Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 159–172

CONDITIONAL HOMOGENEITY AND TRANSLATIVITY OF MAKÓ-PÁLES MEANS

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Dedicated to Professor Zoltán Daróczy and Professor Imre Kátai on the occasion of theirs 75th birthdays

Communicated by Antal Járai

(Received April 21, 2013; accepted May 25, 2013)

Abstract. In this paper we characterize the homogeneous and translative members of the class of Makó-Páles means. This is a common generalization of the classes of weighted quasi-arithmetic means and Lagrangian means. So, as an application we get the description of homogeneous and translative means also within these classes.

1. Introduction

The theory of means is an intensively investigated area of mathematics. One of the typical hot topics of this is the characterization of such members of the examined class of means which posses some special additional property within the class in question.

Here we determine the homogeneous and translative members of a certain class of means. These types of questions are not new, they have already been examined by several authors (see e.g. [2], [5], [3], [6], [4], [7], [11], [15], [16], [17], [18]).

Key words and phrases: Mean, Makó-Páles mean, homogeneity, translativity.
2010 Mathematics Subject Classification: Primary 26E60, Secondary 39B22.
The research of the first author has been supported by the Hungarian Scientific Research Fund OTKA "Mobility" call HUMAN-MB8A-84581 and OTKA NK-81402.
https://doi.org/10.71352/ac.40.159

Firstly, we introduce the most important definitions in the second section. In the third one we solve the homogeneity problem in the class of Makó-Páles means and in the fourth we investigate in the same class the translativity too. Finally, as an application of our theorems, we characterize the homogeneous and translative means in the classes of weighted quasi-arithmetic means and Lagrangian means.

2. Definitions and tools

The following notations are used throughout this work.

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean on* I if it is continuous and fulfills the undermentioned pair of inequalities

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \qquad x, y \in I.$$

If, in addition, these inequalities are sharp whenever $x \neq y$, the mean M is called *strict*. Note that if $M : I^2 \to \mathbb{R}$ is a mean, then for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M\left(I^2\right) = I$; moreover M is *reflexive*, that is $M(x, x) = x$ for all $x \in I$.

M is called *homogeneous* if $I = \mathbb{R}_+$ or $\mathbb{R}$ and

$$M(tx, ty) = tM(x, y) \qquad \text{for all } x, y, t \in I.$$

M is called *translative* if $I = \mathbb{R}$ and

$$M(t + x, t + y) = t + M(x, y) \qquad \text{for all } x, y, t \in I.$$

M is said to be *conditionally homogeneous or translative* if M is defined only on a subinterval, and the beforementioned inequalities are fulfilled only on this, such that the products and the sums make sense. That is, $tx, ty, t + x, t + y$ are in this subinterval.

In this paper we are interested in the conditional homogeneity and transitivity of Makó-Páles means. This class was introduced in [19] by the authors (for further information about this class see [20]).

Given a strictly monotonic, continuous function $\varphi : I \to \mathbb{R}$ and a probability Borel measure μ on $[0, 1]$, the Makó-Páles mean $A_{\varphi,\mu} : I^2 \to I$ is defined by

$$A_{\varphi,\mu}(x, y) = \varphi^{-1}\left(\int_0^1 \varphi\left(tx + (1 - t)y\right) d\mu(t)\right).$$

It is clear that $A_{\varphi,\mu}$ is a strict mean on I unless μ is the Dirac measure concentrated to 0 or 1. It is quite straightforward, with a special choice of φ and μ, that this class is a common generalization of Lagrangian means and weighted quasi-arithmetic means (the definition of these classes can be found in the last section, for more information about these means see e.g. [9], [13], [21]).

In our investigation [19, Theorem 7] is of key importance. For the readers convenience we recall this here. For this we also need a definition from [19].

The kth moment of μ is defined in the following way

$$\hat{\mu}_k := \int\limits_0^1 t^k d\mu(t).$$

Moreover, the Dirac measure concentrated to $\tau \in [0,1]$ will be denoted by δ_τ.

