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RATIONAL HERMITE–FEJÉR INTERPOLATION

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Dedicated
to Professors Zoltán Daróczy and Imre Kátai on their 75*th birthday*

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Abstract. In this paper we construct rational orthogonal systems with respect to the normalized area measure on the unit disc. The generating system is a collection of so called elementary rational functions. In the one dimensional case an explicit formula exists for the corresponding Malmquist–Takenaka functions involving the Blaschke functions. Unfortunately, this formula has no generalization for the case of the unit disc, which justifies our investigations. We focus our attention for such special pole combinations, when an explicit numerical process can be given. In [4] we showed, among others, that if the poles of the elementary rational functions are of order one, then the orthogonalization is naturally related with an interpolation problem. Here we take systems of poles which are of order both one and two. We show that this case leads to an Hermite–Fejér type interpolation problem in a subspace of rational functions. The orthogonal projection onto this subspace is calculated and also the basic Hermite–Fejér interpolation functions are provided. We show that by means of the given process effective algorithms can be constructed for approximations of surfaces.

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1. Introduction

In this paper we will investigate systems of rational functions that are analytic on the closed unit disc $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ of the complex plane $\mathbb{C}$. Rational functions turned to be very useful in several areas including system and control theories [9], and signal and image processing [5], [6], [2]. The members of these systems are those rational functions the poles of which are located outside $\overline{\mathbb{D}}$. Let the set of analytic functions on $\overline{\mathbb{D}}$ be denoted by $\mathfrak{A}$, and the set of proper rational functions belonging to $\mathfrak{A}$ by $\mathfrak{R}$. Clearly, $\mathfrak{R}$ is the set of linear combinations of the elementary rational functions

$$r_{a,m}(z) := \frac{1}{(1-\overline{a}z)^m}$$

($a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $z \in \overline{\mathbb{D}}$, $m \in \mathbb{N}^* := \{1, 2, \dots\}$), i.e.

$$\mathfrak{R} = \operatorname{span}\{r_{a,m} : a \in \mathbb{D},\, a \neq 0,\, m \in \mathbb{N}^*\}. \tag{1.1}$$

The mirror image of a with respect to the unit circle is denoted by $a^* := 1/\overline{a} \notin \overline{\mathbb{D}}$. Then a^* is the pole of $r_{a,m}$, of order m. It explains that the parameter a is called the *inverse pole* of order m of the elementary rational function $r_{a,m}$. Taking the normalized area measure $d\sigma(z) = dx\,dy/\pi$ $(z = x + iy \in \mathbb{D})$ the scalar product is defined as follows

$$\langle f, g\rangle := \int_{\mathbb{D}} f(z)\overline{g}(z)\, d\sigma(z) \qquad (f, g \in \mathfrak{A})\,. \tag{1.2}$$

For the general theory we refer to the works [1], [8], and [14] on Bergman spaces.

We showed in [4] (Theorem 2.1) that the scalar product of an analytic function $f \in \mathfrak{A}$, and an elementary rational function $r_{a,m}$ $(a \in \mathbb{D}, m = 1, 2)$ can be expressed in an explicit form:

$$\langle f, r_{a,1}\rangle = f^{(-1)}(a),\ \langle f, r_{a,2}\rangle = f(a) \qquad (f \in \mathfrak{A},\, a \in \mathbb{D})\,, \tag{1.3}$$

where

$$f^{(-1)}(0) := f(0),\ f^{(-1)}(z) := \frac{1}{z}\int_0^z f(\zeta)\, d\zeta \qquad (z \in \mathbb{D},\, z \neq 0,\, f \in \mathfrak{A})\,. \tag{1.4}$$

Let the finite system of distinct nodes $a_1, a_2, \dots, a_N \in \mathbb{D}$ be fixed. Set

$$\begin{aligned} \mathcal{R}_2 &:= \operatorname{span}\{r_{a_j,2} : j = 1, 2, \dots, N\}, \\ \mathcal{R}_{1,2} &:= \operatorname{span}\{r_{a_j,1}, r_{a_j,2} : j = 1, 2, \dots, N\}\,. \end{aligned} \tag{1.5}$$

