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ON SOME MEAN SQUARE ESTIMATES FOR THE ZETA-FUNCTION IN SHORT INTERVALS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion their 75th birthday

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Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2}+it)|$. If $E^*(t) = E(t) - 2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - -\frac{1}{2}\Delta(4x)$ and we set $\int_0^T E^*(t)\,\mathrm{d}t = 3\pi T/4 + R(T)$, then we obtain

$$\int_T^{T+H} (E^*(t))^2\,\mathrm{d}t \gg HT^{1/3}\log^3 T$$

and

$$HT\log^3 T \ll \int_T^{T+H} R^2(t)\,\mathrm{d}t \ll HT\log^3 T,$$

for $T^{2/3+\varepsilon} \leqslant H \leqslant T$.

1. Introduction and statement of results

This paper is the continuation of the author's works [6], [7], where the analogy between the Riemann zeta-function $\zeta(s)$ and the divisor problem was

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investigated. As usual, let the error term in the classical Dirichlet divisor problem be

$$\Delta(x) = \sum_{n\leqslant x} d(n) - x(\log x + 2\gamma - 1), \tag{1.1}$$

and

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,{\rm d}t - T\left(\log\bigl(\frac{T}{2\pi}\bigr) + 2\gamma - 1\right), \tag{1.2}$$

where $d(n)$ is the number of divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. In view of F.V. Atkinson's classical explicit formula for $E(T)$ (see [1] and [3, Chapter 15]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, if one wants to stress the analogy between $\zeta^2(s)$ and the divisor function, then instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [8], [9] and T. Meurman [10]), where

$$\begin{aligned}\Delta^*(x) :&= -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x) = \\ &= \tfrac{1}{2}\sum_{n\leqslant 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1),\end{aligned} \tag{1.3}$$

since it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$. Namely, M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi\Delta^*\bigl(\frac{t}{2\pi}\bigr),$$

and in particular in [9] he proved that

$$\int_T^{T+H} (E^*(t))^2\,{\rm d}t \ll_\varepsilon HT^{1/3}\log^3 T + T^{1+\varepsilon} \quad (1 \leqslant H \leqslant T). \tag{1.4}$$

Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ (same as $a = O_\varepsilon(b)$) means that the $\ll$–constant depends on ε. The significance of (1.4) is that, in view of (see e.g., [3])

$$\int_0^T (\Delta^*(t))^2\,{\rm d}t \sim AT^{3/2}, \tag{1.5}$$

$$\int_0^T E^2(t)\,\mathrm{d}t \sim BT^{3/2} \quad (A, B > 0, T \to \infty), \tag{1.6}$$

it transpires that $E^*(t)$ is in the mean square sense of a lower order of magnitude than either $\Delta^*(t)$ or $E(t)$.

In [7] the author sharpened (1.4) (in the case when $H = T$) to the asymptotic formula

$$\int_0^T (E^*(t))^2\,\mathrm{d}t \;=\; T^{4/3}P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{1.7}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be evaluated explicitly. This, in particular, shows that (1.4) may be complemented with the lower bound

$$\int_T^{T+H} (E^*(t))^2\,\mathrm{d}t \;\gg\; HT^{1/3}\log^3 T \quad (T^{5/6+\varepsilon} \leqslant H \leqslant T). \tag{1.8}$$

It seems likely that the error term in (1.7) is $O_\varepsilon(T^{1+\varepsilon})$, but this seems difficult to prove.

In [6] the author investigated higher moments of $E^*(t)$, and e.g., in the second part of [6] he proved that

$$\int_0^T (E^*(t))^5\,\mathrm{d}t \;\ll_\varepsilon\; T^{2+\varepsilon}; \tag{1.9}$$

but neither (1.4) nor (1.9) seem to imply each other.

In part III of [6] the error-term function $R(T)$ was introduced by the relation

$$\int_0^T E^*(t)\,\mathrm{d}t = \frac{3\pi}{4}T + R(T). \tag{1.10}$$

It was shown, by using an estimate for two-dimensional exponential sums, that

$$R(T) = O_\varepsilon(T^{593/912+\varepsilon}), \qquad \frac{593}{912} = 0.6502129\ldots. \tag{1.11}$$

