REGULARITY OF FUNCTIONAL EQUATIONS WITH FEW VARIABLES

Antal Járai (Budapest, Hungary)

Dedicated to Professors Zoltán Daróczy and Imre Kátai to their 75th birthday

Communicated by Ferenc Schipp

(Received May 31, 2013; accepted September 26, 2013)

Abstract. Under certain conditions weak regularity property of solutions f of the functional equation

$$f(x) = h\Big(x, y, f\big(g_1(x,y)\big), \dots, f\big(g_n(x,y)\big)\Big),$$
$$(x,y) \in D \subset \mathbb{R}^r \times \mathbb{R}^s,$$

implies that they are $\mathcal{C}^\infty$, even if $1 \le s \le r$. Corollaries which are easy to apply and examples will be given.

1. Introduction

General regularity theorems for the solutions f of the functional equation

$$f(x) = h\Big(x, y, f\big(g_1(x,y)\big), \dots, f\big(g_n(x,y)\big)\Big), \qquad (x,y) \in D \subset \mathbb{R}^r \times \mathbb{R}^s, \tag{1}$$

Key words and phrases: functional equations, regularity.
2010 Mathematics Subject Classification: Primary: 39B05. Secondary: 26B05, 28C15, 54E52, 60E05.
This work is supported by OTKA NK81402 grant.
https://doi.org/10.71352/ac.40.337

have been proven in [4], [5], [6], [7], [10], [11], [14], [15]. Roughly speaking, my older results prove regularity of an r-place function f that is the solution of a functional equation only if there are at least $2r$ variables in the functional equation, because of the "strong rank condition" that rank $\dfrac{\partial g_i}{\partial y} = r$. Simple examples as the Sincov equation

$$f(x_1, x_2) = f(x_1, y) + f(y, x_2), \qquad x_1, x_2, y \in \mathbb{R}$$

show that the "strong rank condition" cannot simply be omitted and the number of variables cannot be reduced without introducing some additional conditions (see below).

The first general regularity results which overcome this difficulty were given by Światak (see [18]). She applied her distribution method to generalizations of the mean value equation. First she investigated the generalized mean value equation

$$\sum_{i=1}^{n} h_i(x,y) f\big(x + g_i(y)\big) = h_0(x,y), \quad x \in \mathbb{R}^r,\ y \in Y \subset \mathbb{R}^s,$$

and finally (in essence) the equation

$$\sum_{i=1}^{n} h_i(x,y) f\big(g_i(x,y)\big) = h_0(x,y), \quad x \in \mathbb{R}^r,\ y \in Y \subset \mathbb{R}^s.$$

with unknown function f and proved that continuous solutions are in $\mathcal{C}^\infty$. This equation is "almost linear", so, formally, it is much less general than equation (1). However Światak's theorems can be applied even if the rank of $\dfrac{\partial g_i}{\partial y}$ is much less than the dimension of the domain of the unknown function f. The essence of her method consists of applying (in the distribution sense) a linear differential operator in y and substituting $y = 0$ to obtain a hypoelliptic partial differential equation. We have to assume that $g_i(x, 0) \equiv x$ for each i, a very strong condition. The method of Światak can be applied even if f is an r-place function and there are only $r + 1$ variables; this is the minimal number of variables if the equation is not a functional equation in a "single variable". So the results of Światak suggest that the "strong rank condition" is too strong, and the results can be extended for other cases, at least if we add some further conditions.

Regularity results for functional equations with "few" variables, i. e., with an r-place unknown function but with less than $2r$ (but more than r) variables have been proved in papers [8], [9] and [12]. These can be applied to prove that $f \in \mathcal{C}^{q-1} \implies f \in \mathcal{C}^q$ for $q = 0, 1, \dots$ where $\mathcal{C}^{-1}$ is understood as the class of measurable functions or as the class of functions having the property of Baire.

Most of the results can be applied for equation (1) except results having the type $f \in \mathcal{C}^0 \implies f \in \mathcal{C}^1$ which are proved only for special h's linear in the f terms.