$\mathcal{R}_2$ is an N dimensional, and $\mathcal{R}_{1,2}$ is a $2N$ dimensional subspace of rational functions. By (1.3) we have that the orthogonal projection $P_2 : \mathfrak{A} \to \mathcal{R}_2$ can be characterized by Lagrange type interpolation properties. Namely, the following three properties are equivalent

$$\text{i)}\ f - P_2 f \perp \mathcal{R}_2\,, \quad \text{ii)}\ \langle f - P_2 f, r_{a_j,2}\rangle = 0\,, \quad \text{iii)}\ (P_2 f)(a_j) = f(a_j)$$

($f \in \mathfrak{A}, j = 1, 2, \ldots, N$). In [4] we constructed an orthogonal basis in the subspace $\mathcal{R}_2$ by means of which the properties and the applications of the projection P_2 were investigated there. It turned out that the basis functions, that can be considered as the two dimensional analogues of the Malmquist–Takenaka functions, enjoy interesting interpolation properties. In this paper we carry out a similar program for the projection $P_{1,2}$, and the subspace $\mathcal{R}_{1,2}$.

It follows from (1.3) that the orthogonal projections $P_{1,2} : \mathfrak{A} \to \mathcal{R}_{1,2}$ can be characterized by Hermite type interpolation conditions. Indeed, the following conditions are equivalent for any $f \in \mathfrak{A}$

$$\begin{aligned} &\text{i)}\ f - P_{1,2} f \perp \mathcal{R}_{1,2}\,, \\ &\text{ii)}\ \langle f - P_{1,2} f, r_{a_j,1}\rangle = 0,\ \langle f - P_{1,2} f, r_{a_j,2}\rangle = 0\,, \\ &\text{iii)}\ (P_{1,2} f)^{(-1)}(a_j) = f^{(-1)}(a_j),\ (P_{1,2} f)(a_j) = f(a_j) \end{aligned} \tag{1.6}$$

($j = 1, 2, \ldots, N$).

Based on this equivalence we consider Hermite–Fejér type interpolation problems in which the values of $f^{(-1)}$, and f rather than those of f, and $f' = f^{(1)}$ are prescribed at the points $a_1, a_2, \ldots, a_N$. Moreover, rational functions belonging to $\mathcal{R}_{1,2}$ are taken instead of polynomials. Since by (1.6) this interpolation problem can be reformulated for orthogonal projections, we will consider the representations of the projections $P_{1,2}$ first.

In *Section 2.* we construct an orthogonal basis, called planar Malmquist–Takenaka system (PMT), in $\mathcal{R}_{1,2}$ by applying Schmidt orthogonalization for the elementary rational functions that generate the space. In *Section 3.* we are concerned with the construction of the Hermite–Fejér type basic interpolation functions. In *Section 4* we present numerical tests for demonstrating the effectiveness of the interpolation method.

2. Orthogonal projections

Two representations for the orthogonal projections $P_{1,2} : \mathfrak{A} \to \mathcal{R}_{1,2}$ will be provided in this section. One in the basis of the original elementary rational functions, the other one in the corresponding orthogonal basis. These representations will be applied in *Section 3* for the construction of the basic interpolation functions in the basis of the elementary rational functions. Let the functions generating the subspace $\mathcal{R}_{1,2}$ be indexed as follows

$$\mathsf{R}_k := r_{a_k,1}\,, \quad \mathsf{R}_{k+N} := r_{a_k,2} \qquad (k = 1, 2, \dots, N)\,. \tag{2.1}$$

The orthogonal projection

$$P_{1,2}f := \sum_{k=1}^{2N} x_k \mathsf{R}_k \tag{2.2}$$

is characterized by the condition $\langle f - P_{1,2}f, \mathsf{R}_k\rangle = 0$ $(k = 1, 2, \dots, 2N)$. In connection with biorthogonal expansions we refer to [7]. By (2.2) we have that the coefficients x_k of the projection satisfy the linear system of equations

$$\sum_{k=1}^{2N} x_k \langle \mathsf{R}_k, \mathsf{R}_n\rangle = \langle f, \mathsf{R}_n\rangle \qquad (n = 1, 2, \dots, 2N)\,. \tag{2.3}$$

In [4] we solved the corresponding equation in the case when the multiplicity of the poles is 2. The proof presented below is organized in a similar way as the one in [4]. Also, we adapt the notations used there. The reason was to make it easy to compare the two cases.