It was also proved that

$$\int_0^T R^2(t)\,\mathrm{d}t = T^2 p_3(\log T) + O_\varepsilon(T^{11/6+\varepsilon}), \tag{1.12}$$

where $p_3(y)$ is a cubic polynomial in y with positive leading coefficient, whose all coefficients may be explicitly evaluated, and

$$\int_0^T R^4(t)\,\mathrm{d}t \ll_\varepsilon T^{3+\varepsilon}. \tag{1.13}$$

The asymptotic formula (1.12) bears resemblance to (1.7), and it is proved by a similar technique. The exponents in the error terms are, in both cases, less than the exponent of T in the main term by $1/6$. From (1.7) one obtains that $E^*(T) = \Omega(T^{1/6}(\log T)^{3/2})$, which shows that $E^*(T)$ cannot be too small ($f = \Omega(g)$ means that $f = o(g)$ does not hold). Likewise, (1.12) yields

$$R(T) \;=\; \Omega\Big(T^{1/2}(\log T)^{3/2}\Big). \tag{1.14}$$

It seems plausible that the error term in (1.12) should be $O_\varepsilon(T^{5/3+\varepsilon})$, while (1.14) leads one to suppose that

$$R(T) = O_\varepsilon(T^{1/2+\varepsilon}) \tag{1.15}$$

holds.

The aim of this paper to prove the following results.

Theorem 1. *For $T^{2/3+\varepsilon} \leqslant H \leqslant T$ we have*

$$\int_T^{T+H} (E^*(t))^2\,\mathrm{d}t \;\gg\; HT^{1/3}\log^3 T. \tag{1.16}$$

Note that (1.16) improves the range of H for which (1.8) holds.

Theorem 2. *For $T^{2/3+\varepsilon} \leqslant H \leqslant T$ we have*

$$\int_T^{T+H} R^2(t)\,\mathrm{d}t \;\gg\; HT\log^3 T, \tag{1.17}$$

and, for $T^\varepsilon \leqslant H \leqslant T$,

$$\int_T^{T+H} R^2(t)\,\mathrm{d}t \;\ll_\varepsilon\; HT\log^3 T + T^{5/3+\varepsilon}. \tag{1.18}$$

The range for which (1.17) holds improves on the range for which (1.12) holds.

Corollary. *If $H = T^{2/3+\varepsilon}$, then every interval $[T, T+H]$ $(T \geqslant T_0)$ contains points t_1, t_2 such that, for some positive constants $A, B > 0$,*

$$|E^*(t_1)| > At_1^{1/6}\log^{3/2} t_1, \quad |R(t_2)| > Bt_2^{1/2}\log^{3/2} t_2. \tag{1.19}$$

Note that this result follows from the asymptotic formulas (1.7) and (1.12), but in the poorer range $T^{5/6+\varepsilon} \leqslant H \leqslant T$. It would be interesting to find large positive and large negative values for which the analogues of (1.19) hold. This was done in [4] for $E(t)$ and $\Delta(x)$, where it was shown that there exist two positive constants C, D such that, for $T \geqslant T_1$, every interval $[T, T+C\sqrt{T}]$ contains points t_3, t_4, t_5, t_6 such that

$$E(t_3) > D\sqrt{t_3},\ E(t_4) < -D\sqrt{t_4},\ \Delta(t_5) > D\sqrt{t_5},\ \Delta(t_6) < -D\sqrt{t_6}. \tag{1.20}$$

It would be interesting to obtain the analogue of (1.19) for large positive and negative values of $E^*(t)$ and $R(t)$, like we have it in (1.20) for $E(t)$ and $\Delta(t)$, but this seems difficult.

2. The necessary lemmas

In this section we shall state the lemmas which are necessary for the proof of our theorems. The first two are Atkinson's classical explicit formula for $E(t)$ (see e.g., [2] or [3]) and the Voronoï-type formula for $\Delta^*(x)$, which is the analogue of the classical truncated Voronoï formula for $\Delta(x)$ (see [10]). The third is an asymptotic formula involving $d^2(n)$.

Lemma 1. *Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then*

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{2.1}$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4}\sum_{n\leqslant N}(-1)^n d(n) n^{-3/4} e(T,n)\cos(f(T,n)), \tag{2.2}$$

$$\Sigma_2(T) = -2\sum_{n\leqslant N'}\frac{d(n)}{n^{1/2}(\log T/(2\pi n))}\cos\left(T\log\Big(\frac{T}{2\pi n}\Big) - T + \frac{1}{4}\pi\right), \tag{2.3}$$

with

$$f(T,n) = 2T\operatorname{arsinh}\left(\sqrt{\pi n/(2T)}\,\right) + \sqrt{2\pi nT + \pi^2 n^2} - \tfrac{1}{4}\pi = \tag{2.4}$$
$$= -\tfrac{1}{4}\pi + 2\sqrt{2\pi nT} + \tfrac{1}{6}\sqrt{2\pi^3}n^{3/2}T^{-1/2} +$$
$$+ a_5 n^{5/2}T^{-3/2} + a_7 n^{7/2}T^{-5/2} + \dots\,,$$

$$e(T,n) = (1+\pi n/(2T))^{-1/4}\Big\{(2T/\pi n)^{1/2}\operatorname{arsinh}\Big(\sqrt{\pi n/(2T)}\,\Big)\Big\}^{-1} = \tag{2.5}$$
$$= 1 + O(n/T) \qquad (1 \leqslant n < T),$$

and $\operatorname{arsinh} x = \log(x + \sqrt{1+x^2}\,)$.