Theorem 2.1 (Makó-Páles, [19])**.** *Let $I \subset \mathbb{R}$ be a nonempty open interval, $\varphi, \psi\colon I \to \mathbb{R}$ be strictly monotonic, continuous functions and μ, ν Borel probability measures on $[0,1]$. If $\hat{\mu}_k = \hat{\nu}_k$ for all $k \in \mathbb{N}$, then $A_{\varphi,\mu} = A_{\psi,\nu}$ if and only if*

1. *either $\mu = \nu = \delta_\tau$ for some $\tau \in [0,1]$ and φ, ψ are arbitrary,*
   or

2. *$\mu = \nu$ is not a Dirac measure and there exist constants $a \neq 0$ and b such that $\psi = a\varphi + b$.*

3. Homogeneity

In this section we solve the homogeneity problem in the class of Makó-Páles means. We do not assume any regularity on the generators.

Theorem 3.1. *Assume that $I \subset (0,\infty)$. Let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotonic function and μ be a probability Borel measure on $[0,1]$. The mean $A_{\varphi,\mu}$ is conditionally homogeneous if and only if μ is a Dirac measure or there are $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that either*

$$\varphi(x) = ax^p + b, \qquad x \in I, \tag{3.1}$$

with some $p \in \mathbb{R} \setminus \{0\}$, or

$$\varphi(x) = a \log x + b, \qquad x \in I. \tag{3.2}$$

The theorem will be derived from the proposition below. Assuming there that I is an interval of positive numbers we set

$$\alpha = \frac{\inf I}{\sup I}$$

adopting the convention that $\frac{x}{\infty} = 0$ for each $x \in \mathbb{R}$. Then $\alpha \in [0,1]$. Observe that if $\lambda \in \left(\alpha, \alpha^{-1}\right)$ (here $\alpha^{-1} := \infty$ if $\alpha = 0$) and $\mathrm{int} I = (r,s)$ with some $0 \le r < s \le \infty$, then $\alpha = \frac{r}{s}$, hence $r < \frac{s}{\lambda}$ and $\frac{r}{\lambda} < s$, and thus the intervals (r,s) and $\left(\frac{r}{\lambda}, \frac{s}{\lambda}\right)$ intersect, that is $\mathrm{int} I \cap \mathrm{int} \frac{1}{\lambda} I \neq \emptyset$. Conversely, if $\mathrm{int} I \cap \mathrm{int} \frac{1}{\lambda} I \neq \emptyset$, then $\alpha < \lambda < \alpha^{-1}$. Therefore

$$\lambda \in \left(\alpha, \alpha^{-1}\right) \quad \text{if and only if} \quad \mathrm{int}\left(I \cap \frac{1}{\lambda} I\right) \neq \emptyset.$$

Proposition 3.2. *Assume that* $I \subset (0,\infty)$, α *is defined as above, and let* $c, d : \left(\alpha, \alpha^{-1}\right) \to \mathbb{R}$. *If* $\varphi : I \to \mathbb{R}$ *is a continuous function satisfying the equation*

$$\varphi(\lambda x) = c(\lambda)\varphi(x) + d(\lambda) \tag{3.3}$$

for all $\lambda \in \left(\alpha, \alpha^{-1}\right)$ *and* $x \in I \cap \frac{1}{\lambda} I$, *then there are* $a, b \in \mathbb{R}$ *such that either*

(i) φ *is of form* (3.1) *with some* $p \in \mathbb{R} \setminus \{0\}$ *and* $d(\lambda) = b(1 - c(\lambda))$ *for every* $\lambda \in \left(\alpha, \alpha^{-1}\right)$, *and* $c(\lambda) = \lambda^p$ *for every* $\lambda \in \left(\alpha, \alpha^{-1}\right)$ *whenever* $a \neq 0$,

or

(ii) φ *is of form* (3.2), $c(\lambda) = 1$ *and* $d(\lambda) = a \log \lambda$ *for every* $\lambda \in \left(\alpha, \alpha^{-1}\right)$.