The proofs use special function spaces, which — roughly speaking — interpolate between measurability and continuity, between Baire property and continuity and between continuity and continuous differentiability. Although it seems to me that in some complicated cases (see, for example, [13]) the general theory cannot be avoided, reading the 70 pages of the general theory given in papers [8], [9] and [12] is not simple. Therefore it seems to be important to formulate and prove corollaries not referring to any new function spaces but only to simple conditions usual in analysis. This has been done first in my talk at the 42th International Symposium on Functional Equations, Opava, 2004. These corollaries have been applied by István Kovácsvölgyi in [17]. The present paper is an extended form of my talk, containing the proofs, too.

Although the general theory of [8], [9] and [12] can be applied for systems of functional equations with several unknown functions, here only less general but easy to apply corollaries for the case of one unknown function will be given. For a general "transfer principle" to reduce functional equations with several unknown functions to functional equations with one unknown function, see my book [13], 1.23. For simplicity I will refer to the material in this book without mentioning [13]; the whole material of [8], [9] and [12] is contained in there. See the historical details of regularity theory of functional equations and the connections to the fifth problem of Hilbert also there or in the papers [1], [2], [3] and [16].

2. Function spaces interpolating between $\mathcal{C}^0$ and $\mathcal{C}^1$

For convenience we introduce the following notation. Suppose that X is an open subset of $\mathbb{R}^n$, Y is a Banach space, $0 \le k \le n$ an integer, $\mathcal{W}$ is a class of functions w mapping some product $U \times P$ (depending on w) into $\mathbb{R}$, where U is an open subset of $\mathbb{R}^k$ and P is an open subset of some Euclidean space, Φ is a class of functions φ mapping some product $U \times P$ into X where, again U is an open subset of $\mathbb{R}^k$ and P is an open subset of some Euclidean space, and $\mathcal{G}$ is a class of functions mapping some open subset P of some Euclidean space into Y. Let

$$\mathcal{F}_k(X, Y, \mathcal{W}, \Phi; \mathcal{G})$$

denote the class of all continuous functions $f : X \to Y$ for which whenever $w \in \mathcal{W}$ and $\varphi \in \Phi$ have the same domain $U \times P$, the parametric integral

$$p \mapsto \int_U w(u,p) f\big(\varphi(u,p)\big)\, du$$

exists for each $p \in P$ and is in the class $\mathcal{G}$. (Integration is with respect to k-dimensional Lebesgue measure.)

The function classes $\mathcal{W}$, Φ, and $\mathcal{G}$ will be defined via smoothness conditions. Let $0 \le m \le \infty$ and let $\mathcal{C}^m$ denote the class of all functions which are defined on some open subset of some Euclidean space, take values in a Banach space, and are m times continuously differentiable. Let $\mathcal{K}^m$ be the subclass of $\mathcal{C}^m$ consisting of functions that have compact support. Let $\mathcal{I}^m$ denote the class of functions $\varphi \in \mathcal{C}^m$ which map some Cartesian product $U \times P$ of open subsets of Euclidean spaces into a Euclidean space so that $u \mapsto \varphi(u,p)$ is an immersion for each $p \in P$. (Recall, that a $\mathcal{C}^1$ mapping of U into X is an immersion if and only if its derivative is an injective linear mapping for each point of U.) Similarly, let $\mathcal{E}^m$ denote the class of those functions $\varphi \in \mathcal{C}^m$ which map some Cartesian product $U \times P$ of open subsets of Euclidean spaces into a Euclidean space so that $u \mapsto \varphi(u,p)$ is an embedding (i.e., an immersion which is a homeomorphism) for each $p \in P$.

Corollary 1. *Let $X \subset \mathbb{R}^r$ be an open set and $f : X \to \mathbb{R}^m$ be a function. Suppose that*

(LFES) *the linear functional equations*

$$f(x) = h_{i,0}(x,y) + \sum_{j=1}^{n_i} h_{i,j}(x,y) f\big(g_{i,j}(x,y)\big)$$

are satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set);

(S2) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, the functions $h_{i,0} : D_i \to \mathbb{R}^m$ and $h_{i,j} : D_i \to \mathbb{R}$ are in $\mathcal{C}^1$, the functions $g_{i,j} : D_i \to$ $\to X$ are in $\mathcal{C}^2$;*

(D) *for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exists an $i \in I$ and a y such that $(x,y) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^0$ implies $f \in \mathcal{C}^1$.