Lemma 2. (see [3, Chapter 15]). *We have, for* $1 \ll N \ll x$,

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\leqslant N}(-1)^n d(n)n^{-\frac34}\cos(4\pi\sqrt{nx} - \tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}). \tag{2.6}$$

Lemma 3. *For* $a > -\frac12$ *a constant we have*

$$\sum_{n\leqslant x} d^2(n)n^a = x^{a+1}P_3(\log x; a) + O_\varepsilon(x^{a+1/2+\varepsilon}), \tag{2.7}$$

where $P_3(y;a)$ *is a polynomial of degree three in* y *whose coefficients depend on* a, *and whose leading coefficient equals* $1/(\pi^2(a+1))$. *All the coefficients of* $P_3(y;a)$ *may be explicitly evaluated.*

This is a standard result, for a proof see e.g., Lemma 3 of [7].

The next lemma brings forth a formula for $\int_0^T E(t)\,\mathrm{d}t$, which is closely related to F.V. Atkinson's classical explicit formula for $E(T)$ (see [1] or e.g., Chapter 15 of [3] or Chapter 2 of [5]). This is due to J.L. Hafner and the author [2] (see also Chapter 3 of [5]).

LEMMA 4. *We have*

$$\int_0^T E(t)\,\mathrm{d}t = \pi T + \frac12\Big(\frac{2T}{\pi}\Big)^{3/4}\sum_{n\leqslant T}(-1)^n d(n)n^{-5/4}e_2(T,n)\sin f(T,n) - \tag{2.8}$$
$$-2\sum_{n\leqslant c_0T} d(n)n^{-1/2}\Big(\log\frac{T}{2\pi n}\Big)^{-2}\sin\Big(T\log\Big(\frac{T}{2\pi n}\Big) - T + \frac14\pi\Big) +$$
$$+O(T^{1/4}),$$

where $c_0 = \frac{1}{2\pi} + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2\pi}}$, *and for* $1 \leqslant n \ll T$,

$$\begin{aligned} e_2(T,n) &= \Bigl(1+\frac{\pi n}{T}\Bigr)^{-1/4}\left\{\Bigl(\frac{2T}{\pi n}\Bigr)^{1/2}\operatorname{ar\,sinh}\Bigl(\frac{\pi n}{2T}\Bigr)^{1/2}\right\}^{-1/2} = \\ &= 1 + b_1\frac{n}{T} + b_2\Bigl(\frac{n}{T}\Bigr)^2 + \dots, \\ f(T,n) &= 2T\operatorname{ar\,sinh}\Bigl(\sqrt{\pi n/(2T)}\Bigr) + \sqrt{2\pi nT + \pi^2 n^2} - \tfrac14\pi = \\ &= -\tfrac14\pi + 2\sqrt{2\pi nT} + a_3 n^{3/2}T^{-1/2} + \\ &+ a_5 n^{5/2}T^{-3/2} + a_7 n^{7/2}T^{-5/2} + \dots . \end{aligned} \tag{2.9}$$

We also need a formula for the integral of $\Delta^*(x)$. From a classical result of G.F. Voronoï [10] (this also easily follows from pp. 90-91 of [3]) we have

$$\int_0^X \Delta(x)\,\mathrm{d}x = \frac{X}{4} + \frac{X^{3/4}}{2\sqrt{2}\pi^2}\sum_{n=1}^\infty d(n)n^{-5/4}\sin(4\pi\sqrt{nX} - \tfrac14\pi) + O(X^{1/4}).$$

To relate the above integral to the one of $\Delta^*(x)$ we proceed as on pp. 472–473 of [3], using (1.3) and (1.10). In this way we are led to

Lemma 5. *We have*

$$\int_0^T \Delta^*(t)\,\mathrm{d}t = -\frac{T}{8} + \frac{T^{3/4}}{2\sqrt{2}\pi^2}\sum_{n\leqslant T^2}(-1)^n d(n)n^{-5/4}\sin(4\pi\sqrt{nT} - \tfrac14\pi) + O(T^{1/4}). \tag{2.10}$$

