Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 123–127

ON SOME ORTHOGONALLY ADDITIVE FUNCTIONS ON INNER PRODUCT SPACES

Karol Baron (Katowice, Poland)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

Communicated by Antal Járai

(Received March 26, 2013; accepted June 10, 2013)

Abstract. Let E be a real inner product space of dimension at least 2. If $f : E \to E$ satisfies

$$f(x+y) = f(x) + f(y) \quad \text{for all orthogonal } x, y \in E$$

and

$$f(f(x)) = x \quad \text{for } x \in E,$$

then f is additive.

Let E be a real inner product space of dimension at least 2.

A function f mapping E into on abelian group is called orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [3, Corollary 10] and [1, Theorem 1], that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

where a and b are additive functions uniquely determined by f.

Our main result says that every involutory orthogonally additive function is additive.

Key words and phrases: Orthogonally additive and additive function, inner product space, involution.

2010 Mathematics Subject Classification: 39B55, 39B12, 46C99.

The research was supported by the Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).

https://doi.org/10.71352/ac.40.123

Theorem 1. *If $f : E \to E$ is orthogonally additive and*

$$f(f(x)) = x \quad \text{for } x \in E, \tag{2}$$

then f is additive.

Proof. Let $(u|v)$ denote the inner product of $u, v \in E$.

As mentioned above f has form (1) with additive functions $a : \mathbb{R} \to E$ and $b : E \to E$. It follows from (1) that

$$\|f(x)\|^2 = \|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2 \quad \text{for } x \in E,$$

which jointly with (2) and (1) gives

$$\begin{aligned} x &= a(\|f(x)\|^2) + b(f(x)) = \\ &= a(\|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2) + b(a(\|x\|^2) + b(x)) \end{aligned}$$

for $x \in E$. Hence, if $x \in E$ and $r \in \mathbb{Q}$, then

$$\begin{aligned} rx = r^4 a(\|a(\|x\|^2)\|^2) + 2r^3 a((a(\|x\|^2)|b(x))) + r^2 a(\|b(x)\|^2) + \\ + r^2 b(a(\|x\|^2)) + rb(b(x)). \end{aligned}$$

Consequently,

$$b(b(x)) = x \quad \text{and} \quad a((a(\|x\|^2)|b(x))) = 0 \quad \text{for } x \in E. \tag{3}$$

In particular for all $x, y \in E$ we have

$$\begin{aligned} 0 &= a((a(\|x+y\|^2)|b(x+y))) = \\ &= a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x+y)) + (a(\|y\|^2)|b(x))), \end{aligned}$$

i.e.,

$$a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x))) = -a((a(\|y\|^2)|b(x)) + 2(a((x|y))|b(y))).$$

As the function of $x \in E$, the left–hand side is even, whereas the right–hand side is odd, and so on each side we have zero for every $x, y \in E$. Hence

$$a((a(\|x\|^2)|b(y))) = 0 \quad \text{for all orthogonal } x, y \in E. \tag{4}$$

Now, if $z \in E$ and $\alpha \in (0, \infty)$, then finding an $x \in E$ such that $x \perp b(z)$ and $\|x\|^2 = \alpha$ and applying (3) and (4) we see that

$$a((a(\alpha)|z)) = a((a(\|x\|^2)|b(b(z)))) = 0.$$

This shows that

$$(5) \qquad a((a(\alpha)|x)) = 0 \quad \text{for } \alpha \in \mathbb{R} \text{ and } x \in E.$$

Suppose $a(\alpha) \neq 0$ for some $\alpha \in \mathbb{R}$. Then

$$\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right) = \alpha$$

and by (5) we have

$$a(\alpha) = a\left(\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right)\right) = 0.$$

The contradiction obtained proves that $a = 0$ and (1) gives $f = b$. ∎

Remark 1. Let H_0 be a basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$ and let H be a basis of the vector space E over $\mathbb{Q}$. Then (cf. [2, Theorem 4.2.3])

$$\mathfrak{c} = \text{card } H_0 \leq \text{ card } H.$$

If H_1 and H_2 are disjoint subsets of H such that

$$1 \leq \text{card } H_1 \leq \mathfrak{c} \quad \text{and} \quad \text{card } H_2 = \text{ card } H,$$

and $a : \mathbb{R} \to E$ and $b : E \to E$ are additive functions such that

$$a(H_0) = H_1, \quad b(H) = H_2$$

and b is injective, then the function $f : E \to E$ given by (1) is orthogonally additive, injective and it is not additive.