Since the functions R_k $(1 \le k \le 2N)$ are linearly independent we have that the self adjoint Gram–matrix

$$\mathbf{C}_n := \begin{pmatrix} \langle \mathsf{R}_1, \mathsf{R}_1\rangle & \langle \mathsf{R}_2, \mathsf{R}_1\rangle & \cdots & \langle \mathsf{R}_n, \mathsf{R}_1\rangle \\ \langle \mathsf{R}_1, \mathsf{R}_2\rangle & \langle \mathsf{R}_2, \mathsf{R}_2\rangle & \cdots & \langle \mathsf{R}_n, \mathsf{R}_2\rangle \\ \vdots & & \cdots & \vdots \\ \langle \mathsf{R}_1, \mathsf{R}_n\rangle & \langle \mathsf{R}_2, \mathsf{R}_n\rangle & \cdots & \langle \mathsf{R}_n, \mathsf{R}_n\rangle \end{pmatrix}$$

is regular for every $n \in \mathbb{N}^*$. By (1.3), (1.4), and (2.1) it is easy to see that the equations

$$\begin{aligned} &\langle f, \mathsf{R}_n\rangle = f^{(-1)}(a_n)\,, && \langle f, \mathsf{R}_{n+N}\rangle = f(a_n)\,, \\ &\mathsf{R}_k^{(-1)}(z) = -\frac{\log(1-\overline{a}_k z)}{\overline{a}_k z}\,, && \mathsf{R}_{k+N}^{(-1)} = \mathsf{R}_k \end{aligned} \tag{2.4}$$

hold for every $n = 1, 2, \dots, N$ and $z \in \mathbb{D}$. Introducing the functions

$$s_0(z) := -\frac{\log(1-z)}{z}\,, \quad s_k(z) = \frac{1}{(1-z)^k} \qquad (z \in \mathbb{D}, k \in \mathbb{N}^*) \tag{2.5}$$

the entries $\gamma_{kn} := \langle \mathsf{R}_k, \mathsf{R}_n \rangle$ $(1 \le k, n \le 2N)$ of the matrix $\mathbf{C}_{2N}$ can be written in the following convenient form

$$\gamma_{k+iN,n+jN} = \langle \mathsf{R}_{k+iN}, \mathsf{R}_{n+jN} \rangle = s_{i+j}(\overline{a}_k a_n) \tag{2.6}$$

$(1 \le k, n \le N,\ i, j = 0, 1)$.
Then projection $P_{1,2}f$ can be received by solving the linear system of equations (2.3). In the space $\mathcal{R}_{1,2}$ we can construct an orthonormal basis by using Schmidt–orthogonalization or the Householder–algorithm. Then the projection is nothing but the Fourier–partial sum with respect to the given orthonormal system.

The result of the Schmidt orthogonalization process applied for the linearly independent system $(\mathsf{R}_k,\ 1 \le k \le 2N)$ with the scalar product (1.2) is $\mathcal{M}_n^{\mathfrak{a}} = \mathcal{M}_n$ $(1 \le n \le 2N)$ which is the two dimensional analogue of the Malmquist–Takenaka system. Therefore this system will be referred to as PMT system, where P stands for planar. It is unfortunate that unlike the one dimensional case the members of the PMT system can not be given in an explicit form. It is known that the PMT system generated from $(\mathsf{R}_n,\ n \in \mathbb{N})$ by Schmidt–orthogonalization can be characterized by the following two properties

$$\begin{aligned} &\text{i)} \quad \operatorname{span}\{\mathsf{R}_i : 1 \le i \le n\} = \operatorname{span}\{\mathcal{M}_i : 1 \le i \le n\}, \\ &\text{ii)} \quad \langle \mathcal{M}_i, \mathcal{M}_j \rangle = \delta_{ij} \qquad (1 \le i, j \le 2N). \end{aligned} \tag{2.7}$$

