ON THE SUM OF DIVISORS FUNCTION

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Dedicated to Professors Zoltán Daróczy and Imre Kátai
on their 75th anniversary

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Abstract. The following assertion is proved. Let Q_1, Q_2 be odd primes, $A_{Q_1,Q_2}(x)$ be the number of those $n \leq x$ for which $Q_1 \nmid \sigma(n), Q_2 \nmid \sigma(n+1)$ simultaneously hold. Then $A_{Q_1,Q_2}(x) \geq \frac{cx}{(\log x)^5}$, if $x > X_0$. c, X_0 are positive constants.

1. Introduction

1.1. Notation

$\mathcal{P}$ = set of primes. $\varphi(n)$ = Euler's totient function, $\sigma(n)$ = sum of divisors function, $\tau(n)$ = number of divisors.

Let $x_1 = \log x$, $x_2 = \log x_1$, ..., $x_{k+1} = \log x_k$.

1.2. Formulation of the theorems

In his paper [4] Kátai (Theorem 4) proved the following assertion:

Let $\lambda > 2$, $I_x = \left[\frac{\lambda x_2}{x_3}, x_2\right]$, $Q_1, Q_2 \in \mathcal{P}$,

$$E_{Q_1,Q_2}(x) := \#\left\{n \leq x \ : \ Q_1 \nmid \varphi(n),\ Q_2 \nmid \varphi(n+1)\right\}.$$

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Then, uniformly for $Q_1, Q_2 \in I_x$,

$$\frac{1}{x} E_{Q_1,Q_2}(x) = \big(1 + O_x(1)\big)\, \frac{B}{2}\, \kappa_1\, \kappa_2,$$

where

$$\kappa_j = \exp\left(-\frac{x_2}{Q_j - 1}\right), \qquad (j = 1, 2),$$

and

$$B = \prod_{\substack{p \geq 3 \\ p \in \mathcal{P}}} \left(1 - \frac{2}{p(p-1)}\right).$$

Kátai notes that similar theorem can be proved for σ instead of φ, and that he is unable to count the asymptotic of those n for which $3 \nmid \varphi(n)$, $3 \nmid \varphi(n+1)$ simultaneously holds.

In this paper we shall investigate the function

$$A_{Q_1,Q_2}(x) := \#\left\{n \leq x \ : \ Q_1 \nmid \sigma(n),\ Q_2 \nmid \sigma(n+1)\right\},$$

where Q_1, Q_2 are arbitrary odd primes, $Q_1 = Q_2$ is included.

Theorem 1. *If* $Q_1, Q_2 \in \mathcal{P}$, $Q_1 \neq 2$, $Q_2 \neq 2$, *then there are constants* $c > 0$ *and* X_0 *such that*

$$A_{Q_1,Q_2}(x) \geq \frac{cx}{(\log x)^5},$$

if $x > X_0$.

Theorem 2. *Let* $Q \in \mathcal{P}$, $Q \neq 2$,

$$B_Q(x) = \#\left\{p \leq x \ : \ p \in \mathcal{P},\ Q \nmid \sigma(p+1)\right\}.$$

Then there are constants $c > 0$ *and* X_0 *such that*

$$B_Q(x) \geq \frac{cx}{(\log x)^5}.$$

Our theorems follow from some variants of known, deep theorems.

2. Main auxiliary results

2.1. Let $Q \in \mathcal{P}$, $Q \neq 2$, $\chi(n)$ be a character mod Q, such that $\chi(-1) = -1$. Let $r(n) = \sum_{d|n} \chi(d)$,

$$T(x) = \sum_{\substack{p \le x \\ p+1 \not\equiv 0 \pmod{Q}}} r(Qp+1)\,|\mu(Qp+1)|.$$

Theorem 3. *We have*

$$T(x) = A_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right),$$

where δ and A_0 are positive constants.

2.2. Let $Q_1 \neq Q_2$, Q_1, Q_2 be odd primes, $\chi(n)$ be a character mod Q_2, such that $\chi(-1) = -1$. Let $r(n) = \sum_{d|n} \chi(d)$. Let $A = Q_1^a Q_2^b$, $a \in \{1,2\}$, $b \in \{1,2\}$ such that $Q_1 \nmid \sigma(A)$. Let

$$S(x) = \sum_{\substack{p \le x \\ p+1 \not\equiv 0 \pmod{Q_1}}} r(Ap+1)\,|\mu(Ap+1)|.$$

Theorem 4. *We have*

$$S(x) = B_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right),$$

where δ and B_0 are suitable positive constants.

3. Deduction of Theorem 1 and 2 from Theorem 3 and 4

In the case $Q_1 = Q_2 = Q$ let $A = Q$, χ mod Q be a Dirichlet character such that $\chi(-1) = -1$, and $r(n) = \sum_{d|n} \chi(d)$.

