that equation* (4) *has a continuous solution ξ. Assume further that the function σ is such that*

$$\lim_{x\to\infty} \int_0^x \frac{dy}{\sigma(t,\, y)} = \infty$$

and the derivatives σ'_t and σ'_x exist. Put

$$\tilde{g}(t,x) = -\int_0^x \frac{\sigma'_t(t,y)}{\sigma^2(t,y)}\, dy + \frac{a\,(t,x)}{\sigma(t,x)} - \frac{1}{2}\sigma'_x(t,x).$$

Then

$$\lim_{t\to\infty} \eta(t) = \infty \qquad \text{a.s.}$$

if at least one of the following two conditions hold:

$$\liminf_{T\to\infty} \frac{1}{\sqrt{2T\ln\ln T}} \int_0^T u(t)\, dt > 1, \qquad u(t) = \inf_{x\in\mathbf{R}} \left[\tilde{g}(t,x)\right], \tag{5}$$

or

$$\int_{-\infty}^0 e^{-2v(x)}\, dx = +\infty \quad \text{and} \quad \int_0^\infty e^{-2v(x)}\, dx < +\infty, \tag{6}$$

where

$$v(x) = \int_0^x \inf_{t>0} \left[\tilde{g}(t,z)\right]\, dz.$$

Proof. Put

$$\gamma(t) = f(t, \xi(t)), \qquad t > 0.$$

Then

$$\xi(t) = f^{-1}(t,\ \gamma(t)),$$

where

$$f(t,x)=\int\limits_0^x \frac{dy}{\sigma(t,y)},$$

and f^{-1} is the inverse function for f with respect to the argument x.

Using the Itô formula for equation (4) (see, for example, Theorem 4, §3, in [1]), we obtain:

$$\begin{aligned}
d\gamma(t) &= [f'_t(t,\xi(t)) + f'_x(t,\xi(t))a(t,\xi(t)) + \frac{1}{2}f''_{xx}(t,\xi(t))\sigma^2(t,\xi(t))]\,dt+ \\
&\quad + f'_x(t,\xi(t))\sigma\,(t,\xi(t))\;dw(t) = \\
&= [f'_t(t,f^{-1}(t,\gamma(t))) + f'_x(t,f^{-1}(t,\gamma(t)))a(t,f^{-1}(t,\gamma(t)))+ \\
&\quad + \frac{1}{2}f''_{xx}(t,f^{-1}(t,\gamma(t)))\sigma^2(t,f^{-1}(t,\gamma(t)))]\,dt+ \\
&\quad + f'_x(t,f^{-1}(t,\gamma(t)))\sigma\left(t,\;f^{-1}(t,\gamma(t))\right)dw(t),
\end{aligned}$$

where

$$f'_x(t,x)=\frac{1}{\sigma(t,\;x)},\qquad f'_t(t,x)=-\int\limits_0^x \frac{\sigma'_t(t,\;y)}{\sigma(t,\;y)}\,dy,$$

$$f''_{xx}(t,x)=-\frac{\sigma'(t,\;x)}{\sigma^2(t,\;x)}$$

.

Thus, the process γ is a solution of the stochastic differential equation

$$d\gamma(t)=\tilde{a}(t,\gamma(t))dt+dw(t),\qquad t\ge 0,$$

where

$$\tilde{a}(t,x)=-\int\limits_0^x \frac{\sigma'_t(t,y)}{\sigma^2(t,y)}\,dy+\frac{a\,(t,x)}{\sigma(t,x)}-\frac{1}{2}\sigma'_x(t,x).$$

Now Theorem 1 follows from Theorem 2, §16 in [1]. ■

Remark 2. It is known that equation (4) has a unique continuous solution if coefficients a and σ are continuous and such that

(i) for any $T\in(0;\infty)$, there exists a constant $K=K(T)$ such that

$$|a\,(t,x)|^2+|\sigma\,(t,x)|^2\le K^2\left(1+|x|^2\right);$$

for $t\in[0;\,T]$ and $x\in(-\infty;+\infty)$;

(ii) for all $C,T\in(0;\infty)$, there exists a constant $L=L(C,T)$ such that

$$|a\,(t,x)-a\,(t,y)|+|\sigma\,(t,x)-\sigma\,(t,y)|\le L\,|x-y|$$

for $t\in[0;\,T]$ and $(x,y)\in(-C;+C)\times(-C;+C)$.

