ABOUT POSITIVE LINEAR FUNCTIONALS ON SPACES OF ARITHMETICAL FUNCTIONS

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Dedicated to Professor Zoltán Daróczy and Professor Imre Kátai on the occassion of their 75th birthday

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Abstract. Let $\mathcal{F}$ be an algebra of real-valued bounded functions on $\mathbb{N}$ which separates the points, which contains the constants and which is complete in the sup-norm. If L is a positive linear functional on $\mathcal{F}$, then, for each $f \in \mathcal{F}$, $L(f)$ can be represented as an integral of $\overline{f}$ on $\beta\mathbb{N}$ where $\overline{f}$ is the unique extension of f to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$.

1. Introduction

A general problem of probabilistic number theory is to find appropriate probability spaces where large classes of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$ can be considered as random variables. In particular, is it possible to write the mean-value

$$M(f) = \lim_{x\to\infty} x^{-1} \sum_{n\le x} f(n)$$

of a function f (if the limit exists) as an integral

$$M(f) = \int_X \overline{f}\, d\mu(x)$$

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where the space X and the integrable function $\overline{f}$ is uniquely determined by $\mathbb{N}$ and f, respectively?
The main difficulties concerning the immediate application of probabilistic tools to the investigation of the above mentioned questions arise from the fact that the asymptotic density

$$\delta(A) = M(1_A) \qquad (A \in \mathcal{A})$$

defines only a finitely additive measure on the family $\mathcal{A}$ of subsets of $\mathbb{N}$ having an asymptotic density.
In the sixties, E. Novoselov built up a theory of polyadic numbers (see [4]), the background of which is as follows. The ring $\mathbb{Z}$ of the integers is embedded into the compact ring S of the *polyadic numbers*. Then, on the additive group of the ring S, as a compact group, there exists a normalized Haar measure P defined on a σ-algebra $\mathcal{A}$, which contains the Borel sets in S such that $(S, \mathcal{A}, P)$ is a probability space, and P is the extension of the asymptotic density. This enabled Novoselov to develop an „integration theory" for the space of *limit periodic functions*, i.e. for arithmetic functions f, which can be approximated by periodic functions with integer period.
A different approach to the mentioned problem of probabilistic number theory was given by K.-H. Indlekofer in the nineties (see [1] and [2]). The underlying idea can be described as follows: $\mathbb{N}$, endowed with the discrete topology, will be embedded in a compact space $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$, and then any algebra $\mathcal{A}$ in $\mathbb{N}$ with an arbitrary finitely additive set function (pseudomeasure) δ on $\mathcal{A}$ can be extended to an algebra $\overline{\mathcal{A}}$ in $\beta\mathbb{N}$, together with an extension $\overline{\delta}$ on $\overline{\mathcal{A}}$ of the pseudomeasure which turns out to be a premeasure on $\overline{\mathcal{A}}$, and to a corresponding integration theory.
For example, the algebra of all residue classes in $\mathbb{N}$ together with the asymptotic density leads to the space of limit periodic functions of Novoselov.
Further, when we apply the above described theory to the algebra $\mathcal{A}$ generated by the sets

$$A_{p^k} := \{m : p^k || m\}$$

and the asymptotic density we arrive at the space of *almost-even functions* (see W. Schwarz and J. Spilker [5]) and the mean-value $M(f)$ of such a function f can be represented as an integral of $\overline{f}$ on $\beta\mathbb{N}$ where $\overline{f}$ is the unique extension of f to a continuous function on $\beta\mathbb{N}$. (Schwarz and Spilker proved such a result by using Gelfand's theory and applying Riesz' representation theorem (see [5]).)
In a recent paper R. Wagner [6] could show that if $\mathcal{F}$ is an algebra of real-valued bounded functions on $\mathbb{N}$ such that

(I) $\mathcal{F}$ separates the points,

(II) $\mathcal{F}$ contains the constants,

(III) $\mathcal{F}$ is complete in the sup-norm and each $f \in \mathcal{F}$ posseses a mean-value $M(f)$ then a suitable algebra $\mathcal{A}$ of sets may be found such that every $A \in \mathcal{A}$ possesses an asymptotic density and $M(f)$ is equal to the integral of $\overline{f}$ on $\beta\mathbb{N}$.