3. Proof of Theorem 1

We use Lemma 1 and Lemma 2 with $N = T$ to deduce that, for $T \leqslant t \leqslant T + H$, $T^{2/3+\varepsilon} \ll H \leqslant T$,

$$E^*(t) := S_1(t) + S_2(t) + S_3(t), \tag{3.1}$$

where

$$S_1(t) := \sqrt{2}\left(\frac{t}{2\pi}\right)^{1/4}\sum_{n\leqslant T}(-1)^n d(n)n^{-3/4}\Biggl\{e(t,n)\cos f(t,n)-$$

$$-\cos(\sqrt{8\pi nt}-\tfrac14\pi)\Biggr\}, \tag{3.2}$$

$$S_2(t) := -2\sum_{n\leqslant N'} d(n)n^{-1/2}\left(\log\frac{t}{2\pi n}\right)^{-1}\cos\left(t\log\frac{t}{2\pi n}-t+\frac{\pi}{4}\right),$$

$$S_3(t) := O_\varepsilon(T^\varepsilon),$$

and $N' = t/(2\pi)+t/2-\sqrt{T^2/4+tT/(2\pi)}$. We have, similarly as in [7],

$$\int_T^{T+H}\left\{S_2^2(t)+S_3^2(t)\right\}\mathrm{d}t \ll_\varepsilon HT^\varepsilon, \tag{3.3}$$

since $S_2(t)$ is in fact quite analogous to the sum representing $\zeta^2(\frac12+it)$. Therefore

$$\begin{aligned}\int_T^{T+H}(E^*(t))^2\,\mathrm{d}t &= \\ &= \int_T^{T+H}\Bigl\{S_1^2(t)+S_2^2(t)+S_3^2(t)+2S_1(t)S_2(t)+ \\ &\qquad + 2S_1(t)S_3(t)+2S_2(t)S_3(t)\Bigr\}\,\mathrm{d}t = \\ &= \int_T^{T+H}S_1^2(t)\,\mathrm{d}t + O_\varepsilon(HT^{1/6+\varepsilon}).\end{aligned} \tag{3.4}$$

Here we used (3.3), (1.4) and the Cauchy-Schwarz inequality for integrals to deduce that

$$\begin{aligned}\int_T^{T+H}S_1(t)S_2(t)\,\mathrm{d}t &\ll \left\{\int_T^{T+H}S_1^2(t)\,\mathrm{d}t\int_T^{T+H}S_2^2(t)\,\mathrm{d}t\right\}^{1/2} \ll \\ &\ll_\varepsilon \left(HT^\varepsilon HT^{1/3}\log^3 T\right)^{1/2} \ll_\varepsilon HT^{1/6+\varepsilon}.\end{aligned}$$

Now we write

$$S_1(t) = S_4(t) + S_5(t),$$

$$S_4(t) := \sqrt{2}\left(\frac{t}{2\pi}\right)^{1/4} \sum_{n\leqslant T^{1/2-\varepsilon}} (-1)^n d(n) n^{-3/4}\Big\{e(t,n)\cos f(t,n) - \cos(\sqrt{8\pi nt} - \tfrac14\pi)\Big\},$$

$$S_5(t) := \sqrt{2}\left(\frac{t}{2\pi}\right)^{1/4} \sum_{T^{1/2-\varepsilon}<n\leqslant T} (-1)^n d(n) n^{-3/4}\Big\{e(t,n)\cos f(t,n) - \cos(\sqrt{8\pi nt} - \tfrac14\pi)\Big\}.$$

We obtain, following the proof of (1.4),

$$\int_T^{T+H} S_1^2(t)\,\mathrm{d}t = \int_T^{T+H} \left\{S_4^2(t) + S_5^2(t) + 2S_4(t)S_5(t)\right\}\,\mathrm{d}t.$$

In view of (1.4) we have

$$\int_T^{T+H} S_4^2(t)\,\mathrm{d}t \ll_\varepsilon HT^{1/3+\varepsilon} \qquad (T^{2/3+\varepsilon} \leqslant H \leqslant T). \tag{3.5}$$

To estimate the mean square of $S_5(t)$, we split the sum into subsums with the range of summation $K < n \leqslant K' \leqslant 2K, T^{1/2-\varepsilon} \leqslant K \leqslant T$. Note that the mean square bound ($c \neq 0$)

$$\int_T^{T+H} \Big|\sum_{K<k\leqslant K'\leqslant 2K} (-1)^k d(k) \mathrm{e}^{\sqrt{ckt}i}\Big|^2 \,\mathrm{d}t = \tag{3.6}$$

$$= H\sum_{K<k\leqslant 2K} d^2(k) + \sum_{K<m\neq n\leqslant 2K} (-1)^{m+n} d(m)d(n) \int_T^{T+H} \mathrm{e}^{\sqrt{ct}(\sqrt{m}-\sqrt{n})i}\,\mathrm{d}t \ll$$

$$\ll HK\log^3 T + \sqrt{T} \sum_{K<m\neq n\leqslant 2K} \frac{d(m)d(n)}{|\sqrt{m}-\sqrt{n}|} \ll_\varepsilon$$