The relation in i) implies that there exist unique numbers

$$\alpha_{nj}, \beta_{nj} \quad (1 \le j \le n), \quad \alpha_{nn} > 0, \quad \beta_{nn} > 0 \qquad (n \in \mathbb{N}^*)$$

such that

$$\mathcal{M}_n = \sum_{j=1}^{n} \alpha_{nj} \mathsf{R}_j, \quad \mathsf{R}_n = \sum_{j=1}^{n} \beta_{nj} \mathcal{M}_j \qquad (n \in \mathbb{N}). \tag{2.8}$$

Hence we have

$$\overline{\beta}_{nk} = \langle \mathcal{M}_k, \mathsf{R}_n \rangle = \sum_{j=1}^{k} \alpha_{kj} \gamma_{jn} \qquad (1 \le k \le n). \tag{2.9}$$

Let us introduce the triangular matrices $\mathbf{A}_n, \mathbf{B}_n \in \mathbb{C}^{n\times n}$ as follows

$$\mathbf{A}_n := \begin{pmatrix} \alpha_{11} & 0 & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \cdots & 0 \\ \vdots & . & . & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn} \end{pmatrix},$$

$$\mathbf{B}_n := \begin{pmatrix} \beta_{11} & 0 & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & 0 & \cdots & 0 \\ \vdots & . & . & \cdots & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \cdots & \beta_{nn} \end{pmatrix}.$$

By definition we have $\mathbf{A}_n^{-1} = \mathbf{B}_n$. We note that this sequence of matrices can be calculated recursively by means of (2.9). Indeed, we have

$$\mathcal{M}_1 = \mathsf{R}_1/\|\mathsf{R}_1\| \,, \quad \|\mathsf{R}_1\| = \sqrt{s_0(|a_1|^2)} = \beta_{11} = 1/\alpha_{11} \,,$$

and let us suppose that the matrices $\mathbf{A}_{n-1}, \mathbf{B}_{n-1}$ have already been determined. For indices $k < n$ the numbers β_{nk} can be calculated by (2.9). Then $\beta_{nn} > 0$ follows from the decomposition of R_n in (2.8):

$$|\beta_{nn}|^2 = \langle \mathsf{R}_n, \mathsf{R}_n\rangle - \sum_{k=0}^{n-1} |\beta_{nk}|^2 > 0 \,.$$

This way we obtain the nth row of the matrix $\mathbf{B}_n$. Hence $\mathbf{A}_n$ can be received by inversion.

We note that the solution of the equations

$$\alpha_{nn} = 1/\beta_{nn} \,, \quad \sum_{j=k}^{n} \alpha_{nj}\beta_{jk} = 0 \qquad (k = n-1, n-2, \ldots, 1) \tag{2.10}$$

yields the nth row of $\mathbf{A}_n$. These equations can be deduced from $\mathbf{A}_n\mathbf{B}_n = \mathbf{E}_n$, where $\mathbf{E}_n \in \mathbb{C}^{n\times n}$ is the unit matrix.

It is clear that (2.9) is equivalent to the condition

$$\mathbf{B}_n^* = \mathbf{A}_n\mathbf{C}_n \,, \quad \text{or in other form} \quad \mathbf{C}_n = \mathbf{B}_n\mathbf{B}_n^* \qquad (n \in \mathbb{N}) \,. \tag{2.11}$$

Consequently, the inverse of $\mathbf{C}_n$ can be expressed by $\mathbf{A}_n = \mathbf{B}_n^{-1}$:

$$\mathbf{C}_n^{-1} = \mathbf{A}_n^*\mathbf{A}_n \,. \tag{2.12}$$

Hence we can infer that the solution of the equation (2.3) can explicitly be given. Then by (2.2) we have the decomposition of the projection $S_n^{\mathfrak{a}} f$ in the basis $\mathsf{R}_k^{\mathfrak{a}}$ $(0 \le k \le 2N)$. For the scalar products on the right sides of the equations in (2.3) we can use the formulas in (2.4) that involve the values of f and $f^{(-1)}$ taken only at the points a_j $(j = 1, 2, \ldots, N)$. This implies that no integration is needed for the calculation of the orthogonal projection $S_n^{\mathfrak{a}} f$.