In this paper we prove that such a representation is valid for any positive linear functional on $\mathcal{F}$. To be more precise we shall use the following

Notations: We write $\ell^\infty = \ell^\infty(\mathbb{N})$ for the set of bounded functions on $\mathbb{N}$ and denote by $||\cdot||_u$ the sup-norm on ℓ^∞. If $\mathcal{A}$ is an algebra of subsets of $\mathbb{N}$ then

$$\overline{\mathcal{A}} := \{\overline{A} : A \in \mathcal{A}\},$$

where $\overline{A} = cl_{\beta\mathbb{N}} A$ is the closure of A in $\beta\mathbb{N}$, is an algebra in $\beta\mathbb{N}$. If δ is a content on $\mathcal{A}$ then the map

$$\begin{aligned} &\overline{\delta} : \overline{\mathcal{A}} \to [0, \infty) \\ &\overline{\delta}(\overline{A}) = \delta(A) \end{aligned}$$

is σ-additive on $\overline{\mathcal{A}}$ and its extension to $\sigma(\overline{\mathcal{A}})$ will be denoted by $\overline{\delta}$, too. We shall write

$$\begin{aligned} \mathcal{E}(\mathcal{A}) := \{s \in \ell^\infty : s = \sum_{j=1}^{m} \alpha_j 1_{A_j}; \alpha_j \in \mathbb{C}, A_j \in \mathcal{A}, j = 1, \ldots, m, \\ \mathbb{N} = \bigcup_{j=1}^{m} A_j \text{ with } A_i \cap A_j = \emptyset, \text{ if } i \neq j\} \end{aligned}$$

for the algebra of simple functions on $\mathcal{A}$. If $f \in \ell^\infty$ then $\overline{f}$ will denote its unique extension to $\beta\mathbb{N}$ ($\overline{f}$ is continuous on $\beta\mathbb{N}$).
With these notations we will prove the following results.

Theorem 1. *Let $\mathcal{F}$ be an algebra of real-valued bounded functions on $\mathbb{N}$ satisfying (I), (II) and (III). Let L be a positive linear functional on $\mathcal{F}$ with $L(1_{\mathbb{N}}) = 1$. Then there exist an algebra $\mathcal{A}$ of subsets of $\mathbb{N}$ and a content δ on $\mathcal{A}$ such that*

(i) *each $f \in \mathcal{F}$ belongs to the $||\cdot||_u$-closure of $\mathcal{E}(\mathcal{A})$ and*

(ii) *for each $f \in \mathcal{F}$ the relation*

$$L(f) = \int_{\beta\mathbb{N}} \overline{f}\, d\overline{\delta}$$

holds.

For an arbitrary subset B of $\mathbb{N}$ we define the number $\delta^*(B)$ by the equation

$$\delta^*(B) = \inf \sum_{i=1}^{m} \delta(A_i)$$

where the infimum is taken over all finite sequences $\{A_i\}_{i=1}^m$ of sets A_i from $\mathcal{A}$ whose union contains B.
Obviously

$$\delta^*(B) = \lim_{B \subset A} \delta(A)$$

where A is restricted to sets from $\mathcal{A}$.

Putting for $f : \mathbb{N} \to \mathbb{C}$

$$||f|| := \inf_{\alpha > 0} \{\alpha + \delta^*(\{n \in \mathbb{N} : |f(n)| > \alpha\})\}$$

then $||f|| = 0$ if and only if $\delta^*(\{n \in \mathbb{N} : |f(n)| > \alpha\}) = 0$ for each $\alpha > 0$. Further,

$$\lim_{n\to\infty} ||f_n - f|| = 0 \quad \text{if and only if} \lim_{n\to\infty} \delta^*(\{m \in \mathbb{N} : |f_n(m) - f(m)| > \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

If there exists a sequence $\{s_n\}$ from $\mathcal{A}$ such that $\lim ||s_n - f|| = 0$ and $\lim \int_{\beta\mathbb{N}} |\overline{s_n} - \overline{s_m}| d\overline{\delta} = 0$ then we say that f belongs to $\mathcal{L}^*(\mathcal{A}, \delta)$.