Proof. We may confine ourselves to the case when $\mathrm{int} I \neq \emptyset$, that is $\alpha < 1$. First assume that φ is constant on a subinterval of I, with non-empty interior. Let J be a maximal one and suppose that $J \subsetneq I$. Then we can find a $\lambda \in (0,\infty)$ such that $\lambda J \subset I$, $\lambda J \setminus J \neq \emptyset$ and $J \cap \lambda J \neq \emptyset$. Then $J \cup \lambda J$ is an interval and $J \subsetneq J \cup \lambda J$. It follows from (3.3) that φ is constant also on λJ, and thus on $J \cup \lambda J$ because of the condition $J \cap \lambda J \neq \emptyset$. This contradiction shows that $J = I$, i.e. φ is constant. In particular, it is of form (3.1) with $a = 0$, and the condition $d(\lambda) = b(1 - c(\lambda))$ for each $\lambda \in \left(\alpha, \alpha^{-1}\right)$ follows from (3.3).

Now consider the complementary case when φ is constant on no interval with non-empty interior. We prove that the functions c and d are continuous. Take any $\lambda_0 \in \left(\alpha, \alpha^{-1}\right)$. Then the interval $I \cap \frac{1}{\lambda_0} I$ has non-empty interior, so there are $x_1, x_2 \in \mathrm{int}\left(I \cap \frac{1}{\lambda_0} I\right)$ with $\varphi(x_1) \neq \varphi(x_2)$. Choose a $\delta \in (0,\infty)$ such that $(\lambda_0 - \delta, \lambda_0 + \delta) \subset \left(\alpha, \alpha^{-1}\right)$ and

$$x_1, x_2 \in \mathrm{int}\left(I \cap \frac{1}{\lambda} I\right), \qquad \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta).$$

For every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ equality (3.3) gives

$$\varphi(\lambda x_1) - \varphi(\lambda x_2) = c(\lambda)\left(\varphi(x_1) - \varphi(x_2)\right),$$

and thus, since φ is continuous and $\varphi(x_1) \neq \varphi(x_2)$, it follows that c is continuous in $(\lambda_0 - \delta, \lambda_0 + \delta)$. Consequently, c is continuous and, according to (3.3), so is d.

We verify that c satisfies the Cauchy equation

$$c(\kappa\lambda) = c(\kappa)c(\lambda) \tag{3.4}$$

for all $\kappa, \lambda \in \left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right)$. Take any such κ, λ. Then $\kappa, \lambda, \kappa\lambda \in \left(\alpha, \alpha^{-1}\right)$, hence $\operatorname{int}\left(\frac{1}{\lambda}I \cap \frac{1}{\kappa\lambda}I\right) \neq \emptyset$, $\operatorname{int}\left(I \cap \frac{1}{\lambda}I\right) \neq \emptyset$ and $\operatorname{int}\left(I \cap \frac{1}{\kappa\lambda}I\right) \neq \emptyset$. Observe that if J_1, J_2, J_3 are open intervals pairwise intersecting, then $J_1 \cap J_2 \cap J_3$ is a non-empty interval. Thus, setting $J_1 = \operatorname{int} I$, $J_2 = \operatorname{int}\frac{1}{\lambda}I$, $J_3 = \operatorname{int}\frac{1}{\kappa\lambda}I$ we see that $I \cap \frac{1}{\lambda}I \cap \frac{1}{\kappa\lambda}I$ is an interval with non-empty interior. Take any of its element x. Then, as φ satisfies (3.3), we get

$$\varphi(\kappa\lambda x) = c(\kappa\lambda)\,\varphi(x) + d(\kappa\lambda)$$

and

$$\begin{aligned} \varphi(\kappa\lambda x) &= c(\kappa)\,\varphi(\lambda x) + d(\kappa) = c(\kappa)\left[c(\lambda)\varphi(x) + d(\lambda)\right] + d(\kappa) = \\ &= c(\kappa)\,c(\lambda)\varphi(x) + c(\kappa)\,d(\lambda) + d(\kappa). \end{aligned}$$

Therefore, since φ is not constant on the interval $I \cap \frac{1}{\lambda}I \cap \frac{1}{\kappa\lambda}I$, we obtain (3.4) and the equality

$$d(\kappa\lambda) = c(\kappa)\,d(\lambda) + d(\kappa) \tag{3.5}$$

for all $\kappa, \lambda \in \left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right)$. If $c(\lambda) = 0$ for a $\lambda \in \left(\alpha, \alpha^{-1}\right)$, then, by virtue of (3.4), we would get $c(1) = c(\lambda)c\left(\lambda^{-1}\right) = 0$, hence, because of (3.3), $\varphi(x) = d(1)$ for each $x \in I$, i.e. φ would be constant. Thus, c does not vanish, which, due to the continuity of c and equality (3.4), means that c is positive. Now the function $f : (\log\alpha, -\log\alpha) \to \mathbb{R}$, given by $f(x) = \log c(\mathrm{e}^x)$, satisfies the Cauchy equation