Using the PMT system the orthogonal projection $S_n^{\mathfrak{a}} f$ can be understood as the partial sum of the Fourier series:

$$S_n^{\mathfrak{a}} f = \sum_{k=1}^{n} \langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle \mathcal{M}_k^{\mathfrak{a}} \qquad (1 \le k \le 2N)\,.$$

By (2.8) we have that the Fourier coefficients with respect to the PMT system can be written as

$$\langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle = \sum_{j=0}^{k} \overline{\alpha}_{kj} \langle f, \mathsf{R}_j^{\mathfrak{a}} \rangle\,.$$

Then it follows from (2.4)

$$\langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle = \sum_{j=1}^{k} \overline{\alpha}_{kj}\, f^{(-1)}(a_j) \qquad (1 \le k \le N)\,,$$

$$\langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle = \sum_{j=1}^{N} \overline{\alpha}_{kj}\, f^{(-1)}(a_j) + \sum_{j=N+1}^{k} \overline{\alpha}_{kj} f(a_j) \qquad (N \le k \le 2N)\,.$$

3. Basic interpolation polynomials

It is known (see e.g. [3], [10]) that the classic Hermite-Fejér interpolation polynomials are of special importance in both the theory and the application of approximation theory. In this section we are concerned with the following Hermite–Fejér type interpolation problem with respect to rational functions: Let the nodes $a_1, a_2, \ldots, a_N \in \mathbb{D}$ and numbers $y_1, y_2, \ldots, y_{2N} \in \mathbb{C}$ be given. Find the function $f \in \mathcal{R}_{1,2}$ for which

$$f^{(-1)}(a_n) = y_n\,, \quad f(a_n) = y_{N+n} \qquad (n = 1, 2, \ldots, N) \tag{3.1}$$

holds. If the function is written in the form

$$f = \sum_{k=1}^{2N} x_k \mathsf{R}_k \tag{3.2}$$

then it follows from (2.4) that (3.1) is equivalent to

$$y_n = f^{(-1)}(a_n) = \langle f, \mathsf{R}_n\rangle = \sum_{k=1}^{2N} x_k \langle \mathsf{R}_k, \mathsf{R}_n\rangle \qquad (1 \le n \le N),$$

$$y_{n+N} = f(a_n) = \langle f, \mathsf{R}_{n+N}\rangle = \sum_{k=1}^{2N} x_k \langle \mathsf{R}_k, \mathsf{R}_{n+N}\rangle \qquad (1 \le n \le N).$$

Set

$$\mathbf{x}_{2N} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2N} \end{pmatrix}, \quad \mathbf{y}_{2N} := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N} \end{pmatrix}.$$

By the definition and of $\mathbf{C}_{2N}$ and by (2.12) this system of equation can be written in the following form

$$\mathbf{x}_{2N} = \mathbf{C}_{2N}^{-1}\mathbf{y}_{2N} = \mathbf{A}_{2N}^*\mathbf{A}_{2N}\mathbf{y}_{2N}\,. \tag{3.3}$$

Let $\mathcal{H}_n$ $(1 \le n \le 2N)$ stand for the Hermite–Fejér basic functions. They are defined by the conditions

$$\mathcal{H}_n^{(-1)}(a_k) = \delta_{kn}\,, \quad \mathcal{H}_{N+n}(a_k) = \delta_{kn} \qquad (1 \le k \le N)\,,$$

where δ_{kn} is the Kronecker symbol. Let $\mathbf{e}_k = (\delta_{k,2N}, k = 1, 2, \dots, 2N)^T$ denote the canonical basis of the space $\mathbb{C}^{2N}$. Then the basic functions $\mathcal{H}_n$ can be decomposed in the basis formed by the elementary rational functions:

$$\mathcal{H}_n = \sum_{k=1}^{2N} h_k^n \mathsf{R}_k \qquad (1 \le n \le 2N)\,,$$

where

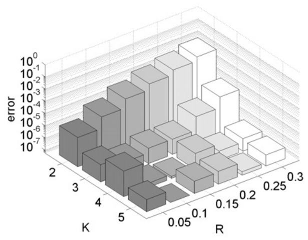
$$\mathbf{h}^n = \mathbf{C}_{2N}^{-1}\mathbf{e}_{2N} \qquad (1 \le n \le 2N)$$

holds true for the coefficients $\mathbf{h}^n := (h_1^n, h_2^n, \dots, h_{2N}^n)^T$. By means of the basic Hermite–Fejér functions we introduce the operators