2.2. Unbounded solutions if $a(t,x) = g(x)\varphi(t)$ and $\sigma(t,x) = \sigma(x)\theta(t)$

Consider a solution $\eta = (\eta(t),\, t \ge 0)$ of stochastic differential equation (2). We assume that $\varphi \in \mathbb{C}$, $\theta \in \mathbb{C}$, $g \in \mathbb{C}_+$, and $\sigma \in \mathbb{C}_+$ are such that equation (2) has a continuous solution η.

Denote

$$B(x) = \int_0^x \frac{dy}{\sigma(y)}. \tag{7}$$

We further assume that

$$\lim_{x\to\infty} B(x) = \infty. \tag{8}$$

Theorem 3. *Let $g \in \mathbb{C}_+$, $\sigma \in \mathbb{C}^1_+$, $\varphi \in \mathbb{C}$, $\theta \in \mathbb{C}^1_+$ be such that equation* (2) *has a continuous solution η. If the function*

$$\tilde{g}_1(t,x) = -\frac{\theta'(t)}{\theta^2(t)} B(x) + \frac{g(x)\,\varphi(t)}{\sigma(x)\theta(t)} - \frac{1}{2}\sigma'(x)\theta(t)$$

satisfies at least one of the conditions (5) *or* (6) *of Theorem 1, where $\tilde{g}(t,x) = \tilde{g}_1(t,x)$, and* (8) *holds, then*

$$\lim_{t\to\infty} \eta(t) = \infty \qquad \text{a.s.}$$

Proof. Denote $\gamma(t) = f(t,\eta(t))$, where

$$f(t,x) = \frac{1}{\theta(t)} \int_0^x \frac{dy}{\sigma(y)} = \frac{1}{\theta(t)} B(x)$$

and where the function B is defined by (7). Since B is a strictly increasing function and condition (8) holds, $B^{-1}(\theta(t)x)$ is the inverse for f with respect to the argument x. This, in particular, means that

$$f(t, f^{-1}(t,x)) = x \quad \text{and} \quad f^{-1}(t, f(t,x)) = x.$$

Thus,

$$f(t,x) = \frac{1}{\theta(t)} B(x)$$

and

$$f(t, f^{-1}(t,x)) = f(t, B^{-1}(\theta(t)x)) = \frac{1}{\theta(t)} B(B^{-1}(\theta(t)x)) = \frac{1}{\theta(t)} x\theta(t) = x.$$

On the other hand

$$f^{-1}(t, f(t,x)) = B^{-1}\left(\frac{1}{\theta(t)}\theta(t)B(x)\right) = B^{-1}(B(x)) = x.$$

Hence $\eta(t) = f^{-1}(t, \gamma(t))$ with $\gamma(t) = f(t, \eta(t))$. Using the Itô formula we obtain:

$$d\gamma(t) = [f_t'(t, \eta(t)) + f_x'(t, \eta(t))g(\eta(t))\varphi(t) + \frac{1}{2}f_{xx}''(t, \eta(t))\sigma^2(\eta(t))\theta^2(t)]dt + \\ + f_x'(t, \eta(t))\sigma(\eta(t))\theta(t)dw(t).$$

Since

$$f_x'(t,x) = \frac{1}{\sigma(x)\theta(t)}, \qquad f_t'(t,x) = -\frac{\theta'(t)}{\theta^2(t)}\int_0^x \frac{dy}{\sigma(y)},$$

$$f_{xx}''(t,x) = -\frac{\sigma'(x)}{\sigma^2(x)\theta(t)},$$

we conclude that

$$d\gamma(t) = \tilde{g}(t, \gamma(t))dt + dw(t), \qquad t \geq 0,$$

where

$$\tilde{g}(t,x) = -\frac{\theta'(t)}{\theta^2(t)}\int_0^x \frac{dy}{\sigma(y)} + \frac{g(x)\varphi(t)}{\sigma(x)\theta(t)} - \frac{1}{2}\sigma'(x)\theta(t). \tag{9}$$

Now Theorem 3 follows from Theorem 1. ■

2.3. Examples

Below there are some useful results for constructing examples where condition (5) holds but (6) does not hold or vice versa.

