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MEAN-VALUE THEOREMS FOR UNIFORMLY SUMMABLE MULTIPLICATIVE FUNCTIONS ON ADDITIVE ARITHMETICAL SEMIGROUPS

Anna Barát and Karl-Heinz Indlekofer

(Paderborn, Germany)

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Abstract. In this paper we give characterizations for uniformly summable multiplicative functions in additive arithmetical semigroups.

1. Introduction

Let (G, ∂) be an additive arithmetical semigroup. By definition G is a free commutative semigroup with identity element 1_G, generated by a countable subset $\mathcal{P}$ of primes and admitting an integer valued degree mapping $\partial : G \to \to \mathbb{N} \cup \{0\}$, which satisfies

(i) $\partial(1_G) = 0$ and $\partial(p) > 0$ for all $p \in \mathcal{P}$,

(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \geq 0$.

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Obviously, $G(0)=1$ and G is countable.
Let

$$\pi(n) := \#\{p \in \mathcal{P} : \partial(p) = n\}$$

denote the total number of primes of degree n in G. We obtain the identity, at least in the formal sense,

$$\hat{Z}(z) := \sum_{n=0}^{\infty} G(n) z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} z^m\right) = \prod_{n=1}^{\infty} (1-z^n)^{-\pi(n)}.$$

$\hat{Z}$ can be considered as the zeta-function associated with the semigroup (G, ∂), the coefficients $\Lambda(n)$ are called the von Mangoldt coefficients.
The von Mangoldt coefficients and the coefficients $\pi(n)$ are related by

$$\sum_{d|n} d\pi(d) = \Lambda(n).$$

In this paper we assume that $\Lambda(n) = O(q^n)$, and the generating function of (G, ∂) has the form

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \frac{\hat{H}(z)}{(1-qz)^{\delta}} \text{ and converges for } |z| < q^{-1}, \tag{1.1}$$

where

$$\hat{H}(z) = O(1) \quad \text{for } |z| < q^{-1}, \text{ and } \lim_{z \to q^{-1}} \hat{H}(z) \quad \text{exists and is positive,} \tag{1.2}$$

and $\delta > 0$. By a recent paper of K.-H. Indlekofer (see [6]), the formal power series $\hat{H}(z)$ is convergent for $z = q^{-1}$ and equals $\lim_{z \to q^{-1}} \hat{H}(z)$, and

$$G(n) \sim \frac{\hat{H}(q^{-1})}{\Gamma(\delta)} q^n n^{\delta - 1} \tag{1.3}$$

holds.
For each arithmetical function $\tilde{f}$ on G, $\tilde{f} : G \to \mathbb{C}$, we associate a power series $\hat{F}$, the *generating function* $\hat{F}$ of $\tilde{f}$, which is defined by

$$\hat{F}(z) = \sum_{a \in G} \tilde{f}(a) z^{\partial(a)} = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a) = n}} \tilde{f}(a) \right) z^n, \tag{1.4}$$

and call the function $f : \mathbb{N}_0 \to \mathbb{C}$, given by

$$f(n) = \sum_{\substack{a \in G \\ \partial(a) = n}} \tilde{f}(a), \tag{1.5}$$

the *summatory function of* $\tilde{f}$.

Further, we introduce the *means*

$$M(n,\tilde f) := \begin{cases} \frac{1}{G(n)} f(n), & \text{if } G(n) \neq 0, \\ 0, & \text{if } G(n) = 0, \end{cases}$$

and say that the function $\tilde f$ possesses an (arithmetical) *mean-value* $M(\tilde f)$, if the limit

$$M(\tilde f) := \lim_{n\to\infty} M(n,\tilde f)$$

exists.
For $1 \le \alpha < \infty$, define

$$||\tilde f||_\alpha := (\limsup_{n\to\infty} M(n,|\tilde f|^\alpha))^{1/\alpha},$$

and let

$$L^\alpha := \{\tilde f : G \to \mathbb{C}, ||\tilde f||_\alpha < \infty\}$$

denote the linear space of functions on G with bounded seminorm $||\cdot||_\alpha$. If

$$\ell^\infty := \{\tilde f : G \to \mathbb{C}, \sup_{g\in G} |\tilde f(g)| < \infty\}$$

is the space of bounded functions on G, we introduce the space $L^*(G)$ of *uniformly summable functions* on G as the $||\cdot||_1$-closure of $\ell^\infty(G)$.
Obviously, $\tilde f \in L^*$ if and only if

$$\lim_{K\to\infty} \sup_{n\ge 1} M(n,|\tilde f_K|) = 0,$$

where

$$\tilde f_K(a) = \begin{cases} \tilde f(a), & \text{if } |\tilde f(a)| \ge K, \\ 0, & \text{otherwise.} \end{cases}$$

