ON A FORMULA OF T. RIVOAL

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

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Abstract. In an unpublished 2005 paper T. Rivoal proved the formula

$$\frac{4}{\pi} = \prod_{k \geq 2} \left(1 + \frac{1}{k+1}\right)^{2\rho(k)\lfloor \log_2(k) - 1 \rfloor}$$

where $\lfloor x \rfloor$ denotes the (lower) integer part of the real number x, and $\rho(k)$ is the 4-periodic sequence defined by $\rho(0) = 1$, $\rho(1) = -1$, $\rho(2) = \rho(3) = 0$. We show how a lemma in a 1988 paper of J. Shallit and the author allows us to prove that formula, as well as a family of similar formulas involving occurrences of blocks of digits in the base-B expansion of the integer k, where B is an integer ≥ 2.

1. Introduction

The author shares probably with many number theorists a kind of fascination for infinite products or series that look simple, but have explicit and somehow unexpected values, such as

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \prod_{n \geq 1} \left(1 - \frac{1}{4n^2}\right) = \frac{2}{\pi}\cdot$$

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In an unpublished paper [12], T. Rivoal proved the formula

$$\frac{4}{\pi} = \prod_{k\geq 2} \left(1 + \frac{1}{k+1}\right)^{2\rho(k)\lfloor \log_2(k)-1\rfloor}$$

where $\lfloor x\rfloor$ denotes the (lower) integer part of the real number x, and where $\rho(k)$ is the 4-periodic sequence defined by $\rho(0) = 1$, $\rho(1) = -1$, $\rho(2) = \rho(3) = 0$. Of course this can also be written

$$\frac{4}{\pi} = \prod_{k\geq 4} \left(1 + \frac{1}{k+1}\right)^{2\rho(k)\lfloor \log_2(k)-1\rfloor}.$$

Grouping terms, this infinite product can also be written

$$\frac{4}{\pi} = \prod_{k\geq 1} \prod_{0\leq r\leq 3} \left(1 + \frac{1}{4k+r+1}\right)^{2\rho(4k+r)\lfloor \log_2(4k+r)-1\rfloor}$$

i.e.,

$$\frac{4}{\pi} = \prod_{k\geq 1} \prod_{0\leq r\leq 1} \left(1 + \frac{1}{4k+r+1}\right)^{2\rho(4k+r)\lfloor \log_2(k)+1\rfloor} =$$

$$= \prod_{k\geq 1} \left(\frac{(4k+2)(4k+2)}{(4k+1)(4k+3)}\right)^{2\lfloor \log_2(k)+1\rfloor}.$$

Now, for $k \geq 1$, the quantity $\lfloor \log_2(k)+1\rfloor$ is the number of digits in the base-2 expansion of k. Hence, letting $N_{0,2}(k)$ (resp. $N_{1,2}(k)$) denote the number of occurrences of 0's (resp. 1's) in the binary expansion of the integer n, we have $\lfloor \log_2(k)+1\rfloor = N_{0,2}(k) + N_{1,2}(k)$. Hence Rivoal's relation reads

$$\prod_{k\geq 1} \left(\frac{(4k+2)(4k+2)}{(4k+1)(4k+3)}\right)^{2(N_{0,2}(k)+N_{1,2}(k))} = \frac{4}{\pi}. \tag{1}$$

2. The main result for base 2

The purpose of this section is to establish a general relation of which Equation (1) is a particular case. We begin with some definitions. In what follows $B \geq 2$ is an integer which will be a numeration base for the integers. The set, or *alphabet*, $\mathcal{D}_B$ is defined by $\mathcal{D}_B := \{0, 1, \cdots, B-1\}$. If w is a *word* over $\mathcal{D}_B$ (i.e., a finite sequence of elements of $\mathcal{D}_B$), we let $L(w)$ denote its *length*: if $w = d_1 d_2 \cdots d_k$, then $L(w) = k$ (the usual notation is $|w|$, but $|\ |$ denotes the absolute value in a few places in this paper). Also w^j stands for the concatenation of j copies of the word w.