Lemma 4. *Assume that*

a_1) *θ is an increasing function for $t > 0$;*

a_2) *there exists $x_0 \geq 0$ such that $\sigma'(x_0) \geq 0$;*

a_3)

$$\liminf_{T\to\infty} \frac{1}{\sqrt{2T\log\log T}}\int_0^T \frac{\varphi(t)}{\theta(t)}\,dt \leq \frac{\sigma(x_0)}{g(x_0)}.$$

Then condition (5) *does not hold.*

Proof. Since

$$u(t) = \inf_{x \in \mathbf{R}} \tilde{g}(t,x) \leq \tilde{g}(t,x_0) \leq \frac{g(x_0)}{\sigma(x_0)} \cdot \frac{\varphi(t)}{\theta(t)},$$

condition (5) does not hold, indeed. ■

Lemma 5. *Assume that*

$b_1)$ $\theta(t) = \theta_0$ *for* $t > 0$;

$b_2)$ $\sigma'(x) \leq 0$ *for* $x \in \mathbf{R}$;

$b_3)$ $\lambda_0 = \inf_{x \in \mathbf{R}} \dfrac{g(x)}{\sigma(x)} > 0$ *and*

$$\liminf_{T \to \infty} \frac{1}{\sqrt{2T \log\log T}} \int_0^T \varphi(t)\, dt > \frac{\theta_0}{\lambda_0}.$$

Then condition (5) *holds.*

Proof. For $x \in \mathbf{R}$,

$$\tilde{g}(t,x) = \frac{g(x)}{\sigma(x)} \cdot \frac{\varphi(t)}{\theta_0} - \frac{1}{2}\sigma'(x)\theta_0 \geq \frac{g(x)}{\sigma(x)} \cdot \frac{\varphi(t)}{\theta_0}$$

whence $u(t) \geq \frac{\lambda_0}{\theta_0}\varphi(t)$ and (5) follows. ■

Next we provide an example of the same kind as in Lemma 5 but with a non-constant function θ.

Example 6. Let $g(x) = \sigma(x) = e^{-x}$, $x \in (-\infty; +\infty)$ and $\varphi(t) = \frac{1}{2} + \cos t$ and $\theta(t) = \frac{1}{t+1}$ for $t \geq 0$. Then

$$\tilde{g}(t,x) = e^{x} + \frac{1}{2}e^{-x}\frac{1}{t+1} + (t+1)\left(\frac{1}{2} + \cos t\right) - 1.$$

Then

$$u(t) = \inf_{x \in \mathbf{R}} \left(e^{x} + \frac{1}{2}e^{-x}\frac{1}{t+1} + (t+1)\left(\frac{1}{2} + \cos t\right) - 1 \right) =$$

$$= \sqrt{\frac{2}{t+1}} + (t+1)\left(\frac{1}{2} + \cos t\right) - 1.$$

Since

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{\sqrt{2T\ln\ln T}} \int_0^T u(t)\,dt = \\
&\quad = \liminf_{T\to\infty} \frac{1}{\sqrt{2T\ln\ln T}} \int_0^T \left(\sqrt{\frac{2}{t+1}} + (t+1)\left(\frac{1}{2} + \cos t\right) - 1 \right) dt = \\
&\quad = \liminf_{T\to\infty} \frac{2\sqrt{2(T+1)} - 2\sqrt{2} + \frac{T^2}{4} - T\left(\frac{1}{2} - \sin T\right) + \sin T + \cos T - 1}{\sqrt{2T\ln\ln T}} = \\
&\quad = +\infty,
\end{aligned}$$

we conclude that (5) holds. Note finally that $v(x) = \inf_{t>0} \tilde{g}(t,x) = -\infty$ and (6) does not apply.