Observe, that if $\dim(Y_i) > 0$, then the dimension condition (D) is satisfied "in general", because "in general"

$$\det\left(\frac{\partial g_{i,j}}{\partial x}(x,y)\right) \neq 0, \quad \dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V)\right) = \dim(V),$$

$$\operatorname{rank}\left(\frac{\partial g_{i,j}}{\partial y}(x,y)\right) = \min\bigl\{r, \dim(Y_i)\bigr\} > 0.$$

Proof. The case $r = 0$ is obvious. Otherwise, we will prove that if

$$f \in \mathcal{F}_{k+1}(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1),$$

then

$$f \in \mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$$

for all $0 \leq k < r$; this proves the statement, because by 21.4 if $f \in \mathcal{C}^0$ then $f \in \mathcal{F}_r(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ and from 21.3 it follows that if $f \in \mathcal{F}_0(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ then $f \in \mathcal{C}^1$.

To prove that

$$f \in \mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$$

we have to prove that if $U \subset \mathbb{R}^k$ is open, P is an open subset of some Euclidean space, $\varphi : U \times P \to X$ is a $\mathcal{I}^2$ function, then for any function $w : U \times P \to X$ belonging to $\mathcal{K}^1$ the parametric integral

$$p \mapsto \int_U w(u,p) f\bigl(\varphi(u,p)\bigr)\, du$$

is continuously differentiable. Let us fix U, P and φ. By the locality principle mentioned in the definition 21.1 it is enough to prove that for any $u_0 \in U$ and $p_0 \in P$ there exists an open neighbourhood U_0 of u_0 and an open neighbourhood P_0 of p_0 such that this parametric integral is continuous whenever the support of $w : U \times P \to \mathbb{R}$ is contained in $U_0 \times P_0$. To prove this we shall use theorem 21.8. Let us fix a $u_0 \in U$ and a $p_0 \in P$.

First, let us observe that using the notation $x_0 = \varphi(u_0, p_0)$ and $V_0 = \operatorname{rng} \dfrac{\partial \varphi}{\partial u}(u_0, p_0)$ by the dimension condition (D) there exist an index $i \in I$ and an $y \in Y_i$ such that $(x_0, y) \in D_i$ and

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x_0,y)(V_0) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x_0,y)\right) > \dim(V_0) = k$$

whenever $1 \leq j \leq n_i$. Let us fix this i, x_0 and y. By the $\mathcal{C}^2$-smoothness condition (S2) and because of $\varphi \in \mathcal{I}^2$ there exist neighbourhoods U_0 of u_0 and

P_0 of p_0 such that using the notation $x = \varphi(u,p)$ and $V = \operatorname{rng} \dfrac{\partial\varphi}{\partial u}(u,p)$ we have $(x,y) \in D_i$ and

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V) = k.$$

Let us choose an arbitrary weight function $w : U \times P \to \mathbb{R}$ from $\mathcal{K}^1$ having its support contained in $U_0 \times P_0$. Let $p_0' \in P_0$ be arbitrary. From the linear functional equation system (LFES) let us choose the equation indexed by i and let us apply theorem 21.8 with the following substitutions: Let $f_j = f$ for $j = 1, 2, \ldots, n_i$ and let $g_j = g_{i,j}$ for $j = 1, 2, \ldots, n_i$. For $-m \le j < 0$ let f_j be a constant function equal to the j'th vector from an arbitrary (but fixed) base of $\mathbb{R}^m$, let h_j be equal to the corresponding coordinate of $h_{i,0}$; for $-m \le j < 0$ let g_j be any of the functions $g_{i,t}$, $t = 1, 2, \ldots, n_i$. Let $k_j = k + 1$ for all j. Applying theorem 21.8 we obtain that for any $p_0' \in P_0$ the parametric integral above is continuous at p_0'. This means that the parametric integral is continuous on P. This completes the proof. ■