If w is a word over $\mathcal{D}_B$, we let $N_{w,B}(n)$ denote the number of possibly overlapping occurrences of w in the B-ary expansion of the integer $n > 0$ if w begins in a 1 or is of the form $w = 0^j$ for some $j \geq 1$, and the number of possibly overlapping occurrences of w in the B-ary expansion of the integer $n > 0$ preceded by an arbitrarily large number of 0's if w begins in a 0, but is not of the form $w = 0^j$ for some $j \geq 1$. Finally we define $N_{w,B}(0) = 0$ for any w (which means that 0 is represented by the empty word in base B).

If w and B are as above, we let $v_B(w)$ denote the "value" of w when w is interpreted as the base B-expansion (possibly with leading 0's) of an integer.

Example 1. To make the above definitions clear we give the following examples: $N_{11,2}(15) = 3$, $N_{001,2}(4) = 1$ (write 4 in base 2 as $0 \cdots 0100$), while $N_{0,4}(4) = 2$. Also $v_2(0010) = 2$.

Now we state a general lemma from [3]. A proof is given in [3] (also see [4], where this lemma is used for proving families of relations involving the quantities $N_{w,B}(k)$).

Lemma 2 ([3]). *Fix an integer $B \geq 2$, and let w be a non-empty word over $\{0, 1, \cdots, B-1\}$. If $f : \mathbb{N} \to \mathbb{C}$ is a function such that $\sum_{n \geq 1} |f(n)| \log n < \infty$, then*

$$\sum_{n \geq 1} N_{w,B}(n) \left(f(n) - \sum_{0 \leq k \leq B-1} f(Bn + k) \right) = \sum f(B^{L(w)} n + v_B(w)),$$

where the last summation is over $n \geq 1$ if $w = 0^j$ for some $j \geq 1$, and over $n \geq 0$ otherwise.

Remark 3. Note that the relation in Lemma 2 above does not involve the value $f(0)$.

The next classical lemma will prove useful (see, e.g., [17, Section 12-13]).

Lemma 4. *Let d be a positive integer. Let $(a_i)_{1 \leq i \leq d}$ and $(b_j)_{1 \leq j \leq d}$ be complex numbers such that no a_i and no b_j belongs to $\{0, -1, -2, \ldots\}$. If $a_1 + a_2 + \cdots + a_d = b_1 + b_2 + \cdots + b_d$, then*

$$\prod_{n \geq 0} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_d)} = \frac{\Gamma(b_1) \cdots \Gamma(b_d)}{\Gamma(a_1) \cdots \Gamma(a_d)}.$$

Theorem 5. *Let w be a word over the alphabet $\{0,1\}$, and $N_{w,2}$ as defined previously. Then*

- *if $w = 0^j$ for some $j \geq 1$,*

$$\prod_{n \geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)} \right)^{2N_{w,2}(n)} = \frac{2^{j+2}\Gamma\left(\frac{1}{2^j}\right)}{\Gamma\left(\frac{1}{2^{j+1}}\right)^2};$$

- *if w is not of the form 0^j for some $j \geq 1$,*

$$\prod_{n \geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)} \right)^{2N_{w,2}(n)} = \frac{\Gamma\left(\frac{v_2(w)}{2^{L(w)}}\right)\Gamma\left(\frac{v_2(w)+1}{2^{L(w)}}\right)}{\Gamma\left(\frac{2v_2(w)+1}{2^{L(w)+1}}\right)^2}.$$

