Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 135–142

ON TRANSLATIONS IN HYPERBOLIC GEOMETRY OF ARBITRARY (FINITE OR INFINITE) DIMENSION > 1

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday, in friendship

Communicated by Antal Járai

(Received March 01, 2013; accepted June 10, 2013)

Abstract. Hyperbolic geometry of the plane was discovered by J. Bolyai (1802–1860), C.F. Gauß (1777–1855), and N. Lobachevski (1793–1856). – In our book [3] we associate to every real vector space X of finite or infiinite dimension > 1, and equipped with a fixed inner product $\delta : X \times X \to \mathbb{R}$, a hyperbolic geometry such that $(X, \delta), (X', \delta')$ are isomorphic if, and only if, the associated hyperbolic geometries are isomorphic. – In this paper we present a common treatment of translations in euclidean and hyperbolic geometry of arbitrary (finite or infinite) dimension greater than one.

1. Introduction

Let $X = (X, \delta)$ be a real inner product space of arbitrary (finite or infinite) dimension greater than one. Here $\delta : X \times X \to \mathbb{R}$ designates a fixed *real inner product* of X. The main result of chapter 1 of our book [3], namely Theorem 7, p. 21, is a common characterization of euclidean and hyperbolic geometry over X: let T be a separable translation group of X with axis $e \in X$ (see sections 7, 8

Key words and phrases: Hyperbolic geometry of arbitrary dimensional real inner product spaces, hyperbolic translations.
2010 Mathematics Subject Classification: 39 B 22, 39 72, 51 M 10.
https://doi.org/10.71352/ac.40.135

of chapter 1 of [3]) and let d be a function, not identically zero, from $X \times X$ into the set $\mathbb{R}_{\geq 0}$ of all non–negative real numbers, satisfying $d(x,y) = d\big(\varphi(x), \varphi(y)\big)$ and, moreover, $d(\beta e, 0) = d(0, \beta e) = d(0, \alpha e) + d(\alpha e, \beta e)$ for all $x, y \in X$, all $\varphi \in T \cup O(X)$ where $O(X)$ is the group of orthogonal bijections of X, and for all real α, β with $0 \leq \alpha \leq \beta$. Then, up to isomorphism, there exist exactly two geometries with distance function d in question, namely the euclidean or the hyperbolic geometry over X. The methods of the proof of Theorem 7 in question are based on the solution of special real functional equations (see J. Aczél [1], J. Aczél and J. Dhombres [2], Z. Daróczy [8], [9], M. Kuczma [10], and others).

2. The metric spaces (X, eucl) and (X, hyp)

Let X be a real inner product space of (finite or infinite) dimension greater than one. The metric space (X, eucl) consists of X as the set of *points* and of the *distance function*

$$\mathrm{eucl}\ (x,y) := \|x - y\| := \sqrt{(x-y)^2} \tag{1}$$

for $x, y \in X$. The metric space (X, hyp) is defined by the set X of *points* and by means of $\mathrm{hyp}\ (x,y) \geq 0$ for $x, y \in X$ and

$$\cosh \mathrm{hyp}(x,y) := \sqrt{1+x^2}\sqrt{1+y^2} - xy. \tag{2}$$

A set $S \neq \emptyset$ together with a mapping $d : S \times S \to \mathbb{R}$ is called a *metric space* (S, d) provided

(i) $d(x,y) = 0$ if, and only if, $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$

hold true for all $x, y, z \in S$.

Observe $d(x,y) \geq 0$ for all $x, y \in S$, since (i), (ii), (iii) imply

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y).$$

Suppose that (S, d) is a metric space and that $c \in S$ and $\varrho \geq 0$ is in $\mathbb{R}$. Then

$$B(c, \varrho) := \{x \in S \mid d(c,x) = \varrho\} \tag{3}$$

is the *ball* with *center* c and *radius* ϱ. Observe $B(c,0)=\{c\}$. If a,b are distinct elements of S, then

(4) $$g(a,b) := \big\{x \in S \mid B\big(a,d(a,x)\big) \cap B\big(b,d(b,x)\big) = \{x\}\big\}$$

will be called a *g–line* (see [3]) of (S,d), and

(5) $$\{x \in S \mid d(a,x) = d(b,x)\}$$

a *hyperplane*. Observe $g(a,b)=g(b,a)$ for $a \neq b$.