To see that f is injective it is enough to observe that if $x, y \in E$ and $f(x) = f(y)$, then

$$a(\|x\|^2) - a(\|y\|^2) = b(y) - b(x),$$

the left–hand side belongs to $\text{Lin}_{\mathbb{Q}} H_1$ and the right–hand side is in $\text{Lin}_{\mathbb{Q}} H_2$, whence $b(x) = b(y)$ and, consequently, $x = y$.

Remark 2. Assume

$$E = E_1 \oplus E_2, \quad E_1 \perp E_2 \quad \text{and} \quad \dim E_1 = 1.$$

Fix an $e \in E_1$ with $\|e\| = 1$, let $a_0 : \mathbb{R} \to \mathbb{R}$ and $b_0 : E_2 \to E_2$ be additive functions such that

$$a_0([0, \infty)) = \mathbb{R}, \quad b_0(E_2) = E_2$$

and define $a : \mathbb{R} \to E$ and $b : E \to E$ by

$$a(\alpha) = a_0(\alpha)e, \quad b(\alpha e + x_2) = b_0(x_2) \quad \text{for } \alpha \in \mathbb{R} \text{ and } x_2 \in E_2.$$

Then the function $f : E \to E$ given by (1) is orthogonally additive, $f(E) = E$ and f is not additive.

To see that $f(E) = E$ fix arbitrarily $y \in E$. Then $y = \beta e + y_2$ where $\beta \in \mathbb{R}$, $y_2 \in E_2$ and $y_2 = b_0(x_2)$ for some $x_2 \in E_2$, $\beta - a_0(\|x_2\|^2) = a_0(\alpha)$ for some $\alpha \in [0, \infty)$. Consequently $\|\sqrt{\alpha}e + x_2\|^2 = \alpha + \|x_2\|^2$ and

$$f(\sqrt{\alpha}e + x_2) = a_0(\alpha + \|x_2\|^2)e + b_0(x_2) = \beta e + y_2 = y.$$

We have been unable to find an example of a bijective orthogonally additive function $f : E \to E$ which is not additive.

Remark 3. If $a : \mathbb{R} \to E$ and $b : E \to E$ are linear and the function $f : E \to E$ given by (1) is bijective, then it is linear.

Proof. As for some $x_0 \in E$ we have

$$-a(1) = f(x_0) = \|x_0\|^2 a(1) + b(x_0),$$

it follows that

$$(\|x_0\|^2 + 1)a(1) = -b(x_0)$$

and so

$$a(1) = b(y_0),$$

where $y_0 = -\frac{1}{1+\|x_0\|^2}x_0$. Consequently

$$f(x) = \|x\|^2 a(1) + b(x) = b(\|x\|^2 y_0 + x)$$

for $x \in E$. Suppose $y_0 \neq 0$. Then

$$f\left(-\frac{1}{\|y_0\|^2}y_0\right) = b\left(\frac{1}{\|y_0\|^2}y_0 - \frac{1}{\|y_0\|^2}y_0\right) = 0 = f(0)$$

which contradicts the injectivity of f. Hence $y_0 = 0$ and $f = b$. ■

References

[1] **Baron, K. and J. Rätz,** On orthogonally additive mappings on inner product spaces, *Bull Polish Acad. Sci. Math.*, **43** (1995), 187–189.

[2] **Kuczma, M.**, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, second edition (edited by A. Gilányi), Birkhäuser Verlag, Basel, 2009.

[3] **Rätz, J.**, On orthogonally additive mappings, *Aequationes Math.*, **28** (1985), 35–49.

K. Baron
Uniwersytet Śląski
Instytut Matematyki
Katowice
Poland
baron@us.edu.pl