Proof. Define f by $f(0) = 0$ and for all $n \geq 1$

$$f(n) := \log\left(\frac{(2n+1)^2}{2n(2n+2)}\right).$$

Then, applying Lemma 2 with $B = 2$ and w a word over $\{0,1\}$, yields

$$\sum_{n \geq 1} N_{w,2}(n)\left(f(n) - f(2n) - f(2n+1)\right) = \sum f(2^{L(w)}n + v_2(w))$$

where the last summation is over $n \geq 1$ if $w = 0^j$ for some $j \geq 1$, and over $n \geq 0$ otherwise. Since

$$f(n) - f(2n) - f(2n+1) = \log\left(\frac{(4n+2)^4}{(4n+1)^2(4n+3)^2}\right) = 2\log\left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right),$$

we get

$$\sum_{n \geq 1} 2N_{w,2}(n)\log\left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right) =$$

$$= \sum \log\left(\frac{(2^{L(w)+1}n + 2v_2(w) + 1)(2^{L(w)+1}n + 2v_2(w) + 1)}{(2^{L(w)+1}n + 2v_2(w))(2^{L(w)+1}n + 2v_2(w) + 2)}\right)$$

where again the last summation is over $n \geq 1$ if $w = 0^j$ for some $j \geq 1$, and over $n \geq 0$ otherwise. Exponentiating yields

$$\prod_{n \geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)} \right)^{2N_{w,2}(n)} =$$

$$= \prod \left(\frac{(2^{L(w)+1}n + 2v_2(w) + 1)(2^{L(w)+1}n + 2v_2(w) + 1)}{(2^{L(w)+1}n + 2v_2(w))(2^{L(w)+1}n + 2v_2(w) + 2)}\right)$$

where the last product is over $n \geq 1$ if $w = 0^j$ for some $j \geq 1$, and over $n \geq 0$ otherwise. Using Lemma 4 (recall that the range of summation for the sum on the right of the formula in that lemma is not the same for $w = 0^j$ and for $w \neq 0^j$), we then get the statement of the theorem. ∎

Corollary 6. *Equation* (1) *holds.*

Proof. Applying Theorem 5 first with $w = 0$, then with $w = 1$, we obtain (note that $v_2(0) = 0$, $v_2(1) = 1$, and remember that $\Gamma(1 + x) = x\Gamma(x)$)

$$\prod_{n\geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right)^{2N_{0,2}(n)} = \frac{8\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)^2}$$

and

$$\prod_{n\geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right)^{2N_{1,2}(n)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)^2}.$$

Thus

$$\prod_{n\geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right)^{2(N_{0,2}(n)+N_{1,2}(n))} = \frac{8\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2\Gamma\left(\frac{3}{4}\right)^2}.$$

But, using Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ (see, e.g., [17, Section 12-14]), we get the classical relations $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1/4)\Gamma(3/4) = = \pi\sqrt{2}$, which finally yield

$$\prod_{n\geq 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)}\right)^{2(N_{0,2}(n)+N_{1,2}(n))} = \frac{4}{\pi},$$

i.e., Equation (1). ∎

Remark 7. We note that the proof of Theorem 5 gives a companion formula to Equation (1), namely

$$(2) \qquad \prod_{k\geq 1} \left(\frac{(4k+2)(4k+2)}{(4k+1)(4k+3)}\right)^{2(N_{0,2}(k)-N_{1,2}(k))} = \frac{8\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} = \frac{16\pi^2}{\Gamma\left(\frac{1}{4}\right)^4},$$

but that we were unable to compute the infinite "alternate" product (see Section 4.1 for a motivation)

$$\prod_{k\geq 1} \left(\frac{(4k+2)(4k+2)}{(4k+1)(4k+3)}\right)^{2(-1)^k(N_{0,2}(k)+N_{1,2}(k))}.$$

3. A few words about generalizations to base B

It is actually possible to obtain formulas similar to Rivoal's for bases B, where $B > 2$. For example Theorem 5 can be generalized as follows.

Theorem 8. *Let w be a word over the alphabet $\{0, 1, \ldots, B-1\}$. Let $(a_i)_{1\le i\le d}$ and $(b_j)_{1\le j\le d}$ be nonnegative real numbers. If $a_1+a_2+\cdots+a_d = b_1+b_2+\cdots+b_d$, then*