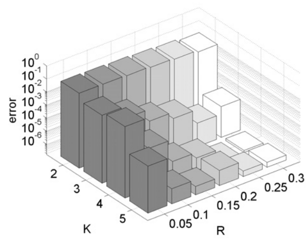
$$\mathcal{F}_n^1 f = \sum_{k=1}^{N} f^{(-1)}(a_k)\mathcal{H}_k\,, \quad \mathcal{F}_n^2 f = \sum_{k=1}^{N} f(a_k)\mathcal{H}_{N+k} \qquad (f \in \mathfrak{A})\,,$$

which are the analogues of the operators used in the original Hermite–Fejér interpolation process.

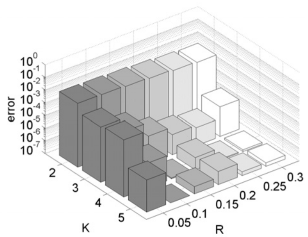
It is known that with special choice of the nodes, for example with Chebyshev abscissas, the Hermite–Fejér interpolation process converges uniformly for every continuous function. This implies the question: Under what combinations of nodes will the $\mathcal{F}_n^1 f$ process have such good approximation properties?



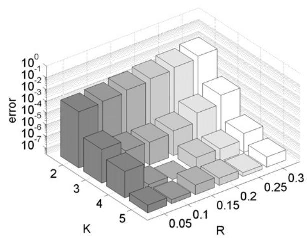
(a) $f(z) = (1 - \bar{b}z)^2$



(b) $f(z) = B_b(z)$



(c) $f(z) = \log(1 - \bar{b}z)$



(d) $f(z) = \cos(z)$

Figure 1: Overview of $\|.\|_\infty$ approximation errors.

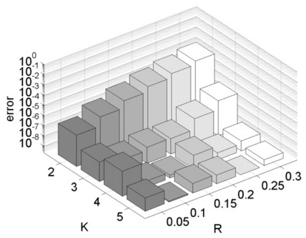
4. Numerical tests

In this section we show some test results. Our aim was twofold. Namely, we wanted to demonstrate the good approximation properties of the projections generated by rational function systems considered above, and to make comparison with the construction given in [4]. Recall that in [4] every pole was of order two while in this paper the poles that generate the system had poles both of order one and two. According to this purpose we used the same test functions and basic pole configurations as in [4].

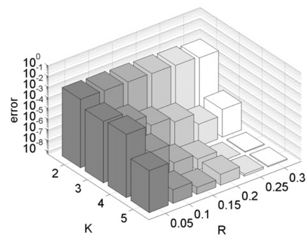
Let the function $\mathcal{R}_0(z) := 1 \ (z \in \mathbb{D})$ that corresponds to the inverse pole $a_0 := 0$ be added to the system $(\mathcal{R}_k, 1 \le k \le 2N)$ introduced in *Section 2.* Then also the functions that do not vanish at 0 can be represented. By the scalar product (1.2) the orthogonal projection onto the subspace spanned by the system is given in (2.2). The x_k coefficients in (2.2) is calculated numerically from the system of linear equations (2.3). The test functions used are:

$$f_1(z) = (1 - \bar{b}z)^2 \,, \qquad\qquad f_2(z) = \frac{z - b}{1 - \bar{b}z} = B_b(z) \,,$$

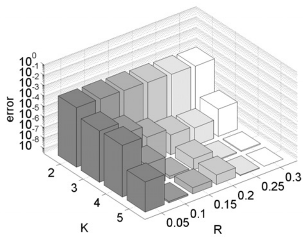
$$f_3(z) = \log(1 - \bar{b}z) \,, \qquad\qquad f_4(z) = \cos z \,,$$



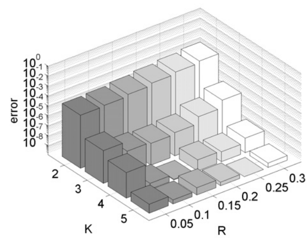
(a) $f(z) = (1 - \bar{b}z)^2$



(b) $f(z) = B_b(z)$



(c) $f(z) = \log(1 - \bar{b}z)$



(d) $f(z) = \cos(z)$

Figure 2: Overview of $\|.\|_\infty$ approximation errors of $f^{(-1)}$.

where $b = 0.5$, and B_b is the Blasckhe-function.