We remark that an arithmetical funtion $\tilde f$ is uniformly summable if and only if
(1.6)

$$\forall \varepsilon > 0 : \exists \gamma > 0 : \forall n \in \mathbb{N} : \forall S \subseteq G : (M(n,\mathbf{1}_S) < \gamma \Rightarrow M(n,\mathbf{1}_S|\tilde f|) < \varepsilon),$$

which yields that from $M(n,\tilde f) \asymp 1$ $(n \ge n_1)$ follows $M(n,\tilde f\mathbf{1}_{G\setminus S}) \asymp 1$ for $n \ge n_1$, if $\varepsilon > 0$ is small enough, and if S is as in (1.6). It is easy to show that, if $1 < \alpha < \infty$,

$$\ell^\infty(G) \subsetneqq L^\alpha \subsetneqq L^* \subsetneqq L^1.$$

The class of uniformly summable functions has been defined by Indlekofer (see [3]) for functions defined on $\mathbb{N}$, and he has given a complete characterization of uniformly summable *multiplicative* functions (see Indlekofer [4]).

The aim of this paper is to deal with analogous questions for additive arithmetical semigroups, and to improve results obtained in the thesis of the first author ([1]).

Here, as in the classical case, an arithmetical function $\tilde{f} : G \to \mathbb{R}$ is called *multiplicative* if $\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b)$ whenever $a, b \in G$ are coprime, and an arithmetical function $\tilde{g}$ on G is called *additive* if $\tilde{g}(ab) = \tilde{g}(a) + \tilde{g}(b)$ for all coprime $a, b \in G$.

If $\tilde{f}$ is a multiplicative function on G, then $\sum\limits_{\substack{a\in G\\ \partial(a)=0}} \tilde{f}(a) = 1 \ (\neq 0)$, and we assume that its generating function $\hat{F}$ converges in some neighborhood of $z = 0$ and satisfies

$$\hat{F}(z) = \sum_{n=0}^{\infty} \left(\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) \right) z^n = \tag{1.7}$$
$$= \prod_{p} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)} \right) =:$$
$$=: \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m} z^m \right).$$

Our modus procedendi is double tracked. On the one hand we want to weaken the conditions imposed on the generating function of G. At the same time we endeavor to deal with the greatest possible class of multiplicative functions.

Wehmeier [8] and Barát [1] considered multiplicative functions $\tilde{f} \in L^*$ which possess a mean-value $M(\tilde{f})$ different from zero, whereas Zhang could only deal with multiplicative functions $\tilde{f}$ $(M(\tilde{f}) \neq 0)$ from L^α $(\alpha > 1)$. The assumptions about G are (see [8])

$$G(n) = Aq^n + r(n) \quad \text{with some specific } r(n) = o(q^n)$$

and (see [9])

$$G(n) = q^{-n} \sum_{j=1}^{\nu} A_j n^{\rho_j - 1} + O(q^n n^{-\gamma}), A_\nu > 0,$$

with $\gamma > \rho + 1 \geq 2$, and $0 < \rho_1 < \ldots < \rho_\nu = \rho$. Then

$$\hat{Z}(z) = \hat{H}(z)(1 - qz)^{-\rho} \quad (\rho \geq 1),$$

where

$$\hat{H}(z) = A_\nu + \sum_{j=1}^{\nu} A_j (1-qz)^{\rho-\rho_j} + (1-qz)^\rho \sum_{n=1}^{\infty} O(n^{-\gamma} q^n) z^n.$$

Barát [1] assumed that, in addition to the conditions (1.1) and (1.2), the coefficients of the generating function satisfy

$$G(n) \asymp n^{\delta-1} q^n \qquad (\delta > 0). \tag{1.8}$$

In this paper we weaken the assumptions about G by omitting the requirement (1.8), and characterize multiplicative function $\tilde{f} \in L^*$ the means of which satisfy $M(n, \tilde{f}) \asymp 1$ for $n \geq n_1$.

In the next section we introduce our results.

2. Results

Theorem 2.1. *Let (G, ∂) be an additive arithmetical semigroup satisfying $\Lambda(n) = O(q^n)$, (1.1), and (1.2) with $\delta > 0$. Let $\tilde{f}$ be a multiplicative function, and $\alpha \geq 1$. If $\tilde{f} \in L^* \cap L^\alpha$, and if $M(n, \tilde{f}) \asymp 1$ for $n \geq n_1$, then the following assertions hold:*

$$\sum_{\substack{p\in P, \partial(p)\leq n \\ |\tilde{f}(p)|\leq \frac{3}{2}}} \frac{Re\, \tilde{f}(p) - 1}{q^{\partial(p)}} = O(1), \qquad \sum_{\substack{p\in P, \partial(p)\leq n \\ |\tilde{f}(p)|\leq \frac{3}{2}}} \frac{|\tilde{f}(p)| - 1}{q^{\partial(p)}} = O(1), \tag{2.1}$$