- *if $w = 0^j$ for some $j \ge 1$,*

$$\prod_{n\ge 1}\left(\prod_{1\le i\le d}\left(\left(\frac{Bn+a_i}{Bn+b_i}\right)\prod_{0\le k\le B-1}\left(\frac{B^2n+Bk+b_i}{B^2n+Bk+a_i}\right)\right)\right)^{N_{w,B}(n)} =$$
$$= \prod_{1\le i\le d}\frac{\Gamma\left(1+\frac{b_i}{B^{j+1}}\right)}{\Gamma\left(1+\frac{a_i}{B^{j+1}}\right)};$$

- *if w is not of the form 0^j for some $j \ge 1$,*

$$\prod_{n\ge 1}\left(\prod_{1\le i\le d}\left(\left(\frac{Bn+a_i}{Bn+b_i}\right)\prod_{0\le k\le B-1}\left(\frac{B^2n+Bk+b_i}{B^2n+Bk+a_i}\right)\right)\right)^{N_{w,B}(n)} =$$
$$= \prod_{1\le i\le d}\frac{\Gamma\left(\frac{v_B(w)}{B^{L(w)}}+\frac{b_i}{B^{L(w)+1}}\right)}{\Gamma\left(\frac{v_B(w)}{B^{L(w)}}+\frac{a_i}{B^{L(w)+1}}\right)}.$$

Proof. Apply Lemma 2 with f defined by $f(0) = 0$ and for all $n \ge 1$

$$f(n) := \log\prod_{1\le i\le d}\frac{Bn+a_i}{Bn+b_i}\,. \qquad \blacksquare$$

Remark 9. Theorem 8 contains Theorem 5 (take $B = 2$, $a_1 = a_2 = 1$, $b_1 = 0$, and $b_2 = 2$).

4. Conclusion

4.1. The "alternate" Euler constant

When he obtained Equation (1), or more precisely the formula

$$\frac{4}{\pi} = \prod_{k\geq 2}\left(1+\frac{1}{k+1}\right)^{2\rho(k)\lfloor \log_2(k)-1\rfloor}$$

Rivoal was inspired by Catalan's and Vacca's identities for the Euler-Mascheroni constant γ

$$\gamma = \int_0^1 \frac{\sum_{n\geq 1} x^{2^n}}{x(1+x)}\mathrm{d}x \quad \text{and} \quad \gamma = \sum_{k\geq 1}(-1)^k \frac{\lfloor \log_2(k)\rfloor}{k}$$

(Catalan's identity dates back to 1875, see [5], while Vacca's identity was proved in 1925, see [16]; for a history of the second formula, see [14]). An analogy between γ and $\log\frac{4}{\pi}$ occurs when writing the above relations as

$$\gamma = \sum_{k\geq 1}(-1)^k \frac{\lfloor \log_2(k)\rfloor}{k} \quad \text{and} \quad \log\frac{4}{\pi} = \sum_{k\geq 1}(2\rho(k)\lfloor \log_2(k)-1\rfloor)\log\left(1+\frac{1}{k+1}\right)\cdot$$

Another similarity is given by the formulas

$$\gamma = \sum_{j\geq 2}\frac{(-1)^j}{j}\zeta(j) \quad \text{and} \quad \log\frac{4}{\pi} = \sum_{j\geq 2}\frac{(-1)^j}{j}\eta(j)$$

where $\eta(j) := (1-2^{1-j})\zeta(j)$ (we use the same notation as, e.g., in [8] where several formulas of the same kind can be found; also see [6] and [13]): these formulas can be obtained by taking $z=1$ and $z=1/2$ in the relation

$$\log\Gamma(1+z) = -\log(1+z) + z(1-\gamma) + \sum_{n\geq 2}\frac{(-1)^n(\zeta(n)-1)}{2^n n}$$

valid for $|z|<2$, see [1, 6.1.33, p. 256], which gives respectively

$$\gamma = \sum_{j\geq 2}\frac{(-1)^j}{j}\zeta(j) \quad \text{and} \quad \gamma = \log\frac{4}{\pi} + 2\sum_{j\geq 2}\frac{(-1)^j\zeta(j)}{2^j j}\cdot$$