Lemma 7. *Let*

c_1) *the function θ is non-decreasing in $t > 0$;*

c_2) *the derivative $\theta'(t)$ is uniformly bounded in $t > 0$;*

c_3) $\dfrac{\sigma(x)}{x} \to 0$ *and* $\dfrac{g(x)}{\sigma(x)} \to 0$ *as* $x \to -\infty$;

c_4) $\displaystyle\int_{-\infty}^{0} \frac{dy}{\sigma(y)} < \infty.$

Then the first condition in (6) *holds.*

Proof. It is clear that, for $x < 0$,

$$\tilde{g}(t,x) \le \frac{\sup_{t>0} \theta'(t)}{\theta^2(0)} \int_{-\infty}^{0} \frac{dy}{\sigma(y)} + \frac{\varphi(t)}{\theta(0)} \cdot \frac{g(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x)\theta(0),$$

whence

$$\begin{aligned}
v(x) &\le \int_0^x \left(c_1 + c_2 \frac{g(z)}{\sigma(z)} - c_3\sigma'(z) \right) dz = \\
&= x \left(c_1 + \frac{c_2}{x} \int_0^x \frac{g(z)}{\sigma(z)}\, dz - c_3 \frac{\sigma(x) - \sigma(0)}{x} \right)
\end{aligned}$$

with

$$c_1 = \frac{\sup_{t>0}\theta'(t)}{\theta^2(0)} \int\limits_{-\infty}^{0} \frac{dy}{\sigma(y)}, \qquad c_2 = \frac{1}{\theta(0)} \inf_{t>0} \varphi(t), \qquad c_3 = \frac{\theta(0)}{2}.$$

Since $c_1 > 0$, the expression in brackets is positive for large $|x|$, that is $e^{-2v(x)} \geq$ $\geq e^{-\alpha x}$ for large $|x|$ and some $\alpha > 0$. This implies the first condition in (6). ■

Lemma 8. *Let*

d_1) $\theta(t)$ *is a non-decreasing bounded function for* $t > 0$;

d_2) *the derivative* θ' *is a bounded function for* $t > 0$;

d_3) σ *is a regularly varying function at* ∞ *of an index* $0 < \rho < 1$;

d_4) $\dfrac{g(x)}{x} \to 0$ *as* $x \to \infty$.

Then the second condition in (6) *holds.*

Proof. It is clear that, for $x > 0$,

$$\tilde{g}(t,x) \geq -\frac{\sup_{t>0}\theta'(t)}{\theta^2(0)} \int\limits_{0}^{x} \frac{dy}{\sigma(y)} + \frac{\inf_{t>0}\varphi(t)}{\sup_{t>0}\theta(t)} \cdot \frac{g(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x) \sup_{t>0}\theta(t).$$

Therefore, with

$$c_1 = \frac{\sup_{t>0}\theta'(t)}{\theta^2(0)}, \qquad c_2 = \frac{\inf_{t>0}\varphi(t)}{\theta(0)}, \qquad c_3 = \frac{\sup_{t>0}\theta(t)}{2},$$

we have

$$\tilde{g}(t,x) \geq c_1 \int\limits_{0}^{x} \frac{dy}{\sigma(y)} + c_2 \frac{g(x)}{\sigma(x)} - c_3 \sigma'(x).$$

By Karamata's theorem, as $x \to \infty$,

$$\frac{g(x)}{\sigma(x)} = o\left(\int\limits_{0}^{x} \frac{dy}{\sigma(y)} \right), \qquad \sigma'(x) = o\left(\int\limits_{0}^{x} \frac{dy}{\sigma(y)} \right).$$

Without loss of generality we may assume that

$$\tilde{g}(t,x) \geq \alpha \int\limits_{0}^{x} \frac{dy}{\sigma(y)}$$

for all $x > 0$ and some $\alpha > 0$. Then

$$v(x) \geq \alpha \int_0^x \left(\int_0^z \frac{dy}{\sigma(y)} \right) dz$$

for all $x > 0$. The asymptotics of the inner integral on the right hand side is given by $\dfrac{z}{\sigma(z)}$ as $z \to \infty$ by Karamata's theorem, thus the asymptotics of the whole right hand side is $\dfrac{x^2}{\sigma(x)}$ as $x \to \infty$. Therefore the second condition in (6) follows from

$$\int_0^\infty e^{-\delta x^2/\sigma(x)}\, dx < \infty \qquad \text{for all } \delta > 0. \qquad \blacksquare$$

Combining Lemmas 4–8 we obtain various cases where only one of conditions (5)–(6) holds.

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