$$\sum_{\substack{p\in P \\ |\tilde{f}(p)|\leq 3/2}} \frac{|\tilde{f}(p) - 1|^2}{q^{\partial(p)}} \quad \textit{converges,} \tag{2.2}$$

$$\sum_{p\in P; n\geq 2} \frac{|\tilde{f}(p^n)|^\lambda}{(q^{\partial(p)})^n} \quad \textit{converges,} \tag{2.3}$$

$$\sum_{\substack{p\in P \\ ||\tilde{f}(p)|-1|>1/2}} \frac{|\tilde{f}(p)|^\lambda}{q^{\partial(p)}} \quad \textit{converges for } 1 \leq \lambda \leq \alpha, \tag{2.4}$$

and for each prime p

$$\sum_{n=1}^{\infty} \frac{\tilde{f}(p^n)}{q^{n\partial(p)}} + 1 \neq 0. \tag{2.5}$$

In the converse direction we deal with two cases: $1 \leq \delta$ and $0 < \delta < 1$. In the first case we prove the following.

Theorem 2.2. *Let (G, ∂) be an additive arithmetical semigroup satisfying the conditions of Theorem 2.1 with $\delta \geq 1$. Let $\tilde{f}$ be a multiplicative function, and let $\alpha \geq 1$. Assume that the conditions (2.1)–(2.5) hold. Then*

$$M(n, \tilde{f}) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) + o(1), \tag{2.6}$$

and $\tilde{f} \in L^* \cap L^\alpha$, and

$$M(n, |\tilde{f}|^\lambda) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda q^{-k\partial(p)}\right) + o(1) \tag{2.7}$$

for $1 \leq \lambda \leq \alpha$.

For $0 < \delta < 1$ we need a further assumption on the multiplicative function $\tilde{f}$ in order to prove our assertion.

Theorem 2.3. *Let an additive arithmetical semigroup (G, ∂) fulfill the conditions of Theorem 2.1, where $0 < \delta < 1$. Let $\alpha \geq 1$, and let $\tilde{f}$ be a multiplicative function satisfying the following condition*

$$\forall \varepsilon > 0 : \exists K > 0 : \forall n \in \mathbb{N} : \tag{2.8}$$

$$S = \{a \in G : \exists p^k || a, p \in P; |\tilde{f}(p^k)|^\alpha > K\} \Rightarrow M(n, \mathbf{1}_S |\tilde{f}|^\alpha) < \varepsilon.$$

Assume that (2.1) holds, and the series (2.2)–(2.4) converge. Then

$$M(n, \tilde{f}) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) + o(1), \tag{2.9}$$

and $\tilde{f} \in L^* \cap L^\alpha$, and

$$M(n, |\tilde{f}|^\lambda) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda q^{-k\partial(p)}\right) + o(1) \tag{2.10}$$

for $1 \leq \lambda \leq \alpha$.

3. Proof of Theorem 2.1

Since $M(n,\tilde{f}) \asymp 1$ $(n \geq n_1)$ and $\tilde{f} \in L^* \cap L^\alpha$ with $\alpha \geq 1$, we obtain, if $\varepsilon > 0$ is small enough, and with suitable $K > 0$,

$$\frac{1}{G(n)} \sum_{\substack{a\in G,\partial(a)=n \\ \varepsilon<|\tilde{f}(a)|\leq K}} 1 \asymp 1.$$

Define an additive function $\tilde{g}$ by

$$\tilde{g}(p^k) = \begin{cases} \log|\tilde{f}(p^k)|, & \text{if } \tilde{f}(p^k) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{G(n)} \sum_{\substack{a\in G,\partial(a)=n \\ \log\varepsilon<\tilde{g}(a)\leq \log K}} 1 \asymp 1, \tag{3.1}$$

and $\tilde{g}$ is finitely distributed. This implies, by Lemma 2.17 in [1],

$$\tilde{g}(a) = c\partial(a) + \tilde{h}(a), \tag{3.2}$$

where the series $\sum\limits_{\substack{p \\ |\tilde{h}(p)|>1}} \frac{1}{q^{\partial(p)}}$ and $\sum\limits_{\substack{p \\ |\tilde{h}(p)|<1}} \frac{\tilde{h}(p)^2}{q^{\partial(p)}}$ converge.

Further, by (1.13), $c = 0$ (for details see [1]).