The lines of $(X,\text{eucl}),(X,\text{hyp})$ are given by the sets

(6) $$\{p+\xi q \mid \xi \in \mathbb{R}\} \text{ with } p,q \in X \text{ such that } q \neq 0,$$

(7) $$\{pC_\xi + qS_\xi \mid \xi \in \mathbb{R}\} \text{ with } p,q \in X, pq=0, q^2=1,$$

respectively, where we wrote $\cosh\xi =: C_\xi$ and $\sinh\xi =: S_\xi$. The hyperplanes of $(X,\text{eucl}),(X,\text{hyp})$ are given by the sets

(8) $$\{x \in X \mid ax=\alpha\} \text{ with } a \in X\backslash\{0\}, \alpha \in \mathbb{R},$$

(9) $$\{\gamma p C_\xi + yS_\xi \mid \xi \in \mathbb{R}, y \in p^\perp, y^2=1\} \text{ with } p \in X, p^2=1, \gamma \in \mathbb{R}_{\geq 0},$$

respectively (see [3]).

Let now (X,d) be one of the metric spaces (X,eucl) or (X,hyp) where $X=(X,\delta)$ is an arbitrary (finite or infinite) dimensional real vector space, $\dim X > 1$, with a fixed real inner product δ. Observe that there exist infinite dimensional real vector spaces X with real inner products δ,δ' such that $(X,\delta) \not\cong (X,\delta')$.

A bijection f of X is called a *motion* of (X,d) (see [3], p. 76) provided, i.e. if, and only if,

$$d(x,y) = d\big(f(x),f(y)\big)$$

holds true for all $x,y \in X$.

The following statement is important.

Proposition 1. *Motions of (X,d) map g–lines onto g–lines.*

Proof. a) If f is a motion of (X,d), then f^{-1} as well, since

$$d\big(f^{-1}(x),f^{-1}(y)\big) = d\Big(f\big(f^{-1}(x)\big),f\big(f^{-1}(y)\big)\Big)$$

for all $f^{-1}(x),f^{-1}(y) \in X$, i.e. for all $x,y \in X$.

b) From (3) we get for a motion f of (X, d),

$$f\big(B(c,\varrho)\big) = \{f(x) \in X \mid d(c,x) = \varrho\} = \big\{f(x) \in X \mid d\big(f(c), f(x)\big) = \varrho\big\},$$

i.e.

$$f\big(B(c,\varrho)\big) = \{y \in X \mid d\big(f(c), y\big) = \varrho\} = B\big(f(c), \varrho\big).$$

c) $f\big(g(a,b)\big)$ (see (4)) consists of all $f(x) \in X$ satisfying

$$B\big(a, d(a,x)\big) \cap B\big(b, d(b,x)\big) = \{x\}.$$

This last equation is equivalent with

$$B\Big(f(a), d\big(f(a), f(x)\big)\Big) \cap B\Big(f(b), d\big(f(b), f(x)\big)\Big) = \{f(x)\}.$$

Put $f(a) =: p, f(b) =: q$. Hence

$$f\big(g(a,b)\big) = \big\{y \in X \mid B\big(p, d(p,y)\big) \cap B\big(q, d(q,y)\big) = \{y\}\big\} = g(p,q). \qquad \blacksquare$$

Remark. If we define the g–lines of (X, d) equivalently as lines of L.M. Blumenthal (see [3], section 2.2), Proposition 1 can be derived as shown on p. 42, [3], along the rows before Proposition 5.