With these notations we prove

Theorem 2. *Let $f \in \mathcal{L}^*(\mathcal{A}, \delta)$. Then there exist $f_n \in \mathcal{F}$ such that*

$$L(f) := \lim_{n\to\infty} L(f_n) = \int_{\beta\mathbb{N}} \overline{f} d\overline{\delta}$$

where $\overline{f} : \beta\mathbb{N} \to \mathbb{C}$ is unique modulo $\overline{\delta}$-null function.

2. Construction of the algebra $\mathcal{A}$ and the content δ

Let $\mathcal{F}$ and L be as in Theorem 1 and observe that L is continuous on $\mathcal{F}$. Then, for each $B \subset \mathbb{N}$ we put

$$\overline{I}(B) := \inf L(f) \qquad \text{for } f \geq 1_B \text{ and } f \in \mathcal{F} \tag{2.1}$$

and

$$\underline{I}(B) := \sup L(f) \qquad \text{for } f \leq 1_B \text{ and } f \in \mathcal{F}, \tag{2.2}$$

and call $A \subset \mathbb{N}$ to be *regular* if

$$\overline{I}(A) = \underline{I}(A).$$

Let $\mathcal{A}$ be the family of all regular sets and put

$$\delta(A) := \overline{I}(A)(= \underline{I}(A)) \qquad \text{for } A \in \mathcal{A}. \tag{2.3}$$

An obvious characterization of regular sets is given by

Lemma 1. *$A \in \mathcal{A}$ if and only if there exist sequences $\{\tilde{f}_n\}$ and $\{f_n\}$ satisfying*

(i) $\tilde{f}_n, f_n \in \mathcal{F}$,

(ii) *$\{\tilde{f}_n\}$ is increasing, $\{f_n\}$ is decreasing,*

(iii) *$0 \leq \tilde{f}_n \leq 1_A$ and $f_n \geq 1_A$*

such that

$$\lim_{n\to\infty} L(\tilde{f}_n) = \lim_{n\to\infty} L(f_n) =: \delta(A).$$

Now the following result holds.

Lemma 2. *The family of regular sets is an algebra and δ is a content on $\mathcal{A}$.*

Proof. We shall show

- $\mathbb{N} \in \mathcal{A}$,
- if $A \in \mathcal{A}$ then $\mathbb{N} \setminus A \in \mathcal{A}$

and

- if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$ and $\delta(A \cup B) = \delta(A) + \delta(B)$ in case $A \cap B = \emptyset$.

Obviously $\mathbb{N} \in \mathcal{A}$ since $1_{\mathbb{N}} \in \mathcal{F}$.
Now, if $A \in \mathcal{A}$ let $\{\tilde{f}_n\}$ and $\{f_n\}$ as in Lemma 1. Then

$$1_{\mathbb{N}} - f_n \leq 1_{\mathbb{N}} - 1_A = 1_{\mathbb{N}\setminus A}$$

and

$$1_{\mathbb{N}} - \tilde{f}_n \geq 1_{\mathbb{N}} - 1_A = 1_{\mathbb{N}\setminus A}$$

which implies

$$\lim_{n\to\infty} L(1_{\mathbb{N}} - f_n) = \lim_{n\to\infty}(1 - L(f_n)) = \lim_{n\to\infty}(1 - L(\tilde{f}_n)) = \lim_{n\to\infty} L(1_{\mathbb{N}} - \tilde{f}_n).$$

Further, let $A, B \in \mathcal{A}$ and associate to A and B, according to Lemma 1, the sequences $\{\tilde{f}_n\}$, $\{f_n\}$ and $\{\tilde{g}_n\}$, $\{g_n\}$, respectively. Putting

$$c_n := \sup_{m\in\mathbb{N}} f_n(m)$$

then obviously $c_n \leq c_1$ for all $n \geq 1$.
Now we consider $1_{A\cap B} = 1_A \cdot 1_B$. Then

$$\begin{aligned}
0 \leq &\overline{I}(A\cap B) - \underline{I}(A\cap B) \leq \\
\leq &L(f_n \cdot g_n) - L(\tilde{f}_n \cdot \tilde{g}_n) = \\
= &L(f_n \cdot g_n) - L(f_n \cdot \tilde{g}_n) + L(f_n \cdot \tilde{g}_n) - L(\tilde{f}_n \cdot \tilde{g}_n) = \\
= &L(f_n(g_n - \tilde{g}_n)) + L(\tilde{g}_n(f_n - \tilde{f}_n)) \leq \\
\leq &L(c_1(g_n - \tilde{g}_n)) + L(f_n - \tilde{f}_n) = \\
= &c_1(L(g_n) - L(\tilde{g}_n)) + L(f_n) - L(\tilde{f}_n)
\end{aligned}$$