Therefore the series

$$\sum_{\substack{p\in P \\ |\tilde{g}(p)|<1}} \frac{(\tilde{g}(p))^2}{q^{\partial(p)}} \quad \text{and} \quad \sum_{\substack{p\in P \\ |\tilde{g}(p)|>1}} \frac{1}{q^{\partial(p)}} \tag{3.3}$$

converge.
If $||\tilde{f}(p)| - 1| \leq \eta_1$, then the series expansion of the logarithm yields

$$\log|\tilde{f}(p)| = \log(1 + (|\tilde{f}(p)| - 1)) = |\tilde{f}(p)| - 1 + O((|\tilde{f}(p)| - 1)^2),$$

so that, for $\eta_1 = 1/2$,

$$||\tilde{f}(p)| - 1| \leq 2|\log|\tilde{f}(p)|| = 2|\tilde{g}(p)|$$

and

$$|\tilde{g}(p)| \le 2||\tilde{f}(p)| - 1| \le 1.$$

Obviously,

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|<1/2}} \frac{(|\tilde{f}(p)|-1)^2}{q^{\partial(p)}} \ll \sum_{\substack{p\in P\\ |\tilde{g}(p)|>|\log(1/2)|}} \frac{1}{q^{\partial(p)}} < \infty$$

and

$$\sum_{\substack{p\in P\\ 1/2\le|\tilde{f}(p)|\le 3/2}} \frac{(|\tilde{f}(p)|-1)^2}{q^{\partial(p)}} \ll \sum_{\substack{p\in P\\ |\tilde{g}(p)|\le 1}} \frac{(\tilde{g}(p))^2}{q^{\partial(p)}} < \infty.$$

Thus the series

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|\le 3/2}} \frac{(|\tilde{f}(p)|-1)^2}{q^{\partial(p)}}$$

converges. Furthermore

$$|\tilde{f}(p)-1|^2 = (|\tilde{f}(p)|-1)^2 + 2(|\tilde{f}(p)|-1) - 2(Re\,(\tilde{f}(p)) - 1). \tag{3.4}$$

We define

$$P_1 := \{p \in P; e^{\tilde{h}(p)} < 1 - \eta_1\}$$

and

$$P_2 := \{p \in P; e^{\tilde{h}(p)} > 1 + \eta_1\}$$

with $0 < \eta_1 < 3/4$.
Let, for some parameters k_0 and n_0,

$$S_1 := \{a \in G; \exists p \in P_1 \cup P_2 : p|a, \partial(p) \ge n_0\},$$

$$S_2 := \{a \in G; \exists p \in P : p^2|a, \partial(p) \ge n_0\},$$

and

$$S_3 := \{a \in G; \exists p \in P : p^{k_0}|a, \partial(p) \le n_0\}.$$

Put

$$S := S_1 \cup S_2 \cup S_3.$$

Let ε be an arbitrary fixed positive number. Choose $K > 0$ large enough, and let k_0, n_0 be parameters, such that $M(n, \mathbf{1}_S) < \gamma$ (cf. (1.6)) holds.
Concerning the second term on the right hand side of (3.4), we show that the

sum $\sum_{\substack{\partial(p)\le N\\ |\tilde f(p)|\le K}} \frac{|\tilde f(p)|-1}{q^{\partial(p)}}$ is bounded. Let the multiplicative function $\tilde f^*$ be defined as

$$\tilde f^* := \tilde f \mathbf{1}_{G\setminus S}. \tag{3.5}$$

Then the function $\tilde f^*$ is bounded on the set of the prime powers. Since $M(n,\tilde f)\asymp 1$ $(n\ge n_1)$ and $\tilde f\in L^*$, there exists a natural number n_1', $n_1'\ge n_0$ and $n_1'\ge n_1$, such that

$$|M(n,\tilde f^*)|\asymp 1 \quad \text{for all}\, n\ge n_1', \text{ and uniformly for large } k_0. \tag{3.6}$$

Then, with Theorem 6 of [5], we obtain

$$\sum_{n\le N}\sum_{\substack{p,\partial(p)=n\\ |\tilde f(p)|\le K}} \frac{|\tilde f(p)|-1}{q^{\partial(p)}} = O(1). \tag{3.7}$$

Further (see Theorem 7, [5]), we conclude

$$\sum_{n\le N}\sum_{\substack{p\in P,\partial(p)=n\\ |\tilde f(p)|\le 3/2}} \frac{Re\,(\tilde f(p))-1}{q^{\partial(p)}} = O(1),$$

and this together with (3.7) shows that (2.1) holds.
Therefore the finite sums over the terms on the right hand side of (3.4), for which $\partial(p)\le N$ and $|\tilde f(p)|\le K$, are bounded, and this implies the convergence of the series

$$\sum_{\substack{p\in P\\ |\tilde f(p)|\le 3/2}} \frac{|\tilde f(p)-1|^2}{q^{\partial(p)}},$$

i.e the convergence of (2.2).