Corollary 2. *Let $X \subset \mathbb{R}^r$ be an open set and $f : X \to \mathbb{R}^m$ a function. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x,y)\big), \ldots, f\big(g_{i,n_i}(x,y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x,y)\big), \ldots, f\big(g_{i,n_i}(x,y)\big)\Big)$$

is satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set);

(SI) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$, all the partial derivatives $\partial_t^{\alpha_0}\partial_{z_1}^{\alpha_1}\ldots\partial_{z_{n_i}}^{\alpha_{n_i}} h_i$ of the functions $h_i : W_i \to \mathbb{R}^m$ are continuously differentiable, the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^\infty$;*

(D) *for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exist an $i \in I$ and a y such that $(x,y) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^1$ implies $f \in \mathcal{C}^\infty$.

Proof. The proof of this corollary is very similar to the proof of the previous corollary. The only important difference is that we have to apply 21.9 instead of 21.8 to prove that if all order ℓ partial derivatives of f are in

$$\mathcal{F}_{k+1}(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1),$$

then they are also in

$$\mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$$

for all $0 \le k < r$; this proves the statement, because by 21.4 if $f \in \mathcal{C}^\ell$ then all of its order ℓ partial derivatives are in $\mathcal{F}_r(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ and from 21.3 it follows that if all partial derivatives of order ℓ of f are in $\mathcal{F}_0(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ then $f \in \mathcal{C}^{\ell+1}$. Unfortunately, Theorem 21.9 is not formulated in a general enough form: We need it in the case when function h is defined on an open subset W of $D \times Z_1 \times \cdots \times Z_n \to Z$ (and not the whole of this set) such that

$$\Big(x, y, f_1\big(g_1(x,y)\big), \dots, f_n\big(g_n(x,y)\big)\Big) \in W$$

and

$$f(x) = h\Big(x, y, f_1\big(g_1(x,y)\big), \dots, f_n\big(g_n(x,y)\big)\Big)$$

whenever $(x, y) \in D$. Fortunately, the proof of the original version works word-by-word in this somewhat more general case, too.

Now Corollary 2 can be proved the same way as we proved Corollary 1 above. ■

3. Critical subspaces

Let $X \subset \mathbb{R}^r$ be an open set and for each $i \in I$ let $D_i \subset X \times Y_i$ be an open set, where Y_i is a Euclidean space and let the functions $g_{i,j} : D_i \to X$, $1 \le j \le n_i$ be in $\mathcal{C}^1$. Suppose that for each $x \in X$ there is an $i \in I$ and a y such that $(x, y) \in D$. For a proper linear subspace V of $\mathbb{R}^r$ we will say that it is a critical subspace at x if for each $i \in I$ and for each y for which $(x, y) \in D_i$ we have

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) \le \dim(V)$$

for some $1 \le j \le n_i$.

It is clear, that the dimension condition (D) can be formulated in a way that there is no critical subspace for any $x \in X$. It is also clear that if a linear subspace V is critical then any proper linear subspace of $\mathbb{R}^r$ containing V is critical too. Hence it is enough to consider minimal critical linear subspaces.

Example 1. Let us consider equation

$$f(x+t,y)+f(x,y+t^2)=2f(x,y), \quad x,y,t\in\mathbb{R}. \tag{2}$$

For this equation a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V)+\big\{(u,0):u\in\mathbb{R}\big\}\Big)\le 1,$$

i. e., $V=V_1=\big\{(u,0):u\in\mathbb{R}\big\}$, or

$$\dim\Big(\mathbb{I}(V)+\big\{(2tu,0):u\in\mathbb{R}\big\}\Big)\le 1$$

for all $t\in\mathbb{R}$, i. e., $V=V_2=\big\{(0,u):u\in\mathbb{R}\big\}$.

Substituting $x+t$ with x, for the new equation

$$f(x,y)=2f(x-t,y)-f(x-t,y+t^2), \quad x,y,t\in\mathbb{R}$$

a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V)+\big\{(-u,2tu):u\in\mathbb{R}\big\}\Big)\le 1$$

for all $t\in\mathbb{R}$ or

$$\dim\Big(\mathbb{I}(V)+\big\{(-u,0):u\in\mathbb{R}\big\}\Big)\le 1.$$

Hence for these two equations only V_2 remains critical.