$$\ll_\varepsilon HK\log^3 T + T^{1/2+\varepsilon} \sum_{K<m\neq n\leqslant 2K} \frac{K^{1/2}}{|m-n|} \ll_\varepsilon$$

$$\ll_\varepsilon T^\varepsilon(HK + T^{1/2}K^{3/2})$$

holds for $1 \ll K \ll T^C$ $(C > 0)$, where we used the standard first derivative test for exponential sums (see Lemma 2.1 of [3]) and Lemma 3. The same bound also holds if in the exponential we have $f(t,k)$ (cf. (2.4)) instead of $\sqrt{ctk}$, as shown e.g., in the derivation of the mean square formula for $E(t)$ in Chapter 15 of [3]. Using (3.6) it follows that

$$\int_T^{T+H} S_5^2(t)\,\mathrm{d}t \ll_\varepsilon T^{1/2+\varepsilon}(HT^{-1/4} + T^{1/2}) = HT^{1/4+\varepsilon} + T^{1+\varepsilon} \tag{3.7}$$

holds for $T^{2/3+\varepsilon} \leqslant H \leqslant T$. Consequently using (3.5), (3.7) and the Cauchy-Schwarz inequality we obtain

$$\int_T^{T+H} S_4(t)S_5(t)\,\mathrm{d}t \ll_\varepsilon T^\varepsilon(HT^{7/24} + H^{1/2}T^{2/3}).$$

Therefore, for $T^{2/3+\varepsilon} \leqslant H \leqslant T$, we have shown that

$$\int_T^{T+H} S_1^2(t)\,\mathrm{d}t = \int_T^{T+H} S_4^2(t)\,\mathrm{d}t + O_\varepsilon\Big(T^\varepsilon(HT^{7/24} + H^{1/2}T^{2/3})\Big). \tag{3.8}$$

The integral on the right-hand side of (3.8) is equal to

$$\sqrt{\frac{2}{\pi}}\sum_{n\leqslant T^{1/2-\varepsilon}} d^2(n)n^{-3/2}\int_T^{T+H} t^{1/2}\Big(e(t,n)\cos f(t,n) - \cos(\sqrt{8\pi nt} - \pi/4)\Big)^2\,\mathrm{d}t +$$

$$+\sqrt{\frac{2}{\pi}}\sum_{1\leqslant m\neq n\leqslant T^{1/2-\varepsilon}} (-1)^{m+n}d(m)d(n)(mn)^{-3/4}\times$$

$$\times\int_T^{T+H} t^{1/2}\Big(e(t,m)\cos f(t,m) - \ldots\Big)\Big(e(t,n)\cos f(t,n) - \ldots)\Big)\,\mathrm{d}t.$$

In this expression we first replace the factors $e(t,m)$ and $e(t,n)$ by 1, and it is seen that the total error made in this process is $O_\varepsilon(HT^{1/4+\varepsilon})$, since $e(t,n) = 1 + O(n/t)$ and $m, n \leqslant T^{1/2-\varepsilon}$. Consider now the sum over $m \neq n$. If both m and n are $\leqslant T^{1/3-\varepsilon}$, then observe that Taylor's formula gives

$$\sin f(t,m) - \sin(2\sqrt{2\pi mt} - \pi/4) = \sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\sin(y_0 + \tfrac12 k\pi) \tag{3.9}$$

$$y = f(t,m),\ y_0 = 2\sqrt{2\pi mt} - \pi/4,\ y - y_0 = d_3m^{3/2}t^{-1/2} + d_5m^{5/2}t^{-3/2} + \ldots,$$

and similarly for $\sin f(t,n)$. Therefore the total contribution of these terms, by the first derivative test, will be

$$\ll T \sum_{m \neq n \leqslant T^{1/3-\varepsilon}} d(m)d(n)(mn)^{3/4}T^{-1} \cdot \frac{1}{|\sqrt{m}-\sqrt{n}|} \ll_\varepsilon$$

$$\ll_\varepsilon T^\varepsilon \sum_{m \neq n \leqslant T^{1/3-\varepsilon}} \frac{(mn)^{3/4}(\sqrt{m}+\sqrt{n})}{|m-n|} \ll_\varepsilon T^\varepsilon \sum_{m \leqslant T^{1/3-\varepsilon}} m^2 \ll_\varepsilon T^{1+\varepsilon}.$$