3. Translations of $(X, d), d \in \{\text{eucl, hyp}\}$

Let $e \in X$ be given with $e^2 = 1$. Put $H := e^\perp$, i.e. $H = \{x \in X \mid xe = 0\}$, and $\varrho : H \times \mathbb{R} \to \mathbb{R}$ by means of

(10) $$\varrho(h, \lambda) = \sinh \lambda \cdot \sqrt{1 + h^2} \text{ for } d = \text{hyp},$$

(11) $$\varrho(h, \lambda) = \lambda \text{ for } d = \text{eucl},$$

and all $(h, \lambda) \in H \times \mathbb{R}$, according to section 1.7, [3]. For $t \in \mathbb{R}$ we define the *translation* $T_t^e : X \to X$ of (X, d) with axis e,

(12) $$T_t^e\big(h + \varrho(h,\tau)e\big) = h + \varrho(h, \tau + t)e,$$

by observing that to $x \in X$ there exist uniquely determined $\overline{x} \in H$ and $x_0 \in \mathbb{R}$ with $x = \overline{x} + x_0 e$, namely $xe = (\overline{x} + x_0 e)e = x_0$ and $\overline{x} = x - x_0 e$, and by defining $h \in H$ and $\tau \in \mathbb{R}$ for $x \in X$ by means of $x =: h + \varrho(h,\tau)e$, i.e. by $h = \overline{x}$ and $\varrho(h,\tau) = x_0$. In other words: with

$$x = \overline{x} + x_0 e = \overline{x} + \varrho(\overline{x},\tau)e = \overline{x} + \sinh\tau \cdot \sqrt{1+\overline{x}^2}e$$

the translation (12) reads as

$$T_t^e(\overline{x} + \sinh\tau \cdot \sqrt{1+\overline{x}^2}e) = \overline{x} + \sinh(\tau + t)\sqrt{1+\overline{x}^2}e \tag{13}$$

for $d = \text{hyp}$ and with

$$x = \overline{x} + x_0 e = \overline{x} + \varrho(\overline{x},\tau)e = \overline{x} + \tau e$$

as

$$T_t^e(\overline{x} + \tau e) = \overline{x} + (\tau + t)e, \text{i.e.}$$

$$T_t^e(x) = x + te, \tag{14}$$

for $d = \text{eucl}$.

Remark. $H = e^\perp$ is the euclidean hyperplane (8),

$$\{x \in X \mid ex = 0\},$$

in case $d = \text{eucl}$, and the hyperbolic hyperplane (9),

$$\{0 \cdot e \cdot C_\xi + yS_\xi \mid \xi \in \mathbb{R}, y \in e^\perp, y^2 = 1\},$$

for $d = \text{hyp}$.

Remark. Observe that the functions $\varrho : H \times \mathbb{R} \to \mathbb{R}$ in (10), (11) are characterized by Theorem 7 ([3], p. 21) as kernels of suitable translation groups $\{T_t^e \mid t \in \mathbb{R}\}$ leading to hyperbolic, euclidean geometry, respectively.

According to our definition of the translation $T_t^e : X \to X$ we defined here the set of all translations of (X,d) by

$$TL(X,d) = \left\{T_t^e \mid t \in \mathbb{R} \text{ and } e \in X \text{ with } e^2 = 1\right\}$$

with

$$T_t^e\big(x = h + \varrho(h,\tau)e\big) = h + \varrho(h,\tau + t)e$$

for all $x \in X$, i.e. for all $h \in H = e^\perp$ and all $\tau \in \mathbb{R}$. We also have

$$T_t^e(x) = x + [(xe)(\cosh t - 1) + \sqrt{1+x^2}\sinh t]e$$

(see (1.8) of section 1.7, [3]) in the case $d = \text{hyp}$.

Remark. In [7] we define as hyperbolic translation of X besides $\mu = \mathrm{id}$ every hyperbolic motion $\mu \neq \mathrm{id}$ of X with the existence of an element $a \neq 0$ in X such that $0 \neq \mu(x) - x \in \mathbb{R}a$ holds true for all $x \in X$: this, especially, means that there is no $x \in X$ with $\mu(x) = x$. For the case $\dim X = 2$ compare the book [4] (here p. 163, or even section 3.4) concerning hyperbolic translations and hyperbolische Schubspiegelungen. In comparison with the planar case, observe, that to every $T_t^j, t \in \mathbb{R}, j \in X$ with $j^2 = 1$, there exists a hyperbolic line g remaining fixed, in its entirety, under T_t^j. In fact, take the hyperbolic line

$$g = \{p \cdot \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}, p = 0 \text{ and } q = j.$$

Now

$$T_t^j(j \sinh \xi) = T_t^j(0 + \sinh \xi \cdot \sqrt{1 + 0^2} j) = 0 + \sinh(\xi + t) j$$

implies $T_t^j(g) = g$.