Substituting $x+t^2$ with x, for the new equation

$$f(x,y)=2f(x,y-t^2)-f(x+t,y-t^2), \quad x,y,t\in\mathbb{R}$$

a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V)+\big\{(-u,-2tu):u\in\mathbb{R}\big\}\Big)\le 1$$

or

$$\dim\Big(\mathbb{I}(V)+\big\{(0,-2tu):u\in\mathbb{R}\big\}\Big)\le 1$$

for all $t\in\mathbb{R}$. Hence for all three equations, i. e., for the system of these three equations no critical subspace remains.

Example 2. Sincov equation

$$f(x_1,x_2)=f(x_1,y)+f(y,x_2), \qquad x_1,x_2,y\in\mathbb{R}$$

does not belong to the class of generalized mean value equations, but if we try to apply our corollaries, we find the critical subspaces V_1 and V_2 as above, but in this case we cannot remove these with substitutions. Let us observe the connection of the critical subspaces with the general solution $f(x_1, x_2) = g(x_1) - g(x_2)$.

Example 3. In her paper H. Światak illutrates her regularity results with the following examples:

$$f(x+t,y)+f(x,y+t^2)=2f(x,y), \tag{3}$$

$$f(x+t,y)+f(x,y+t^2)+f(x,y-t^2)=3f(x,y), \tag{4}$$

$$f(x+t,y)+t^2f(x,y+t)-(1+t^2)f(x,y)=t(2x+t), \tag{5}$$

$$\begin{aligned} f(x+t,y)+t^2f(x,y+t)+(t-t^2-1)f(x,y) &= \\ = t(x^2+2x+t), \end{aligned} \tag{6}$$

$$\begin{aligned} f(x+t,y)+t^2f(x,y+t)+(t-t^2-1)f(x,y) &= \\ = t(x^2+2x+t+t^3+2yt+y^2), \end{aligned} \tag{7}$$

$$\begin{aligned} f(x+t,y)+t^2f(x,y+t)-(1+t^2)f(x,y) &= \\ = t(2x+t+t^3+2yt^2), \end{aligned} \tag{8}$$

$$\begin{aligned} f(x+t,y)+f(x-t,y)+f(x,y+t^2)+f(x,y-t^2) &= \\ = 4f(x,y), \end{aligned} \tag{9}$$

$$\begin{aligned} f(x+t,y)+f(x-t,y)+f(x,y+t^3)+f(x,y-t^3) &= \\ = 4f(x,y), \end{aligned} \tag{10}$$

where the unknown function is $f : \mathbb{R}^2 \to \mathbb{R}$.

All these equations can be easily handled by using Corollary 1 and Corollary 2, similarly to the equation of Example 1, which is the same as equation (3).

Differentiation with respect to t and putting $t = 0$ yields

$$\frac{\partial^4 f}{\partial x^4}(x,y)+C\frac{\partial^2 f}{\partial y^2}(x,y)=C^* \tag{11}$$

for equations (3)–(9), and

$$\frac{\partial^6 f}{\partial x^6}(x,y) + C\frac{\partial^2 f}{\partial y^2}(x,y) = C^* \tag{12}$$

for equation (10), where $C > 0$ and $C^* \in \mathbb{R}$. It is easy to verify that equations (11) and (12) are hypoelliptic and therefore we may apply one of the theorems of Światak to obtain that all the continuous solutions of equations (3)–(10) are in $\mathcal{C}^\infty$.

Example 4. Of course, it is very easy to give examples which our theory can be applied to but the theory of Światak cannot. We will give an example which Światak's theory can be applied to but ours cannot.

To the equation

$$\begin{aligned}0 = 4f(x,y) &+ f(x+2t,y) + f(x,y+2t)+\\ &+ f(x-2t,y) + f(x,y-2t)+\\ &+ 2f(x+t,y+t) + 2f(x-t,y+t)+\\ &+ 2f(x-t,y-t) + 2f(x+t,y-t),\end{aligned} \tag{13}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the unknown function, Światak's theory can be applied, and it implies that all continuous solutions are in $\mathcal{C}^\infty$. To solve this equation, let us differentiate with respect to x and y, and put $t = 0$ to obtain $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, respectively, and conlude that $f \equiv 0$ is the only solution.