If, say, $m \leqslant T^{1/3-\varepsilon}, T^{1/3-\varepsilon} < n \leqslant T^{1/2-\varepsilon}$, then the contribution is a multiple of

$$\sum_{m \leqslant T^{1/3-\varepsilon}} (-1)^m d(m) m^{3/4} \sum_{T^{1/3-\varepsilon} < n \leqslant T^{1/2-\varepsilon}} (-1)^n d(n) n^{-3/4} \times$$

$$\times \int_T^{T+H} \mathrm{e}^{\pm i\sqrt{8\pi mt}} \left(\mathrm{e}^{\pm i f(t,n)} - \mathrm{e}^{\pm i\sqrt{8\pi nt}}\right) \mathrm{d}t.$$

The contribution of terms with two square roots in the exponential is, by the first derivative test,

$$\begin{aligned} &\ll \sum_{m \leqslant T^{1/3-\varepsilon}} d(m)m^{3/4} \sum_{T^{1/3-\varepsilon} < n \leqslant T^{1/2-\varepsilon}} d(n)n^{-3/4} \frac{T^{1/2}}{|\sqrt{m}-\sqrt{n}|} \\ &\ll_\varepsilon T^{3/4+\varepsilon} \sum_{T^{1/3-\varepsilon} < n \leqslant T^{1/2-\varepsilon}} d(n)n^{-1/2} \ll_\varepsilon T^{1+\varepsilon}. \end{aligned} \tag{3.10}$$

The remaining case of interest is when we have the exponential factor

$$\exp\Big(iF(t,m,n)\Big), \quad F(t,m,n) := \sqrt{8\pi mt} - f(t,n),$$

when

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F(t,m,n) &= \sqrt{\frac{2\pi m}{t}} - 2\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{2t}} = \\ &= \sqrt{\frac{2\pi}{t}}(\sqrt{m}-\sqrt{n}\,) + a_3 n^{3/2}t^{-3/2} + a_5 n^{5/2}t^{-5/2} + \dots . \end{aligned}$$

But as, for $m \neq n$ and $m, n \leqslant T^{1/2-\varepsilon}$,

$$\frac{|\sqrt{m}-\sqrt{n}|}{\sqrt{t}} = \frac{|m-n|}{|\sqrt{m}+\sqrt{n}|\sqrt{t}} \geqslant \frac{1}{(\sqrt{m}+\sqrt{n})\sqrt{t}} \gg \frac{t^\varepsilon n^{3/2}}{t\sqrt{t}},$$

then again by the first derivative test we obtain a contribution which is, similarly to (3.8), $\ll_\varepsilon T^{1+\varepsilon}$. Finally, the same argument shows that the contribution, when $T^{1/3-\varepsilon} < m \ne n \leqslant T^{1/2-\varepsilon}$, is

$$\begin{aligned}
&\ll T^{1/2} \sum_{T^{1/3-\varepsilon}<m\leqslant T^{1/2-\varepsilon}} d(m)m^{-3/4}\times \\
&\quad\times \sum_{T^{1/3-\varepsilon}<n\leqslant T^{1/2-\varepsilon}, n\ne m} d(n)n^{-3/4}\frac{\sqrt{T}}{|\sqrt{m}-\sqrt{n}|} \ll_\varepsilon \\
&\ll_\varepsilon T^{1+\varepsilon} \sum_{T^{1/3-\varepsilon}<m\ne n\leqslant T^{1/2-\varepsilon}} \frac{(mn)^{-3/4}(\sqrt{m}+\sqrt{n})}{|m-n|} \ll_\varepsilon \\
&\ll_\varepsilon T^{1+\varepsilon} \sum_{T^{1/3-\varepsilon}<m\leqslant T^{1/2-\varepsilon}} m^{-1}\log T \ll_\varepsilon T^{1+\varepsilon}.
\end{aligned}$$

From (3.8) and the preceding estimates it follows that, for $T^{2/3+\varepsilon} \leqslant H \leqslant T$,

$$\begin{aligned}
&\int_T^{T+H} (E^*(t))^2\,\mathrm{d}t = O_\varepsilon(HT^{7/24+\varepsilon} + H^{1/2}T^{2/3+\varepsilon} + T^{1+\varepsilon})+ \\
&+\sqrt{\frac{2}{\pi}} \sum_{n\leqslant T^{1/2-\varepsilon}} \frac{d^2(n)}{n^{3/2}} \int_T^{T+H} t^{1/2}\Big\{\cos f(t,n) - \cos(\sqrt{8\pi nt}-\pi/4)\Big\}^2\,\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Since the integrand on the right-hand side of (3.11) is non-negative, it is not difficult to deduce (1.16) of Theorem 1 from (3.11). To manage the cosines in (3.11) we use the elementary identity

$$(\cos\alpha - \cos\beta)^2 = 1 - \cos(\alpha-\beta) + \{\tfrac12\cos 2\alpha + \tfrac12\cos 2\beta - \cos(\alpha+\beta)\}$$