Next we prove the convergence of the series (2.4). Let

$$S_4 := \{a\in G; \exists p\in P : p|a; ||\tilde f(p)|-1|>1/2, \partial(p)\ge n_0\}.$$

Thus, if n_0 is large enough, we obtain

$$M(n,|\tilde f|\mathbf{1}_{G\setminus S_4})\asymp 1 \quad \text{for all } n\ge n_1'. \tag{3.8}$$

Now choose $1 < \lambda \le \alpha$, and $\beta \in \mathbb{R}$ with $\frac{1}{\lambda} + \frac{1}{\beta} = 1$. Then Hölder's inequality yields

$$1 \ll \frac{1}{G(n)} \sum_{\substack{a \in G \\ \partial(a)=n}} |\tilde{f}(a)| \le \frac{1}{G(n)} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right)^{\frac{1}{\lambda}} G(n)^{\frac{1}{\beta}} =$$

$$= \frac{G(n)^{1-\frac{1}{\lambda}}}{G(n)} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right)^{\frac{1}{\lambda}} =$$

$$= \left(\frac{1}{G(n)} \sum_{\substack{a \in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right)^{\frac{1}{\lambda}} = M(n, |\tilde{f}|^\lambda)^{\frac{1}{\lambda}} \ll 1,$$

since $\tilde{f} \in L^\alpha$. Hence

$$M(n, |\tilde{f}|^\lambda) \asymp 1 \quad \text{for all } n \ge n_1'.$$

Similarly

$$M(n, |\tilde{f}|^\lambda \mathbf{1}_{G \setminus S_4}) \asymp 1 \quad \text{for all } n \ge n_1'.$$

For $0 < r = |z| < 1/q$ we obtain

(3.9)

$$1 \asymp \frac{\hat{Z}(r) \sum\limits_{n=0}^{\infty} \left(\sum\limits_{\substack{a \in G \setminus S_4 \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n}{\hat{Z}(r) \sum\limits_{n=0}^{\infty} \left(\sum\limits_{\substack{a \in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n} = \prod_{\substack{p \in P, \partial(p) \ge n_0 \\ ||\tilde{f}(p)|-1|>1/2}} \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda r^{k\partial(p)} \right)^{-1}.$$

The last product in (3.9) has the form $\prod\limits_{n=1}^{\infty} (1 + b_n)$, where $b_n \ge 0$. Therefore there exists a real constant c_1 such that, for all $r < \frac{1}{q}$,

$$\sum_{p; ||\tilde{f}(p)|-1|>1/2} |\tilde{f}(p)|^\lambda r^{\partial(p)} \le c_1 < \infty.$$

Thus, for $r \to 1/q$,

$$\sum_{p; ||\tilde{f}(p)|-1|>1/2} \frac{|\tilde{f}(p)|^\lambda}{q^{\partial(p)}} < \infty,$$

which yields the convergence of the series (2.4) for all $1 \leq \lambda \leq \alpha$.

Next, we prove the convergence of the series (2.3). Choose

$$S_2 := \{a \in G; \exists p \in P : p^2|a; \partial(p) \geq n_0\}.$$

Then, analogous to what we have seen above, we can prove that there exists a real constant c_2 such that for all $r \in \mathbb{R}$

$$\sum_{\substack{p\in P,k\geq 2\\ \partial(p)\geq n_0}} |\tilde{f}(p^k)|^\lambda r^{k\partial(p)} \leq c_2 < \infty.$$

Thus, for $r \to 1/q$,

$$\sum_{p\in P;k\geq 2} \frac{|\tilde{f}(p^k)|^\lambda}{q^{k\partial(p)}} < \infty$$

holds, and therefore the series (2.3) converges for all $1 \leq \lambda \leq \alpha$.

Next, we show the validity of (2.5) for every $p \in P$. We know (see [5]), that

$$M(n, \tilde{f}^*) = \prod_{\partial(p)\leq n} \left(1 - q^{-\partial(p)}\right) \left(1 + \sum_{k=1}^{\infty} \tilde{f}^*(p^k) q^{-k\partial(p)}\right) + o(1). \tag{3.10}$$

Suppose now that, for some p_1 with $\partial(p_1) < n_0$, we have

$$1 + \sum_{k=1}^{\infty} \tilde{f}({p_1}^k) q^{-k\partial(p_1)} = 0.$$

Since

$$1 + \sum_{k=1}^{\infty} \tilde{f}^*({p_1}^k) q^{-k\partial(p_1)} = 1 + \sum_{k=k_0}^{\infty} \tilde{f}({p_1}^k) q^{-k\partial(p_1)},$$

we achieve a contradiction to (3.6).