which tends to zero as $n \to \infty$. Thus $A \cap B \in \mathcal{A}$.
Obviously

$$\overline{I}(A\cup B) \leq \overline{I}(A) + \overline{I}(B) \tag{2.4}$$

If $A \cap B = \emptyset$ then

$$\underline{I}(A) + \underline{I}(B) \leq \underline{I}(A\cup B) \tag{2.5}$$

thus, by (2.4) and (2.5)

$$\underline{I}(A) + \underline{I}(B) \leq \underline{I}(A\cup B) \leq \overline{I}(A\cup B) \leq \overline{I}(A) + \overline{I}(B). \tag{2.6}$$

Since $\delta(A)$ and $\delta(B)$ exist, $\delta(A\cup B)$ exists by (2.6), too, and the assertions of Lemma 2 hold. ■

In the next step we show that every $f \in \mathcal{F}$ can be approximated in the sup-norm by step functions $s = \sum_{j=1}^{m} \alpha_j 1_{A_j}$ when $\mathbb{N} = \bigcup_{j=1}^{m} A_j$, with $A_i \cap A_j = \emptyset$ if $i \neq j$. Put $\tilde{L}(s) = \sum_{j=1}^{m} \alpha_j \delta(A_j)$. Then, if $s_n \to f$ we shall obtain $\tilde{L}(s_n) \to L(f)$.

For this purpose we denote by $\mathcal{H}$ the space of all uniformly continuous, bounded functions $h : \mathbb{R} \to \mathbb{R}$ and define, if $f \in \mathcal{F}$ is given, for all $a \in \mathbb{R}$

$$V(f)(a) := \inf L(h \circ f) \qquad \text{where } h \in \mathcal{H} \text{ and } h \geq 1_{(-\infty,a]}. \tag{2.7}$$

We observe that $h \circ f \in \mathcal{F}$ since f is bounded and h can be approximated by polynomials on each bounded and closed interval (Theorem of Weierstraß).

First we prove

Lemma 3. *$V(f)$ is a distribution function.*

Proof. Obviously, $V(f)$ is monotone increasing, and there are real numbers $c < d$ such that

$$1_{(-\infty,a]} \circ f = 0 \quad \text{for } a \leq c$$

and

$$1_{(-\infty,a]} \circ f = 1 \quad \text{for } a \geq d.$$

Therefore, we only have to prove that $V(f)$ is continuous on the right. For this let $\varepsilon > 0$. Then there exists $h \in \mathcal{H}$ with $h \geq 1_{(-\infty,a]}$ and $0 \leq L(h \circ f) - -V(f)(a) \leq \varepsilon$. For $\delta > 0$ put

$$s_\delta(h)(t) := h(t - \delta).$$

Then

$$s_\delta(h) \geq 1_{(-\infty,a+\delta]}.$$

We choose $\delta_0 > 0$ such that for all $\delta \leq \delta_0$

$$\sup_{t \in \mathbb{R}} |s_\delta(h)(t) - h(t)| < \varepsilon.$$

Then

$$\begin{aligned} 0 &\leq V(f)(a+\delta) - V(f)(a) \leq \\ &\leq L(s_\delta(h) \circ f) - L(h \circ f) + L(h \circ f) - V(f)(a) \leq \\ &\leq 2\varepsilon \end{aligned}$$

which proves Lemma 3. ■

Lemma 4. *If $a \in \mathbb{R}$ is a point of continuity for $V(f)$ then*

$$A := \{n \in \mathbb{N} : f(n) \leq a\} \in \mathcal{A}.$$

Proof. Obviously

$$V(f) \geq \overline{I}(A).$$

If $a_n \nearrow a$ there exist $h_n \in \mathcal{H}$ satisfying

$$1_{(-\infty,a_n]} \leq h_n \leq 1_{(-\infty,a]}.$$

Then

$$V(f)(a) - V(f)(a_n) \geq V(f)(a) - L(h_n \circ f) \geq \overline{I}(A) - \underline{I}(A),$$

and, since $V(f)(a) - V(f)(a_n) \to 0$ as $n \to \infty$, the assertion of Lemma 4 is true. ■