Easy calculations show that for the equation as it stands critical subspaces are

$$V_1 = \{(u,0) : u \in \mathbb{R}\},$$
$$V_2 = \{(0,u) : u \in \mathbb{R}\},$$
$$V_+ = \{(u,u) : u \in \mathbb{R}\}$$

and

$$V_- = \{(u,-u) : u \in \mathbb{R}\}.$$

Using appropriate substitutions, V_1 and V_2 can be removed but V_+ and V_- cannot and this makes it impossible to apply our theory.

To understand the reason for this phenomenon let us consider the equation

$$\begin{aligned}0 =4f(x,y) &+ f(x+2t,y) + f(x,y+2t)+\\ &+ f(x-2t,y) + f(x,y-2t)-\\ &- 2f(x+t,y+t) - 2f(x-t,y+t)-\\ &- 2f(x-t,y-t) - 2f(x+t,y-t),\end{aligned} \tag{14}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the unknown function. This equation differs from equation (13) only in the signs of some of the terms. Our general theory is not capable of taking care of such minor differences: If it was possible to apply it to equation (13) then it would be possible to apply it to equation (14) as well. But any function $f(x, y) = \varphi(x + y) + \psi(x - y)$ is a solution of equation (14), so no regularity phenomenon holds for equation (14). Observe the role of the critical subspaces.

4. Function spaces interpolating between $\mathcal{C}^{-1}$ and $\mathcal{C}^0$

Let X be a set, Y a metric space, and $f : X \to Y$ a function. Let U be a Hausdorff space with the Radon (outer) measure μ, and P a topological space, the "parameter space" with a given point $p_0 \in P$. Let φ be a function from $U \times P$ into X. We will think of φ as a surface $\varphi_p : u \mapsto \varphi(u, p)$ for each p, depending on the parameter p.

Riesz's theorem suggests the following condition:

(R) *For each sequence $p_m \to p_0$ there exists a subsequence p_{m_i} such that for almost all $u \in U$ we have*

$$f\bigl(\varphi(u, p_{m_i})\bigr) \to f\bigl(\varphi(u, p_0)\bigr).$$

We need some kind of a measurability condition:

(M) $u \mapsto f\bigl(\varphi(u, p_0)\bigr)$ *is μ measurable.*

The theorem of Piccard suggests that the following condition is connected with the Baire property:

(S) *For each sequence $p_m \to p_0$ we have $f\bigl(\varphi(u, p_m)\bigr) \to f\bigl(\varphi(u, p_0)\bigr)$ except for a set of first category of points $u \in U$.*

For our investigations we also need the following property:

(B) $u \mapsto f\bigl(\varphi(u, p_0)\bigr)$ *has the Baire property.*

Let X be an open subset of $\mathbb{R}^n$ and $0 \le k \le n$. The class of all functions f for which condition (R) $\big[$(M), (S), (B)$\big]$ is satisfied whenever U is an open subset of $\mathbb{R}^k$, $\mu = \lambda^k$, P is an open subset of some Euclidean space, $p_0 \in P$, and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$ function for which φ_p is an immersion of U into X for each $p \in P$, will be denoted by $\mathcal{R}_k(X, Y)$ $\big[\mathcal{M}_k(X, Y), \mathcal{S}_k(X, Y), \mathcal{B}_k(X, Y)\big]$. (For $k = 0$, take $\mathbb{R}^0 = \{0\}$ and $\lambda^0\big(\{0\}\big) = 1$, i. e., λ^0 is the counting measure on $\mathbb{R}^0$. A function $\varphi : \{0\} \times P \to X$ is a $\mathcal{C}^1$ function if and only if $p \mapsto \varphi(0, p)$ is a $\mathcal{C}^1$ function. Any function mapping a subset of $\mathbb{R}^0$, i. e., $\emptyset$ or $\{0\}$ into X is considered an immersion.)