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in (\log\alpha, -\log\alpha)$ with $x + y \in (\log\alpha, -\log\alpha)$, in particular for all $x, y \in \left(\frac{1}{2}\log\alpha, -\frac{1}{2}\log\alpha\right)$. Applying [14, Theorem 13.5.2] or [1, Theorem 1. p. 46] and taking into account the continuity of f we find a number $p \in \mathbb{R}$ such that $f(x) = px$ for every $x \in (\log\alpha, -\log\alpha)$, that is

$$c(\lambda) = \lambda^p, \qquad \lambda \in \left(\alpha, \alpha^{-1}\right). \tag{3.6}$$

Assume that $p \neq 0$. Then, on account of (3.5),

$$d(\kappa\lambda) = \kappa^p d(\lambda) + d(\kappa)$$

and, by symmetry,

$$d(\kappa\lambda) = \lambda^p d(\kappa) + d(\lambda),$$

hence $d(1) = 0$ and

$$(1 - \lambda^p)\, d(\kappa) = (1 - \kappa^p)\, d(\lambda)$$

for all $\kappa, \lambda \in \left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right)$. In other words

$$\frac{d(\kappa)}{1-\kappa^p} = \frac{d(\lambda)}{1-\lambda^p}, \qquad \kappa, \lambda \in \left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right) \setminus \{1\},$$

which means that

$$d(\lambda) = b\left(1 - \lambda^{\kappa}\right), \qquad \lambda \in \left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right), \tag{3.7}$$

with some $b \in \mathbb{R}$. Take any $x_0 \in I$ and $x \in I \cap \left(\alpha^{\frac{1}{2}} x_0, \alpha^{-\frac{1}{2}} x_0\right)$. Then, by (3.3), (3.6) and (3.7), we have

$$\begin{aligned} \varphi(x) &= \varphi\left(\frac{x}{x_0} x_0\right) = c\left(\frac{x}{x_0}\right)\varphi(x_0) + d\left(\frac{x}{x_0}\right) = \\ &= \left(\frac{x}{x_0}\right)^p \varphi(x_0) + b\left(1 - \left(\frac{x}{x_0}\right)^p\right) = \frac{\varphi(x_0) - b}{x_0^p} x^p + b. \end{aligned}$$

This means that

$$\varphi(x) = a(x_0)\, x^p + b, \qquad x \in I \cap \left(\alpha^{\frac{1}{2}} x_0, \alpha^{-\frac{1}{2}} x_0\right),$$

with some $a(x_0) \in \mathbb{R}$. If $x \in I \cap \left(\alpha^{\frac{1}{2}} x_0, \alpha^{-\frac{1}{2}} x_0\right)$, then also $x \in I \cap \left(\alpha^{\frac{1}{2}} x, \alpha^{-\frac{1}{2}} x\right)$, so

$$a(x) x^p + b = \varphi(x) = a(x_0)\, x^p + b,$$

hence $a(x) = a(x_0)$. In such a way we see that

$$\varphi(x) = a x^p + b, \qquad x \in I,$$

with some $a \in \mathbb{R}$. This, according to (3.3) and (3.6), gives $d(\lambda) = b\left(1 - \lambda^p\right) = b\left(1 - c(\lambda)\right)$ for each $\lambda \in \left(\alpha, \alpha^{-1}\right)$.