Lemma 5. *Let $f \in \mathcal{F}$. Then f belongs to the $||\cdot||_u$-closure of $\mathcal{E}(\mathcal{A})$.*

Proof. Choose the interval $[a, b]$ such that $a < \inf_{n\in\mathbb{N}} f(n)$ and $\sup_{n\in\mathbb{N}} f(n) < b$. For $\varepsilon > 0$ let $\{t_0, t_1, \ldots, t_n\}$ be a dissection of $[a, b]$ with $t_0 = a$, $t_j < t_{j+1}$, $t_n = b$ such that each t_j $(j = 0, \ldots, n)$ is a point of continuity of $V(f)$ and $t_{j+1} - t_j < \varepsilon$.
Then

$$\sum_{j=0}^{n-1} t_{j+1} 1_{(t_j,t_{j+1}]} \circ f \geq f \geq \sum_{j=0}^{n-1} t_j 1_{(t_j,t_{j+1}]} \circ f$$

where

$$1_{(t_j,t_{j+1}]} \circ f = 1_{\{n\in\mathbb{N}: t_j < f(n) \leq t_{j+1}\}}$$

and

$$\{n \in \mathbb{N} : t_j < f(n) \leq t_{j+1}\} \in \mathcal{A}.$$

Further

$$||\sum_{j=0}^{n-1} t_{j+1} 1_{(t_j,t_{j+1}]} \circ f - \sum_{j=0}^{n-1} t_j 1_{(t_j,t_{j+1}]} \circ f||_u < \varepsilon,$$

which proves Lemma 5. ■

Now we can show

Lemma 6. *Let $f \in \mathcal{F}$. Then there exists a sequence $\{s_n\}$ from $\mathcal{E}(\mathcal{A})$ such that $\lim_{n\to\infty} ||f - s_n||_u = 0$ and*

$$L(f) = \lim_{n\to\infty} \tilde{L}(s_n).$$

Proof. Let $f \in \mathcal{F}$ and $\varepsilon > 0$. Choose, with the notations in the proof of Lemma 5,

$$s = \sum_{j=0}^{n-1} t_j 1_{(t_j, t_{j+1}]} \circ f.$$

Then $s \in \mathcal{E}(\mathcal{A})$ and $||f - s||_u < \varepsilon$. We write s in the form

$$s = \sum_{j=1}^{n} (t_j - t_{j-1}) 1_{(t_j, \infty)} \circ f + t_0 1_{\mathbb{N}}$$

and put

$$B_j := \{n \in \mathbb{N} : t_j < f(n)\}.$$

Then there exist $h_j \in \mathcal{H}$ so that

$$|L(h_j \circ f) - \delta(B_j)| < \varepsilon.$$

The functions h_j can be chosen with values $h_j(t) \in [0, 1]$ satisfying $h_j > 1_{(t_j, \infty)}$ and $h_j(t) = 0$ for $t \leq t_{j-1}$. Putting

$$g = \sum_{j=1}^{n} (t_j - t_{j-1})(h_j \circ f) + t_0 1_{\mathbb{N}}$$

we obtain

$$||f - g||_u < \varepsilon$$

and

$$|\tilde{L}(s) - L(g)| \leq \sum_{j=0}^{n-1} (t_{j+1} - t_j)\varepsilon \leq (b - a)\varepsilon.$$

From this we conclude

$$|L(f) - \tilde{L}(s)| < ((b - a) + 1)\varepsilon$$

and Lemma 6 is valid. ∎

3. Integration on $\beta\mathbb{N}$

Starting from the algebra $\mathcal{A}$ of regular sets together with the content δ we arrive at the algebra

$$\overline{\mathcal{A}} = \{\overline{A} : A \in \mathcal{A}\}$$

and the premeasure $\overline{\delta}$ on $\overline{\mathcal{A}}$,

$$\overline{\delta}(\overline{A}) = \delta(A)$$

in $\beta\mathbb{N}$. Define the outer measure $\overline{\delta}^*$ on the class of all subsets E of $\beta\mathbb{N}$ by

$$\overline{\delta}^*(E) = \inf \sum_{j=1}^{\infty} \overline{\delta}(\overline{A_i}) \tag{3.1}$$

the infimum being taken over all sequences of sets $\{\overline{A_i}\}$ in $\overline{\mathcal{A}}$ such that $E \subset \bigcup\limits_{j=1}^{\infty} \overline{A_j}$.