Corollary 3. *Let $X \subset \mathbb{R}^r$ be an open set, $f : X \to \mathbb{R}^m$ a function and let $K \subset \{0, 1, \dots, r\}$ contain 0 and r. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x, y)\big), \dots, f\big(g_{i,n_i}(x, y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x, y)\big), \dots, f\big(g_{i,n_i}(x, y)\big)\Big)$$

is satisfied whenever $i \in I$ and $(x, y) \in D_i$ (here I is an index set);

(S1) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$ for all $y \in Y_i$, h_i is continuous in the other variables and the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^1$;*

(CD) *for each $x_0 \in X$ and for each proper linear subspace V_0 of $\mathbb{R}^r$ with $k_0 = \dim(V_0) \in K$ there exist an $i \in I$ and a y_0 such that $(x_0, y_0) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x, y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x, y)\right)$$

is the same constant k in K and greater than k_0 for $1 \le j \le n_i$ whenever x is close enough to x_0, y is close enough to y_0 and V is close enough to V_0 in the Grassmann manifold $\mathbb{G}(r, k)$.

Then $f \in \mathcal{C}^{-1}$ implies $f \in \mathcal{C}^0$.

Here again if $\dim(Y_i) > 0$, the "constant dimension" condition (CD) is satisfied "in general" but not if there is a critical subspace for some $x \in X$.

Proof. First let us consider the case $\mathcal{C}^{-1}$ is the class of measurable functions and $f \in \mathcal{C}^{-1}$. Then by 19.6 we have

$$f \in \mathcal{M}_r(X, \mathbb{R}^m) \cap \mathcal{R}_r(X, \mathbb{R}^m).$$

We have to prove that

$$f \in \mathcal{M}_0(X, \mathbb{R}^m) \cap \mathcal{R}_0(X, \mathbb{R}^m),$$

because by 19.4 this intersection is equal to $\mathcal{C}^0(X, \mathbb{R}^m)$. The case $r = 0$ is obvious. Otherwise we shall prove that if $k_0 \in K$ and

$$f \in \mathcal{M}_k(X, \mathbb{R}^m) \cap \mathcal{R}_k(X, \mathbb{R}^m)$$

for each $k > k_0$ for which $k \in K$, then

$$f \in \mathcal{M}_{k_0}(X, \mathbb{R}^m) \cap \mathcal{R}_{k_0}(X, \mathbb{R}^m).$$

To prove this we have to prove that if $U \subset \mathbb{R}^{k_0}$ is open, P is an open subset of some Euclidean space, $\varphi : U \times P \to X$ is a $\mathcal{C}^1$ function for which $u \mapsto \varphi(u, p)$ is an immersion for each $p \in P$, then condition (R) and (M) are satisfied. Let us fix U, P and φ. By the locality principle mentioned in definition 19.1 it is enough to prove that for any $u_0 \in U$ and $p_0 \in P$ there exists an open neighbourhood U_0 of u_0 and an open neighbourhood P_0 of p_0 such that $\varphi|_{U_0 \times P_0}$ satisfies (R) and (M). To prove this we shall use theorem 19.13. Let us fix a $u_0 \in U$ and a $p_0 \in P$.

Using the notation $x_0 = \varphi(u_0, p_0)$ and $V_0 = \operatorname{rng} \dfrac{\partial \varphi}{\partial u}(u_0, p_0)$ let us choose an index $i \in I$ and an $y_0 \in Y_i$ such that $(x_0, y_0) \in D_i$ and for

$$k_j = \dim \left(\frac{\partial g_{i,j}}{\partial x}(x_0, y)(V_0) + \operatorname{rng} \frac{\partial g_{i,j}}{\partial y}(x_0, y) \right) > \dim(V_0) = k_0$$

we have $k_j \in K$ and the constant dimension condition is satisfied at x_0, y_0 and V_0 whenever $1 \le j \le n_i$. Let us fix this i, x_0 and y_0. By the $\mathcal{C}^1$-smoothness condition (S1) and since $\varphi \in \mathcal{I}^1$ there exist neighbourhoods U_0 of u_0 and P_0 of p_0 such that using the notation $x = \varphi(u, p)$ and $V = \operatorname{rng} \dfrac{\partial \varphi}{\partial u}(u, p)$ we have $(x, y) \in D_i$ and

$$k_j = \dim \left(\frac{\partial g_{i,j}}{\partial x}(x, y)(V) + \operatorname{rng} \frac{\partial g_{i,j}}{\partial y}(x, y) \right) > \dim(V) = k_0.$$