Finally, consider the case $p=0$. Then (3.6) and (3.5) become $c(\lambda)=1$ for every $\lambda\in\left(\alpha,\alpha^{-1}\right)$ and

$$d(\kappa\lambda)=d(\kappa)+d(\lambda)$$

for all $\kappa,\lambda\in\left(\alpha^{\frac{1}{2}},\alpha^{-\frac{1}{2}}\right)$, respectively. Using [14, Theorem 13.5.2] or [1, Theorem 1. p. 46] again and taking into account the continuity of d we get

$$d(\lambda)=a\log\lambda,\qquad \lambda\in\left(\alpha,\alpha^{-1}\right),$$

with some $a\in\mathbb{R}$. Now taking any $x_0\in I$ we see that (3.3) gives

$$\begin{aligned}\varphi(x) &= \varphi\left(\frac{x}{x_0}x_0\right)=c\left(\frac{x}{x_0}\right)\varphi\left(x_0\right)+d\left(\frac{x}{x_0}\right)= \\ &= \varphi\left(x_0\right)+a\log\frac{x}{x_0}=a\log x+\left(\varphi\left(x_0\right)-a\log x_0\right)=a\log x+b\left(x_0\right)\end{aligned}$$

for every $x\in I\cap\left(\alpha x_0,\alpha^{-1}x_0\right)$ with some $b\left(x_0\right)\in\mathbb{R}$. As previously, it can be shown that b does not depend on x_0, and thus

$$\varphi(x)=a\log x+b$$

for every $x\in I$. ∎

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Assume that $A_{\varphi,\mu}$ is conditionally homogeneous, that is

(3.8)
$$\varphi^{-1}\left(\int_0^1\varphi\left(t\lambda x+(1-t)\lambda y\right)d\mu(t)\right)=\lambda\varphi^{-1}\left(\int_0^1\varphi\left(tx+(1-t)y\right)d\mu(t)\right)$$

for all $\lambda\in(0,\infty)$ and $x,y\in I$ with $\lambda x,\lambda y\in I$, and thus for all $\lambda\in\left(\alpha,\alpha^{-1}\right)$ and $x,y\in I\cap\frac{1}{\lambda}I$. Defining $\psi_\lambda:\frac{1}{\lambda}I\to\mathbb{R}$ by

$$\psi_\lambda(x)=\varphi\left(\lambda x\right)$$

and inserting it into the left side of equality (3.8) we get

$$\psi_\lambda^{-1}\left(\int_0^1\psi_\lambda\left(tx+(1-t)y\right)d\mu(t)\right)=\varphi^{-1}\left(\int_0^1\varphi\left(tx+(1-t)y\right)d\mu(t)\right)$$

for all $\lambda\in\left(\alpha,\alpha^{-1}\right)$ and $x,y\in I\cap\frac{1}{\lambda}I$. It is clear that all the assumptions of Theorem 2.1 are fulfilled. On account of this we infer that either μ is a

Dirac measure concentrated at a point of $[0,1]$, or there are functions $c, d : \left(\alpha, \alpha^{-1}\right) \to \mathbb{R}$ such that c does not vanish and

$$\psi_\lambda(x) = c(\lambda)\varphi(x) + d(\lambda), \qquad \lambda \in \left(\alpha, \alpha^{-1}\right),\ x \in I \cap \frac{1}{\lambda} I.$$

Now the possible forms of φ, c and d follow directly from Proposition 3.2.

The opposite direction can be verified trivially. ■

Corollary 3.3. *Assume that $I \subset (0, \infty)$. The only conditionally homogeneous Makó-Páles means on I are these of the forms*

$$\left(\int_0^1 (tx + (1-t)y)^p \, d\mu(t)\right)^{\frac{1}{p}}$$

with an $p \in \mathbb{R} \setminus \{0\}$ and

$$\exp\left(\int_0^1 \log\left(tx + (1-t)y\right) d\mu(t)\right),$$

where μ is a probabilistic Borel measure on $[0,1]$.

Proof. If μ is a Dirac measure concentrated at a point $\tau \in [0,1]$, then

$$A_{\varphi,\mu}(x,y) = \tau x + (1-\tau)y = \int_0^1 (tx + (1-t)y)\, d\mu(t),$$

so $A_{\varphi,\mu}$ is of the first form with $p = 1$. The rest follows directly from Proposition 3.2. ■

4. Translativity

Now we deal with the characterization of translative means in the class of Makó-Páles means. Here we do not assume further regularity on the generating functions.