If $p + 1 \not\equiv 0 \pmod{Q}$, and $|\mu(Qp+1)|\, r(Qp+1) \neq 0$, then $Qp+1$ is a squarefree number, and $\pi | Qp+1$, $\pi \in \mathcal{P}$ implies that $1 + \chi(\pi) = 2$, consequently $Q \nmid \pi + 1$, thus $Q \nmid \sigma(Qp+1)$. Furthermore $Q \nmid \sigma(Qp) = (Q+1)(p+1)$.

Assume that $Q_1 \neq Q_2$. Let $A = Q_1^a Q_2^b$, where a and b are such positive integers that $Q_1 \nmid (1 + Q_2 + \ldots + Q_2^b$, $Q_2 \nmid 1 + Q_1 + \ldots + Q_1^a$. Observe that

$a, b \in \{1, 2\}$ is a suitable choice. Let χ be a character mod Q_2, such that $\chi(-1) = -1$, and $r(n) = \sum_{d|n} \chi(d)$.

Let $p \not\equiv -1 \pmod{Q_1}$. Then $Q_1 \nmid \sigma(Ap)$, and if $|\mu(Ap+1)|\, r(Ap+1) \neq 0$, then $Q_2 \nmid \sigma(Ap+1)$.

We have

$$T^2(x) \leq A_{Q_1,Q_2}(x) \sum_{\substack{p \leq x \\ p \not\equiv -1 \pmod{Q}}} r^2(Qp+1)\ |\mu(Qp+1)|.$$

Since $r^2(Qp+1) \leq \tau^2(Qp+1)$, and $\sum_{n \leq x} \tau^2(n) \leq cx \cdot x_1^3$, therefore $A_{Q,Q}(x) \gg \gg \frac{x}{x_1^5}$.

We can obtain similarly that

$$S^2(x) \leq B_Q(x) \sum_{\substack{p \leq x \\ p+1 \not\equiv 0 \pmod{Q}}} r^2(Ap+1)\ |\mu(Ap+1)|,$$

and hence that $B_Q(x) \gg \frac{x}{x_1^5}$.

Remark. We could improve these inequalities by using some sieve results.

4. Sketch of the proof of Theorem 3 and 4

The main ingredient of the proof is the inequality due to E. Bombieri and A.I. Vinogradov which is quoted now as Lemma 1.

Let $\pi(z, D, l) = \#\{p \leq z \ :\ p \equiv l \pmod{D}\}$,

$$\text{li}\ z = \int_2^z \frac{du}{\log u}.$$

Lemma 1. (See Elliott [1], Chapter 7.)

$$\sum_{D \leq \frac{\sqrt{x}}{x_1 B}} \max_{\substack{l \pmod{D} \\ (l,D)=1}} \max_{z \leq x} \left| \pi(z, D, l) - \frac{\text{li}\ z}{\varphi(D)} \right| \ll \frac{x}{x_1^A},$$

where $B \geq 2A + 23$.

I. Kátai considered in [4] the sum

$$T(x)=\sum_{p\le x} r_4(p-1)\ |\mu(p-1)|,$$

where $r_4(n)=\sum_{d|n}\chi_4(d)$, χ (mod 4) is the character satisfying $\chi(-1)=-1$, and proved that

$$T(x)=A_0\frac{x}{\log x}+O\left(\frac{x}{(\log x)^{1+\delta}}\right).$$

To prove Theorem 3 we can follow his argument. Let

$$T(x,k)=\sum_{\substack{p\le x\\ p+1\not\equiv 0 \pmod Q\\ Qp+1\equiv 0 \pmod k}} r(Qp+1).$$

Arguing as in [4], we have

$$T(x)=\sum_{d\le x_1^3}\mu(d)T(x,d^2)+O\left(\frac{x}{x_1^{1,5}}\right).$$

Let $k\le x_1^6$, $(k,Q)=1$. Then

$$T(x,k)=\sum_{\substack{p\le x\\ Qp+1\equiv 0(k)\\ p+1\not\equiv 0(Q)}}\ \sum_{Qp+1=uv}\chi(u).$$

Thus

$$T(x,k)=\sum_{\substack{u\le Qx+1\\ (u,Q)=1}}\chi(u)\sum_{\substack{p\le x\\ p\not\equiv -1(Q)\\ Qp+1\not\equiv 0([k,u])}}1.$$

Since $(Q,u)=1$, therefore the right most sum equals

$$\sum_{j=1}^{Q-2}\pi(x,Q[k,u],l_j),$$

where l_j are those residues for which

$$Ql_j+1\equiv 0 \pmod{[k,u]},\quad l_j+1\not\equiv 0 \pmod Q$$

holds. With this modification we can proceed further. The "enveloping sieve" of C. Hooley can be applied.

The proof of Theorem 4 is similar.

References

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