From the functional equation system (FES) let us choose the equation indexed by i and let us apply theorem 19.13 with the following substitutions: Let $f_j = f$ for $j = 1, 2, \ldots, n_i$ and let $g_j = g_{i,j}$ for $j = 1, 2, \ldots, n_i$. Applying theorem 19.13 we obtain that conditions (R) and (M) are satisfied for f, U_0, P_0, p_0, $\varphi|_{U_0 \times P_0}$ and λ^k. This completes the proof for the measurability case.

In the case when $\mathcal{C}^{-1}$ is understood to be the class of functions having the property of Baire, the proof is completely analogous but we use Theorem 20.10 instead of 19.13. ■

Example 5. All equations in Example 3 can be easily handled by using Corollary 3.

References

[1] **Aczél, J.,** Some unsolved problems in the theory of functional equations II, *Aequationes Math.*, **26** (1984), 255–260.

[2] **Aczél, J.,** The state of the second part of Hilbert's fifth problem, *Bull. Amer. Math. Soc. (N.S.)*, **20** (1989), 153–163.

[3] **Hilbert, D.,** *Gesammelte Abhandlungen Band III*, Springer Verlag, Berlin–Heidelberg–New York, 1970.

[4] **Járai, A.,** On regular solutions of functional equations, *Aequationes Math.*, **30** (1986), 21–54.

[5] **Járai, A.,** Differentiation of parametric integrals and regularity of functional equations, *Grazer Math. Ber.*, **315** (1991), 45–50.

[6] **Járai, A.,** On Lipschitz property of continuous solutions of functional equations, *Aequationes Math.*, **47** (1994), 69–78.

[7] **Járai, A.,** *Regularity Properties of Functional Equations*, Leaflets in Mathematics, Janus Pannonius University, Pécs, 1996.

[8] **Járai, A.,** Baire property implies continuity for solutions of functional equations — even with few variables, *Acta Sci. Math. Szeged*, **66** (2000), 579–601.

[9] **Járai, A.,** Continuity implies differentiability for solutions of functional equations — even with few variables, *Acta Sci. Math. Szeged*, **67**, (2001), 719–734.

[10] **Járai, A.,** Solutions of functional equations of bounded variations, *Aequationes Math.*, **61** (2001), 205–211.

[11] **Járai, A.,** Regularity properties of functional equations on manifolds, *Aequationes Math.*, **64** (2002), 248–262.

[12] **Járai, A.,** Measurability implies continuity for solutions of functional equations — even with few variables, *Aequationes Math.*, **65** (2003), 236–266.

[13] **Járai, A.,** *Regularity Properties of Functional Equations in Several Variables*, Springer, 2005.

[14] **Járai, A.,** On the measurable solutions of a functional equation, *Aequationes Math.*, **80** (2010), 131–139.

[15] **Járai, A. and W. Sander,** A regularity theorem in information theory, *Publ. Math. Debrecen*, **50** (1997), 339–357.

[16] **Járai, A. and L. Székelyhidi,** Regularization and general methods in the theory of functional equations. Survey paper, *Aequationes Math.*, **52** (1996), 10–29.

[17] **Kovácsvölgyi, I.,** Comparison of two regulatory results on functional equations with few variables, *Aequationes Math.*, **81** (2011), 55–64.

[18] **Światak, H.,** The regularity of the locally integrable and continuous solutions of nonlinear functional equations, *Trans. Amer. Math. Soc.*, **221(1)** (9176), 97–118.

A. Járai
Department of Computer Algebra
Eötvös Loránd University
Pázmány P. sétány 1/C.
H-1117 Budapest
Hungary
`ajaraimoon.inf.elte.hu`