Theorem 4.1. *Let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotonic function and μ be a probability Borel measure on $[0,1]$. The mean $A_{\varphi,\mu}$ is conditionally translative, i.e.*

$$A_{\varphi,\mu}(x+\lambda, y+\lambda) = A_{\varphi,\mu}(x,y) + \lambda$$

for all $x, y \in I$ and $\lambda \in \mathbb{R}$ with $x+\lambda, y+\lambda \in I$ if and only if μ is a Dirac measure or there are $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that either

$$\varphi(x) = a\mathrm{e}^{px} + b, \qquad x \in I,$$

with some $p \in \mathbb{R} \setminus \{0\}$, or

$$\varphi(x) = ax + b, \qquad x \in I. \tag{4.1}$$

Proof. Assume that the mean $A_{\varphi,\mu}$ is conditionally translative. Then, for every $\lambda \in \mathbb{R}$, the formula

$$\psi_\lambda(x) = \varphi\left(x+\lambda\right)$$

defines a function $\psi_\lambda : I - \lambda \to \mathbb{R}$. Now, taking into consideration the conditional translativity of $A_{\varphi,\mu}$ and setting $\beta = \sup I - \inf I$, we see that

$$\psi_\lambda^{-1}\left(\int_0^1 \psi_\lambda\left(tx+(1-t)y\right)d\mu(t)\right) = \varphi^{-1}\left(\int_0^1 \varphi\left(tx+(1-t)y\right)d\mu(t)\right)$$

for all $\lambda \in (-\beta, \beta)$ and $x, y \in I \cap (I - \lambda)$. Making use of Theorem 2.1 we conclude that either μ is a Dirac measure concentrated at a point of $[0,1]$, or there are functions $c : (-\beta,\beta) \to \mathbb{R} \setminus \{0\}$ and $d : (-\beta,\beta) \to \mathbb{R}$ such that

$$\psi_\lambda(x) = c(\lambda)\varphi(x) + d(\lambda),$$

or φ satisfies the equation

$$\varphi(x+\lambda) = c(\lambda)\varphi(x) + d(\lambda)$$

for all $\lambda \in (-\beta,\beta)$ and $x \in I \cap (I-\lambda)$. Assume the second possibility. Then, putting $I_0 = \exp(I)$, $\varphi_0 = \varphi \circ \log|_{I_0}$, $c_0 = c \circ \log|_{I_0}$, $d_0 = d \circ \log|_{I_0}$ and $\alpha = \mathrm{e}^{-\beta}$, we see that for all $\lambda \in \left(\alpha, \alpha^{-1}\right)$ and $x \in I_0 \cap \frac{1}{\lambda} I_0$ we have

$$\begin{aligned} \varphi_0(\lambda x) &= \varphi\left(\log x + \log \lambda\right) = c\left(\log \lambda\right)\varphi\left(\log x\right) + d\left(\log \lambda\right) = \\ &= c_0\left(\lambda\right)\varphi_0\left(x\right) + d_0\left(\lambda\right). \end{aligned}$$

Now, by Proposition 3.2, there are $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that either

$$\varphi_0(x) = ax^p + b, \qquad x \in I_0,$$

with some $p \in \mathbb{R} \setminus \{0\}$, or

$$\varphi_0(x) = a \log x + b, \qquad x \in I_0,$$

and we come to the *if* assertion. The converse is obvious. ■

Corollary 4.2. *The only conditionally translative Makó-Páles means on I are these of the forms*

$$\frac{1}{p} \log \left(\int_0^1 \mathrm{e}^{p(tx+(1-t)y)} d\mu(t) \right)$$

with a $p \in \mathbb{R} \setminus \{0\}$, and

$$\tau x + (1-\tau) y$$

with a $\tau \in [0,1]$.

Proof. In view of Theorem 4.1, it is enough to observe only that if φ is of form (4.1), then

$$\begin{aligned} A_{\varphi,\mu}(x,y) &= \int_0^1 (tx + (1-t)y) d\mu(t) = x \int_0^1 t d\mu(t) + y \left(1 - \int_0^1 t d\mu(t)\right) \\ &= \tau x - (1-\tau) y \end{aligned}$$

for all $x, y \in I$, where τ is the first moment of the measure μ. ■

5. Applications

With the help of the previous results, we can determine the conditionally homogeneous and translative weighted quasi-arithmetic means and Lagrangian means, respectively. These results are not new, see [5] regarding the case of weighted quasi-arithmetic means and [10] regarding the case of Lagrangian means. However, using our earlier theorems we can give new and probably shorter proofs of these claims.