Theorem 2. *Take a fixed element $e \in X$ with $e^2 = 1$. Then*

$$TL(X, d) = \{\alpha T_t^e \alpha^{-1} \mid \alpha \in O(X) \text{ and } t \in \mathbb{R}\}. \tag{15}$$

Proof. Given $t \in \mathbb{R}$ and $j \in X$ with $j^2 = 1$. According to step A of the proof of Theorem 7 ([3], section 1.11) there exists $\gamma \in O(X)$ with $\gamma(j) = e$. Put $\gamma^{-1} =: \alpha$. For all $x = h + \varrho(h, \tau)e$ with $h := \overline{x}$ we would like to prove

$$L := \alpha T_t^e(x) = T_t^j \alpha(x) =: R. \tag{16}$$

Obviously, $\alpha(h) \cdot j = \alpha(h)\alpha(e) = he = 0$, and

$$L = \alpha\big(h + \varrho(h, \tau + t)e\big) = \alpha(h) + \varrho(h, \tau + t)j.$$

Moreover, $\alpha(h) \in j^\perp$,

$$R = T_t^j \alpha(x) = T_t^j \big(\alpha(h) + \varrho(h, \tau)j\big),$$

$$\varrho(h, \tau) = \sqrt{1 + h \cdot h} \sinh \tau = \sqrt{1 + \alpha(h) \cdot \alpha(h)} \sinh \tau = \varrho\big(\alpha(h), \tau\big),$$

and $\varrho(h, \tau + t) = \varrho\big(\alpha(h), \tau + t\big)$ as well. Hence

$$L = \alpha(h) + \varrho\big(\alpha(h), \tau + t\big)j = T_t^j \Big(\alpha(h) + \varrho\big(\alpha(h), \tau\big)j\Big),$$

and $R = T_t^j \Big(\alpha(h) + \varrho\big(\alpha(h), \tau\big)j\Big) = L$, i.e. we obtain

$$\alpha T_t^e \alpha^{-1} = T_t^j$$

and (16) for $d = \mathrm{hyp}$. In the case $d = \mathrm{eucl}$, of course, the proof of $\varrho(h, \tau) = \varrho\big(\alpha(h), \tau\big)$ is trivial. – The remaining question is whether every $\alpha T_t^e \alpha^{-1}$

must be a translation of (X,d) in the case $t \in \mathbb{R}$ and $\alpha \in O(X)$? Now given $\alpha T_t^e \alpha^{-1}$, put $\alpha(e) =: j$ and consider T_t^j. Observe

$$\alpha T_t^e \big(x = h + \varrho(h,\tau)e\big) = \alpha\big(h + \varrho(h,\tau+t)e\big) = \alpha(h) + \varrho(h,\tau+t)j,$$

$$T_t^j \alpha\big(x = h + \varrho(h,\tau)e\big) = T_t^j\big(\alpha(h) + \varrho(h,\tau)j\big)$$

together with $\varrho(h,\tau) = \sinh\tau \cdot \sqrt{1+h^2} = \sinh\tau\sqrt{1+[\alpha(h)]^2}$, i.e. $\varrho(h,\tau) = \varrho\big(\alpha(h),\tau\big)$ for $d = \text{hyp}$. Hence

$$T_t^j \alpha(x) = T_t^j \Big(\alpha(h) + \varrho\big(\alpha(h),\tau\big)j\Big) = \alpha(h) + \varrho\big(\alpha(h),\tau+t\big)j,$$

i.e. $\alpha T_t^e(x) = \alpha(h) + \varrho(h,\tau+t)j = T_t^j\alpha(x)$. Thus

$$\alpha T_t^e(x) = T_t^j \alpha(x)$$

for all $x \in X$, i.e. $\alpha T_t^e \alpha^{-1}$ is the translation T_t^j. ■

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