This ends the proof of Theorem 2.1. ■

4. Proof of Theorem 2.2

First we prove that $M(n, \tilde{f}) \asymp 1$ $(n \geq n_1)$. By the convergence of (2.4) and the condition (2.5), there exists some number m_0 sufficiently large such that

$|\tilde{f}(p)q^{-\partial(p)}| < \frac{1}{4}$, and

$$\left|1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)(q^{-1}e^{i\Theta})^{k\partial(p)}\right| > \frac{1}{2} \tag{4.1}$$

holds for all p with $\partial(p) \geq m_0$, and all real Θ with $|\Theta| \leq \pi$. We write

$$\begin{aligned}\hat{F}(z) = &\prod_{p,\partial(p)<m_0} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)z^{k\partial(p)}\right) \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)|<K}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)z^{k\partial(p)}\right) \times \\ &\times \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)|\geq K}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)z^{k\partial(p)}\right) =: \\ &=: \Pi_1(z)\Pi_2(z)\Pi_3(z),\end{aligned}$$

where the first product $\Pi_1(z)$ is absolutely convergent for $|z| \leq q^{-1}$, since each factor of the finite product $\Pi_1(z)$ is convergent by (2.4). The third product $\Pi_3(z)$ is also absolutely convergent for $|z| \leq q^{-1}$. We now estimate the second product $\Pi_2(z)$:

$$\begin{aligned}\Pi_2(z) &= \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)|<K}} \left(1 + \sum_{k=2}^{\infty} \tilde{f}(p^k)z^{k\partial(p)}\right) \frac{1 - \tilde{f}(p)z^{\partial(p)}}{1 - \tilde{f}(p)z^{\partial(p)}} = \\ &= \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)|<K}} (1 - \tilde{f}(p)z^{\partial(p)})^{-1} \times \\ &\quad \times \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)|<K}} \left(1 + \sum_{k=2}^{\infty} \tilde{f}(p)(\tilde{f}(p^k) - \tilde{f}(p^{k-1}))z^{k\partial(p)}\right) =: \\ &=: \Pi_4(z)\Pi_5(z).\end{aligned}$$

By the convergence of the series (2.4) the second product $\Pi_5(z)$ of the last line is absolutely convergent for $|z| \leq q^{-1}$. We apply Theorem 4 of [5] to the product $\Pi_4(z)$, that is a generating function of a completely multiplicative function $\tilde{f}_1$, where $\tilde{f}_1(p) = \tilde{f}(p)$ for $\partial(p) \geq m_0$, and $|\tilde{f}(p)| < K$, and $\tilde{f}_1(p) = 0$ otherwise. We obtain

$$\sum_{a\in G,\partial(a)=n} \tilde{f}_1(a) = \prod_{p\in P} (1 - q^{\partial(p)})(1 - \tilde{f}(p)q^{-\partial(p)})^{-1}G(n) + o(G(n)).$$

Thus we can write

$$\hat{F}(z) = \Pi_4(z)(\Pi_1(z)\Pi_5(z)\Pi_3(z)) =: \Pi_4(z)A(z), \tag{4.2}$$

where $A(z)$ is absolutely convergent for $|z| = q^{-1}$. Applying Lemma 2.21 of [1] it follows

$$M(\tilde{f}) = A(q^{-1})M(n, \tilde{f}_1) + o(1),$$

and therefore

$$M(n, \tilde{f}) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) + o(1).$$

If $\alpha > 1$ and $||\tilde{f}(p)| - 1| < 1/2$, then

$$|\tilde{f}(p)|^\alpha - 1 = \alpha(|\tilde{f}(p)| - 1) + O((|\tilde{f}(p)| - 1)^2)$$

and

$$(|\tilde{f}(p)|^\alpha - 1)^2 = O((|\tilde{f}(p)| - 1)^2) = O(|\tilde{f}(p) - 1|^2).$$

Therefore, in the same way as above, we deduce that

$$M(n, |\tilde{f}|^\lambda) = \prod_{p \in P, \partial(p) \leq n} (1 - q^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda q^{-k\partial(p)}\right) + o(1)$$

for $1 \leq \lambda \leq \alpha$ and $\tilde{f} \in L^\alpha$.