A more striking analogy between the constants γ and $\log\frac{4}{\pi}$ was noted by Sondow in [13] where it is proved that

$$\gamma = \sum_{n\geq 1}\left(\frac{1}{n} - \log\frac{n+1}{n}\right) = \iint_{[0,1]^2}\frac{1-x}{(1-xy)(-\log xy)}\mathrm{d}x\mathrm{d}y$$

and

$$\log \frac{4}{\pi} = \sum_{n \geq 1} (-1)^{n-1} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) = \iint_{[0,1]^2} \frac{1-x}{(1+xy)(-\log xy)} \mathrm{d}x\mathrm{d}y$$

leading Sondow to call "alternating Euler constant" the quantity $\log \frac{4}{\pi}$. In the same spirit Sondow compares in [14] the following two expressions

$$\gamma = \frac{1}{2} + \sum_{n \geq 1, \text{ even}} \frac{N_{1,2}(n) + N_{0,2}(n) - 1}{n(n+1)(n+2)}$$

and

$$\log \frac{4}{\pi} = \frac{1}{4} + \sum_{n \geq 1, \text{ even}} \frac{N_{1,2}(n) - N_{0,2}(n)}{n(n+1)(n+2)}$$

where the first expression is due to Addison [2] and the second is a modification of a formula in [4].

One way of "explaining" the links between γ and $\log 4/\pi$ is the introduction of the "generalized-Euler-constant function" by Sondow and Hadjicostas in [15], or of a similar function introduced by Pilehrood and Pilehrood in [10]: the function $\gamma(z)$ of [15] and the function $f_1(z)$ in [10] are defined by

$$\gamma(z) = \sum_{n \geq 1} z^{n-1} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) \quad \text{and} \quad f_1(z) = \sum_{n \geq 1} z^n \left(\frac{1}{n} - \log \frac{n+1}{n} \right)$$

(so that $f_1(z) = z\gamma(z)$). Namely one has

$$\gamma = \gamma(1) \quad \text{and} \quad \log \left(\frac{4}{\pi} \right) = \gamma(-1)$$

(for more on $\gamma(z)$ see [11]).

4.2. Catalan-type formulas

In his paper [12] Rivoal gives a Catalan-like formula for $4/\pi$ in relation with Equality 1, namely

$$\sum_{k \geq 2} (2\rho(k) \lfloor \log_2(k) - 1 \rfloor) \log \left(1 + \frac{1}{k+1} \right) = \log \frac{4}{\pi} = \int_0^1 \frac{x-1}{\log x} \frac{\sum_{n \geq 2} x^{2^n}}{x(1+x)(1+x^2)} \mathrm{d}x.$$

Comparing with Catalan's identity

$$\gamma = \int_0^1 \frac{\sum_{n \geq 1} x^{2^n}}{x(1+x)} \mathrm{d}x,$$

Rivoal suggested (private communication) that similar relations may exist for logarithms of the infinite products we studied here. However we do not have general results in that direction.

4.3. Two more remarks

We would like to make two more remarks about Lemma 2.

• Which functions can be obtained on the left side of the equality given in that lemma? In other words given a map g from the integers to the real numbers, we want to know when it is possible to find a map f such that

$$g(n) = f(n) - \sum_{0 \le j \le B-1} f(Bn + j).$$

A particular case is addressed in [3], the case where f is a constant multiple of g. In other words what are the eigenvectors of the operator $f :\to Tf$, where $Tf(n) = \sum_{0 \le j \le B-1} f(Bn + j)$, and f is supposed to behave "regularly"? This looks like a functional equation with means: $\sum_{0 \le j \le B-1} f(Bn + j)$ is B times the arithmetic mean of the values of f on $[Bn, Bn + B - 1]$. Looking in the literature for papers with keywords "mean" and "functional equation", we found several papers, in particular by Daróczy and coauthors, e.g., [7], but were not able to find references really related to our question.

• Another question about Lemma 2 is whether the quantities $N_{w,B}(n)$ can be replaced by more general sequences. We think that it is possible to introduce generalizations of B-additive sequences for which a similar lemma holds. We hope to address that question in the near future, possibly including distribution results (see, e.g., the survey of Kátai [9]).

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