The inverse poles $0 = a_0, \ldots, a_N$ were distributed uniformly on concentric circles around the origin according to the pseudo-hyperbolic distance

$$\rho(z, w) = |B_z(w)|$$

as follows

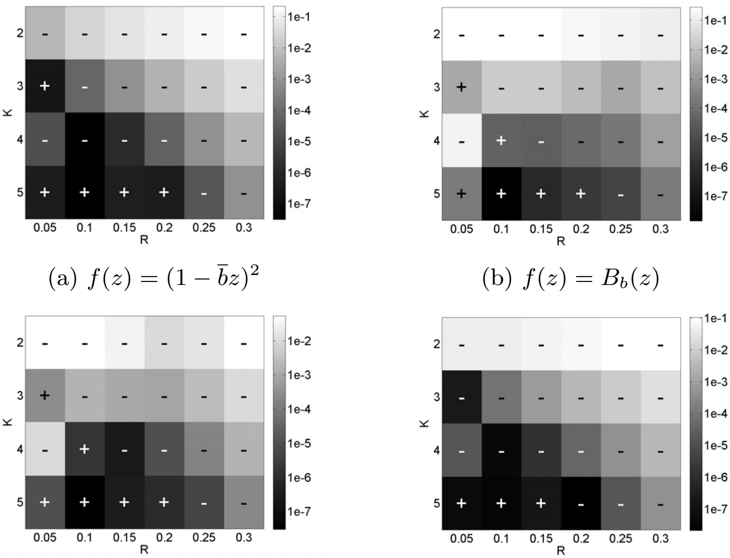
$$a_{2^k-1+j} = r_k e^{\imath 2\pi \frac{j}{2^k}},$$

where $0 \le k < K \in \mathbb{N}^+$, $0 \le j < 2^k$, $N = 2^K - 2$, and $\rho(r_k, r_{k+1}) = R$ $(0 \le k < K - 1)$. These pole combinations are justified by their role in the construction of hyperbolic wavelets [11], [12], [13]. The tests were performed for the values $K = 2, 3, 4, 5$, and $R = 0.05, 0.1, \ldots, 0.3$.

On *Figure 1.* the error

$$\|f_l - \mathcal{S}_{2N}^{\mathfrak{a}} f_l\|_\infty \qquad (l = 1, 2, 3, 4) \tag{4.1}$$

is displayed for the different values of the parameters R, and K. The errors are approximated by uniformly sampling the error function on the unit circle and taking the maximum of the sampled values. We note that by the maximum modulus principle it is sufficient to consider the unit circle.



(a) $f(z) = (1 - \bar{b}z)^2$ (b) $f(z) = B_b(z)$

(c) $f(z) = \log(1 - \bar{b}z)$ (d) $f(z) = \cos(z)$

Figure 3: Comparison of $\|.\|_\infty$ approximation errors.

Similarly, on *Figure 2.* the approximation of the error

$$\|f_l^{(-1)} - (S_{2N}^{\mathfrak{a}} f_l)^{(-1)}\|_\infty \qquad (l = 1, 2, 3, 4) \tag{4.2}$$

of the integral functions are shown.

On *Figure 3.* we compare the method presented here with the one discussed in [4]. Each cell corresponds to a pole configuration. The shading of the cell represents the magnitude, while the sign indicates the direction of the difference. A minus sign means the error of the method proposed here is smaller. We note, that only the function values are compared here, as the system in [4] does not approximate the integrals.

We can conclude that the system generated by poles of order one and two can approximate both the value and the integral of a given analytic function. Using this system compared to the one proposed in [4] can in some cases result in an insignificant increase of the approximation error in terms of the function values, but in most cases it outperforms the one in [4] even in this sense.

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