Next, we prove that $\tilde{f} \in L^*$. Using the equation (4.2) we can write the multiplicative function $\tilde{f}$ as the convolution

$$\tilde{f} = \tilde{f}_1 * \tilde{f}_2, \tag{4.3}$$

where $\tilde{f}_1$ is the completely multiplicative function defined above, and $\tilde{f}_2$ is a multiplicative function, such that its generating function $A(z)$ is absolutely convergent for $|z| \leq q^{-1}$. Thus

$$\sum_{m \in \mathbb{N}} \sum_{b \in G, \partial(b) = m} |\tilde{f}_2(b)| q^{-\partial(b)} < \infty. \tag{4.4}$$

Hence, for an arbitrary ε, there exists a natural number m_0 such that

$$\sum_{m \geq m_0} \sum_{b \in G, \partial(b) = m} |\tilde{f}_2(b)| q^{-\partial(b)} < \frac{\varepsilon}{2}.$$

Using our assumptions (2.1)–(2.4) we deduce by Theorem 6 of [5] that $M(n,|\tilde{f}_1|)\asymp 1$ and $M(n,|\tilde{f}_1|^2)\asymp 1$ $(n\ge n_1)$.
Let $\varepsilon>0$ be arbitrary and fixed. We prove that there exists K_0 such that

$$\sum_{a\in G,\partial(a)=n} |\tilde{f}_{K_0}(a)| < \varepsilon G(n)$$

holds for all $n\in\mathbb{N}$. Consider

$$\begin{aligned}\sum_{a\in G,\partial(a)=n}|\tilde{f}_{K_0}(a)| &= \sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\ge K_0\\ \partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| =\\ &= \sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\ge K_0\\ |\tilde{f}_2(b)|\ge K_1,\partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)|+\\ &\quad+\sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\ge K_0\\ |\tilde{f}_2(b)|< K_1,\partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| =:\\ &=:\Sigma_1+\Sigma_2,\end{aligned}$$

where the parameter K_1 is chosen such that $\partial(b)\ge m_0$ if $|\tilde{f}_2(b)|\ge K_1$. Let us now estimate Σ_1. By our assumptions on the arithmetical semigroup, $G(n)\sim$ $\sim q^n n^{\delta-1}$ $(1\le\delta)$ holds, (see [6]) and we obtain

$$\begin{aligned}\Sigma_1 &= \sum_{\substack{b\in G\\ |\tilde{f}_2(b)|\ge K_1\\ \partial(b)\le n}} |\tilde{f}_2(b)| \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde{f}_1(a)| \le\\ &\le \sum_{\substack{b\in G\\ m_0\le\partial(b)\le n}} |\tilde{f}_2(b)| \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde{f}_1(a)| \ll \sum_{\substack{b\in G\\ m_0\le\partial(b)\le n}} |\tilde{f}_2(b)|q^{-\partial(b)}G(n) <\\ &<\frac{\varepsilon}{2}G(n),\end{aligned}$$

whereby we have used the following

$$G(n-\partial(b))\sim q^{n-\partial(b)}(n-\partial(b))^{\delta-1} = q^n n^{\delta-1}(1-\partial(b)/n)^{\delta-1}q^{-\partial(b)} \ll q^{-\partial(b)}G(n).$$

Afterwards, we estimate Σ_2. We use (4.4) and $G(n)\sim q^n n^{\delta-1}$ to obtain the following

$$\begin{aligned}
\Sigma_2 = & \sum_{\substack{a,b\in G\\ |\tilde f_2(b)|<K_1\\ |\tilde f_1(a)||\tilde f_2(b)|\ge K_0,\partial(a)+\partial(b)=n}} |\tilde f_1(a)||\tilde f_2(b)| = \\
= & \sum_{b\in G,|\tilde f_2(b)|<K_1} \sum_{\substack{a\in G\\ |\tilde f_1(a)||\tilde f_2(b)|\ge K_0\\ \partial(a)=n-\partial(b)}} \frac{|\tilde f_1(a)|^2}{|\tilde f_1(a)|} \le \\
\le & \sum_{b\in G,|\tilde f_2(b)|<K_1} |\tilde f_2(b)|\frac{|\tilde f_2(b)|}{K_0} \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde f_1(a)|^2 \ll \\
\ll & \frac{K_1}{K_0}\sum_{b\in G}|\tilde f_2(b)|G(n-\partial(b)) \le \frac{\varepsilon}{2}G(n),
\end{aligned}$$

since $M(n,|\tilde f_1|^2)\asymp 1$.
Therefore $\tilde f\in L^*$. This ends the proof of Theorem 2.2. ■

5. Proof of Theorem 2.3

Let $\varepsilon>0$ be arbitrary and fixed. Then, by (2.8), there exists $K>0$ with

$$S=\{a\in G:\exists\, p^k||a, p\in P, |\tilde f(p^k)|>K\},$$

such that

$$M(n,|\tilde f|\mathbf{1}_S)<\varepsilon.$$

Let such a K be fixed. It yields

$$\left|\frac{1}{G(n)}\sum_{\substack{a\in G\\ \partial(a)=n}}\tilde f(a)-\frac{1}{G(n)}\sum_{\substack{a\in G\setminus S\\ \partial(a)=n}}\tilde f(a)\right|<\varepsilon.$$