Let $I \subset \mathbb{R}$ be an interval, $\varphi \colon I \to \mathbb{R}$ be a continuous, strictly monotonic function and $\tau \in [0,1]$ be a given real number.

A mean $A_{\varphi,\tau} \colon I \times I \to I$ is called a *weighted quasi-arithmetic mean* if it has the form

$$A_{\varphi,\tau}(x,y) = \varphi^{-1}(\tau\varphi(x) + (1-\tau)\varphi(y)).$$

If $\mu = (1-\tau)\delta_0 + \tau\delta_1$, then $A_{\varphi,\mu} = A_{\varphi,\tau}$. This actually enlightens the role of the measure μ while considering the Makó-Páles means.

A mean $L_\varphi\colon I \times I \to I$ is called a *Lagrangian mean* if it can be written in the form

$$L_\varphi(x,y) = \begin{cases} \varphi^{-1}\left(\frac{1}{y-x}\int\limits_x^y \varphi(t)dt\right), & \text{if} \quad x \neq y, \\ x, & \text{if} \quad x = y. \end{cases}$$

If μ is the Lebesgue measure λ on $[0,1]$, then clearly $A_{\varphi,\lambda} = L_\varphi$.

Corollary 5.1. *Assume that $I \subset (0,\infty)$. The only conditionally homogeneous weighted quasi-arithmetic means on I are the weighted power means of the form*

$$(\tau x^p + (1-\tau)y^p)^{\frac{1}{p}}$$

with a $p \in \mathbb{R}\setminus\{0\}$ and a $\tau \in [0,1]$, and the weighted geometric means of the form

$$x^\tau y^{1-\tau}$$

with a $\tau \in [0,1]$.

Proof. Any weighted quasi-arithmetic mean is a Makó-Páles mean with μ which is a convex combination of the Dirac measures δ_0 and δ_1. Using this, after a short calculation we get our statement from Corollary 3.3. ∎

Corollary 5.2. *The only conditionally translative weighted quasi-arithmetic means on I are these of the forms*

$$\frac{1}{p}\log\left(\tau e^{px} + (1-\tau)e^{py}\right)$$

with a $p \in \mathbb{R}\setminus\{0\}$ and $\tau \in [0,1]$, and the weighted arithmetic means of the form

$$\tau x + (1-\tau)y$$

with a $\tau \in [0,1]$.

Proof. As above we identify μ with a convex combination of the Dirac measures concentrated to 0 and 1. Using Corollary 4.2 we get our result. ∎

Remark 5.3. The class of means occurring in Corollary 5.2 plays an important role in the invariance theory of quasi-arithmetic and weighted quasi-arithmetic means. For the details see e.g. [8] and [12].

Corollary 5.4. *Assume that $I \subset (0,\infty)$. The only conditionally homogeneous Lagrangian means on I are these of the forms*

$$\left(\frac{1}{y-x}\int_x^y s^p ds\right)^{\frac{1}{p}}$$

with a $p \in \mathbb{R}\setminus\{0\}$ and

$$\exp\left(\frac{1}{y-x}\int_x^y \log s ds\right)$$

whenever $x \neq y$.

Proof. Using the fact that the Lagrange mean generated by φ can be derived from a Makó-Páles mean $A_{\varphi,\mu}$ choosing μ as the Lebesgue measure, we get our statement immediately from Corollary 3.3. ■

Corollary 5.5. *The only conditionally translative Lagrangian means on I are the arithmetic mean and this of the form*

$$\frac{1}{p}\log\left(\frac{1}{y-x}\left(\frac{e^{p(y+1)}}{y+1}-\frac{e^{p(x+1)}}{x+1}\right)\right)$$

with a $p \in \mathbb{R}\setminus\{0\}$ whenever $x \neq y$.

Proof. As in the previous proof we get the statement from Corollary 4.2 after taking μ as the Lebesgue measure. ■

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