We extend $\overline{\delta}$ with the help of $\overline{\delta}^*$ to a complete measure, which we denote by $\overline{\delta}$, too, on the σ-algebra of $\overline{\delta}^*$-measurable sets.

Then the integral for simple functions $\overline{s} \in \mathcal{E}(\overline{\mathcal{A}})$, $\overline{s} = \sum\limits_{j=1}^{m} \alpha_j 1_{\overline{A_j}}$ is defined by

$$\int\limits_{\beta\mathbb{N}} \overline{s}\, d\overline{\delta} = \sum_{j=1}^{m} \alpha_j \overline{\delta}(\overline{A_j}).$$

For each $f \in \mathcal{F}$ there are $s_n \in \mathcal{E}(\mathcal{A})$ such that

$$L(f) = \lim_{n\to\infty} \tilde{L}(s_n) = \lim_{n\to\infty} \int\limits_{\beta\mathbb{N}} \overline{s_n}\, d\overline{\delta} = \int\limits_{\beta\mathbb{N}} \lim_{n\to\infty} \overline{s_n}\, d\overline{\delta} = \int\limits_{\beta\mathbb{N}} \overline{f}\, d\overline{\delta}, \tag{3.2}$$

where $\overline{s_n}$ and $\overline{f}$ are the unique extensions of s_n and f to $\beta\mathbb{N}$, respectively, and this proves the second assertion of Theorem 1.

A subset $E \subset \beta\mathbb{N}$ is said to be a $\overline{\delta}$-*null set* if $\overline{\delta}(E) = 0$. The function $\overline{f} : \mathbb{N} \to \mathbb{C}$ is called a *null function* if the set $\{w \in \beta\mathbb{N} : |\overline{f}(w)| > \varepsilon\}$ is a $\overline{\delta}$-null set for each $\varepsilon > 0$.

If we define

$$||\overline{f}|| = \inf_{\alpha > 0} \{\alpha + \overline{\delta}^*(\{w \in \beta\mathbb{N} : |\overline{f}(w)| > \alpha\})\}$$

then $\overline{f}$ is a null function if and only if $||f|| = 0$. A sequence $\{\overline{f_n}\}$ of functions on $\beta\mathbb{N}$ to $\mathbb{C}$ *converges in* $\overline{\delta}$- *measure* to the function f on $\beta\mathbb{N}$ to $\mathbb{C}$ if and only if

$$\lim_{n\to\infty} ||\overline{f_n} - \overline{f}|| = 0.$$

It is clear that such a sequence $\{\overline{f_n}\}$ converges in $\overline{\delta}$- measure to $\overline{f}$ if and only if

$$\lim_{n\to\infty} \overline{\delta}^*(\{w \in \beta\mathbb{N} : |\overline{f_n}(w) - \overline{f}(w)| > \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

We observe that $\overline{f} : \mathbb{N} \to \mathbb{C}$ is *integrable on* $\beta\mathbb{N}$ if there is a sequence $\{\overline{s_n}\}$ of simple functions from $\mathcal{E}(\overline{\mathcal{A}})$ converging to $\overline{f}$ in $\overline{\delta}$- measure on $\beta\mathbb{N}$ and satisfying in addition

$$\lim_{n\to\infty} \int\limits_{\beta\mathbb{N}} |\overline{s_n} - \overline{s_m}| d\overline{\delta} = 0.$$

Such a sequence $\{\overline{s_n}\}$ of simple functions is be said to *determine* $\overline{f}$.
Obviously, if $B \subset \mathbb{N}$,

$$\delta^*(B) = \overline{\delta}^*(\overline{B})$$

since the open cover

$$\overline{B} \subset \bigcup_{i=1}^{\infty} \overline{A_i}$$

possesses a finite subcover. Thus the proof of Theorem 2 follows immediately.

■

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