with $\alpha = f(t,n), \beta = \sqrt{8\pi nt} - \frac14\pi$. By the first derivative test it is seen that the terms coming from curly braces contribute $\ll T$ to (3.11). Furthermore, in view of

$$1 - \cos\gamma = 2\sin^2(\tfrac12\gamma),\ |\sin x| \geqslant \frac{2}{\pi}|x| \quad (|x| \leqslant \pi/2),$$

it is seen that the sum on the right-hand side of (3.11) is

$$\begin{aligned}
&\sqrt{\frac{8}{\pi}} \sum_{n\leqslant T^{1/2-\varepsilon}} d^2(n)n^{-3/2} \int_T^{T+H} t^{1/2}\sin^2\Big(a_3n^{3/2}t^{-1/2} + a_5n^{5/2}t^{-3/2} + \cdots\Big)\,\mathrm{d}t+ \\
&+O_\varepsilon(T^{1+\varepsilon}) \gg
\end{aligned}$$

$$\gg \sum_{n\ll T^{1/3}} d^2(n)n^{-3/2} \int_T^{T+H} t^{1/2}n^3t^{-1}\,\mathrm{d}t + O_\varepsilon(T^{1+\varepsilon}) \geqslant$$
$$\geqslant C_1T^{-1/2}H \sum_{n\ll T^{1/3}} d^2(n)n^{3/2} + O_\varepsilon(T^{1+\varepsilon}) \geqslant$$
$$\geqslant C_2T^{-1/2}HT^{5/6}\log^3 T + O_\varepsilon(T^{1+\varepsilon}) \geqslant \tfrac{1}{2}C_2HT^{1/3}\log^3 T$$

for $T^{2/3+\varepsilon} \leqslant H \leqslant T$. Since all the O-terms in (3.11) are $o(HT^{1/3}\log^3 T)$ in this range, it means that we have proved (1.16) of Theorem 1. ■

4. Proof of Theorem 2

Combining Lemma 4 and Lemma 5 we obtain, with c_0 as in (2.8),

$$R(T) = \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{T<n\leqslant T^2} (-1)^{n+1}d(n)n^{-5/4}\sin(2\pi\sqrt{2nT}-\tfrac{1}{4}\pi)+ \tag{4.1}$$
$$+\frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{n\leqslant T}(-1)^n d(n)n^{-5/4}\Big\{e_2(T,n)\sin f(T,n) - \sin(2\pi\sqrt{2nT}-\tfrac{1}{4}\pi)\Big\} -$$
$$-2\sum_{n\leqslant c_0T} d(n)n^{-1/2}\Big(\log\frac{T}{2\pi n}\Big)^{-2}\sin\Big(T\log\Big(\frac{T}{2\pi n}\Big)-T+\tfrac{1}{4}\pi\Big) + O(T^{1/4}).$$

This gives, since the estimation of $\sum_{n\leqslant N'}\cdots$ is similar (see e.g., [3]) to the estimation of

$$\zeta^2(\tfrac{1}{2}+it) \;=\; O(t^{1/3}),$$

$$R(T) = O(T^{1/2}\log T) + \tag{4.2}$$
$$+\frac{1}{2}\Big(\frac{2T}{\pi}\Big)^{3/4} \sum_{n\leqslant T}(-1)^n d(n)n^{-5/4}\Big\{e_2(T,n)\sin f(T,n) - \sin(2\sqrt{2\pi nT}-\pi/4)\Big\}.$$

We further simplify (4.2) by estimating trivially the portion of the sum for which $n > T^{1/2-\varepsilon}$ and then using $e_2(T,n) = 1 + O(n/T)$. We obtain

$$R(T) = O_\varepsilon(T^{1/2+\varepsilon}) + \tag{4.3}$$
$$+\frac{1}{2}\Big(\frac{2T}{\pi}\Big)^{3/4} \sum_{n\leqslant T^{1/2-\varepsilon}} (-1)^n d(n)n^{-5/4}\Big\{\sin f(T,n) - \sin(2\sqrt{2\pi nT}-\pi/4)\Big\} =$$
$$= \frac{1}{2}\Big(\frac{2T}{\pi}\Big)^{3/4}(s_1(T)+s_2(T)) + O_\varepsilon(T^{1/2+\varepsilon}),$$

say, where in s_1 summation is over $n \leqslant T^{1/3-\varepsilon}$, and in s_2 summation is over n such that $T^{1/3-\varepsilon} < n \leqslant T^{1/2-\varepsilon}$.