By Theorems 4, 6, 7, and Corollary 5 from [5] we obtain

$$
\begin{aligned}
M(n, \mathbf{1}_{G\setminus S}\tilde{f}) &= \frac{1}{G(n)} \sum_{\substack{a\in G\setminus S\\ \partial(a)=n}} \tilde{f}(a) = \\
&= \prod_{\substack{p\in P\\ |\tilde{f}(p^k)|\le K, \partial(p)\le n}} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) + o(1).
\end{aligned}
$$

Write the product on the right side in the form

$$
\begin{aligned}
&\prod_{\substack{p\in P, \partial(p)\le n\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K, k=2,3,\dots}} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) \times \\
&\times \prod_{\substack{p\in P, \partial(p)\le n\\ K\ge|\tilde{f}(p)|>K_2\\ |\tilde{f}(p^k)|\le K}} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) =: \Pi_{1,K}(n)\Pi_{2,K}(n)
\end{aligned}
$$

with some $K_2 > 0$. The product $\Pi_{2,K}(n)$ is absolutely convergent for $|z| \le q^{-1}$, and

$$
\lim_{n\to\infty} \lim_{K\to\infty} \Pi_{2,K}(n) = \prod_{\substack{p\in P\\ |\tilde{f}(p)|>K_2}} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right)
$$

because of (2.3) and (2.4). We derive, where m_0 is large enough,

$$
\begin{aligned}
\Pi_{1,K}(n) = &\prod_{\substack{p\in P, \partial(p)\le m_0\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K}} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) \times \\
&\times \prod_{\substack{p, m_0<\partial(p)\le n\\ |\tilde{f}(p)|\le K_2}} (1-q^{-\partial(p)})(1+\tilde{f}(p)q^{-\partial(p)}) \times \\
&\times \prod_{\substack{p\in P, m_0<\partial(p)\le n\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K}} (1+\tilde{f}(p)q^{-\partial(p)})^{-1} \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) =: \\
&=: \Pi_{3,K}(n)\Pi_4(n)\Pi_{5,K}(n).
\end{aligned}
$$

Since $\Pi_{3,K}(n)$ and $\Pi_{5,K}(n)$ are absolutely convergent for $K \to \infty$ and $n \to \infty$, we arrive at

$$M(n, \mathbf{1}_{G\setminus S}\tilde{f}) = (1+\vartheta\varepsilon) \prod_{p, \partial(p)\le n} (1-q^{-\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right) + o(1)$$

with $|\vartheta| \le 1$, and (1.16) is proven.

Assertion (1.17) follows in the same way, since the corresponding series (1.8)–(1.11) for $|\tilde{f}|^{\lambda}$ are convergent, and thus $\tilde{f} \in L^{\alpha}$

Finally, we prove that $\tilde{f} \in L^*$. For a real number K, $K > 0$ it yields

$$\sum_{\substack{a\in G \\ |\tilde{f}(a)|>K \\ \partial(a)=n}} |\tilde{f}(a)| = \sum_{\substack{a\in G\setminus S \\ |\tilde{f}(a)|>K \\ \partial(a)=n}} |\tilde{f}(a)| + \sum_{\substack{a\in S \\ |\tilde{f}(a)|>K \\ \partial(a)=n}} |\tilde{f}(a)|, \tag{5.1}$$

where the second sum on the right hand side is $< G(n)\varepsilon/2$. Put $\tilde{f}_3 = \tilde{f}\mathbf{1}_{G\setminus S}$. Then $\tilde{f}_3$ is a multiplicative function with $|\tilde{f}_3(p^k)| \le K$, and the mentioned results from [5] give $M(n, |\tilde{f}_3|^2) = O(1)$. Therefore

$$\begin{aligned}\sum_{\substack{a\in G\setminus S \\ |\tilde{f}(a)|>K \\ \partial(a)=n}} |\tilde{f}(a)| &\le \sum_{\substack{a\in G\setminus S \\ |\tilde{f}(a)|>K \\ \partial(a)=n}} |\tilde{f}(a)| \frac{|\tilde{f}(a)|}{K} = \\ &= \frac{1}{K} \sum_{\substack{a\in G \\ |\tilde{f}_3(a)|>K \\ \partial(a)=n}} |\tilde{f}_3(a)|^2 < G(n)\varepsilon/2,\end{aligned}$$

if K is large enough. By (5.1) it follows that $\tilde{f} \in L^*$.

This ends the proof of Theorem 2.3. ■

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A. Barát and K.-H. Indlekofer
Faculty of Computer Science
Electrical Engineering and Mathematics
University of Paderborn
Warburger Strasse 100
D-33098 Paderborn, Germany
barat@math.upb.de
k-heinz@math.uni-paderborn.de