Now we replace T by t and suppose that $T \leqslant t \leqslant T+H, T^{2/3+\varepsilon} \leqslant H \leqslant T$. We prove first (1.18) of Theorem 2. In $s_1(t)$ we use (3.9), and in $s_2(t)$ we consider separately the contributions coming from $\sin f(t,n)$ and $\sin(\sqrt{8\pi nt} - -\pi/4)$. In both cases we use (3.6), since it was mentioned that the argument also works for $f(t,n)$ in the exponential. Thus we are led to the estimation of the integrals ($K \ll T^{1/3-\varepsilon}$)

$$I_1(K) := \int_T^{T+H} T^{1/2}\Big|\sum_{K<n\leqslant K'\leqslant 2K}(-1)^n d(n) n^{1/4} \mathrm{e}^{i\sqrt{8\pi nt}}\Big|^2 \,\mathrm{d}t \ll_\varepsilon$$

$$\ll_\varepsilon \max_{K\ll T^{1/3-\varepsilon}} T^{1/2+\varepsilon}(HK^{3/2} + T^{1/2}K^2) \ll_\varepsilon T^{1+\varepsilon}H + T^{5/3+\varepsilon}$$

and ($T^{1/3-\varepsilon} \ll K \ll T^{1/2-\varepsilon}$)

$$I_2(K) := \int_T^{T+H} T^{3/2}\Big|\sum_{K<n\leqslant K'\leqslant 2K}(-1)^n d(n) n^{-5/4} \mathrm{e}^{i\sqrt{8\pi nt}}\Big|^2 \,\mathrm{d}t \ll_\varepsilon$$

$$\ll_\varepsilon \max_{T^{1/3-\varepsilon}\ll K\ll T^{1/2-\varepsilon}} T^{3/2+\varepsilon}(HK^{-3/2} + T^{1/2}K^{-1}) \ll_\varepsilon T^{1+\varepsilon}H + T^{5/3+\varepsilon},$$

while the integral

$$I_3(K) := \int_T^{T+H} T^{3/2}\Big|\sum_{K<n\leqslant K'\leqslant 2K}(-1)^n d(n) n^{-5/4} \mathrm{e}^{if(t,n)}\Big|^2 \,\mathrm{d}t$$

is estimated analogously as $I_2(K)$. Since

$$\int_T^{T+H} R^2(t)\,\mathrm{d}t \ll_\varepsilon HT^{1+\varepsilon} + \log T \max_{K\ll T^{1/3-\varepsilon}} I_1(K) +$$

$$+ \log T \max_{T^{1/3-\varepsilon}\ll K\ll T^{1/2-\varepsilon}} \Big(I_2(K) + I_3(K)\Big),$$

the bound (1.18) follows.

The proof of (1.17) is carried out by using (4.3) and (1.18), and is analogous to the proof of Theorem 1, only somewhat less involved. The sum corresponding to $S_4(t)$ in the proof of Theorem 1 (cf. (3.5)) is the main term on the right-hand side of (4.3). There is no need to repeat the details.

References

[1] **Atkinson, F.V.,** The mean value of the Riemann zeta-function, *Acta Math.*, **81** (1949), 353–376.

[2] **Hafner, J.L. and A. Ivić,** On the mean square of the Riemann zeta-function on the critical line, *J. Number Theory*, **31** (1989), 151–191.

[3] **Ivić, A.,** *The Riemann Zeta-function*, John Wiley & Sons, New York, 1985 (2nd ed. Dover, Mineola, New York, 2003).

[4] **Ivić, A.,** Large values of certain number-theoretic error terms, *Acta Arithm.*, **56** (1990), 135–159.

[5] **Ivić, A.,** The mean values of the Riemann zeta-function, LNs **82**, Tata Inst. of Fundamental Research, Bombay (distr. by Springer Verlag, Berlin etc.), 1991.

[6] **Ivić, A.,** On the Riemann zeta-function and the divisor problem, *Central European J. Math.*, **(2)(4)** (2004), 1–15, II, ibid. **(3)(2)** (2005), 203–214, III, *Annales Univ. Sci. Budapest, Sect. Comp.*, **29** (2008), 3–23, and IV, *Uniform Distribution Theory*, **1** (2006), 125–135.

[7] **Ivić, A.,** On the mean square of the zeta-function and the divisor problem, *Annales Acad. Scien. Fennicae Mathematica*, **23** (2007), 1–9.

[8] **Jutila, M.,** Riemann's zeta-function and the divisor problem, *Arkiv Mat.*, **21** (1983), 75–96 and II, ibid. **31** (1993), 61–70.

[9] **Jutila, M.,** On a formula of Atkinson, in: Topics in classical number theory, Colloq. Budapest 1981, Vol. I, Colloq. Math. Soc. János Bolyai **34** (1984), 807–823.

[10] **Voronoï, G.F.,** Sur une fonction transcendante et ses applications à la sommation de quelques séries, *Ann. École Normale*, (3) **21** (1904), 2-7-267 and ibid. 459-533.

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