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MIXING OF PROUHET-THUE-MORSE AND RUDIN-SHAPIRO SEQUENCES

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

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Abstract. Let $P_\gamma(N,z) = \sum_{0\le n<N} z^n.\gamma(n)$ be a $\mathbb{C}^2$-valued polynomial where $\gamma(\cdot)$ belongs to a family of sequences in $\mathbb{C}^2$ derived from linear automata that recognize classical Prouhet-Thue-Morse sequence and Rudin-Shapiro sequence in $\{-1,+1\}$. Upper bound on the complex unit disk of the quadratic norms of $P_\gamma(N,z)$ and analogous polynomials, obtained by summing on the intervals $[b, b+N)$, are estimated. Examples of applications and some generalizations are considered.

1. Introduction

The famous Prouhet-Thue-Morse (in short PTM) and Rudin-Shapiro (in short RS) sequences, respectively denoted here by $(\mu_n)_{n\ge0}$ and $(\rho_n)_{n\ge0}$, are classically defined as follows. If the number $s_1(n)$ of "1" in the binary expansion of n is even, then $\mu_n = 1$ otherwise $\mu_n = -1$. If the number $s_{11}(n)$ of

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occurrences of the word "11" in the binary expansion of n is even, then $\rho_n = 1$ otherwise $\rho_n = -1$. Those sequences are automatic. To explain this fact and the way we intend to mix PTM sequence with RS sequence we recall some basic vocabulary and notation. For more details and general investigations on automatic sequences we refer to the monograph of J.-P. Allouche and J. Shallit [6].

Let n be a positive integer and denote by $\tilde{n}$ the binary word $e_{t_n}(n)\cdots e_0(n)$ corresponding to the binary expansion $n = \sum_{i=0}^{t_n} e_i(n)2^i$ with $e_{t_n}(n) = 1$ and $e_i(n) \in \{0,1\}$ for all indices i, $0 \le i < t_n$ and $e_j(n) = 0$ for $j > t_n$. In addition, set $\tilde{0} = 0$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of positive natural numbers. Let $B := \{0,1\}$ and let B^* be the set of all binary words endowed with the concatenation law $(v_1 \dots v_k) \cdot (w_1 \dots w_\ell) = v_1 \dots v_k w_1 \dots w_\ell$; the empty word is denoted by $\wedge$. The weight of a nonempty binary word $w = w_k \dots w_0$ is the integer $\dot{w} = w_k 2^k + \cdots + w_1 2 + w_0$ and by definition $\dot{\wedge} = 0$. A deterministic complete 2-automaton is a quadruplet $\mathcal{A} := (E, \Phi, I_0, \tau)$ where E is a finite set called set of states, $\Phi := \{\phi_0, \phi_1\}$ where the maps $\phi_j : E \to E$ are called instructions, I_0 is an element of E called initial state and τ is the output map defined on E and taking its values in a finite set of symbols S. A sequence u in the set S is said to be 2-automatic (or simply, automatic) and recognized by $\mathcal{A}$ if $u(n) := \tau \circ f_{\mathcal{A}}(\tilde{n})$ where $f_{\mathcal{A}} : B^* \to E$ is the *internal map* defined by

$$f_{\mathcal{A}}(w_k \cdots w_0) := \phi_{w_0} \circ \cdots \circ \phi_{w_k}(I_0)\,, \tag{1.1}$$

and $f_{\mathcal{A}}(\wedge)(J) := J$ for all states J. With this definition, the automaton reads any binary word from left to right. If the output map τ is omitted, the triplet $\mathcal{A}' := (E, \Phi, I_0)$ is called semi-automaton. For any automatic sequence many semi-automata $\mathcal{A}'$ can be used to recognize the sequence and it is always possible to choose one which is left-regular, that means $\phi_0(I_0) = I_0$, and linear *i.e.*, there is a vector space V, called support of the semi-automaton, such that $E \subset V$, the instruction maps are restrictions of endomorphisms of V and moreover, if S is a subset of the scalar field of V, the output function is the restriction of a linear form.

In Section 2, two left-regular and linear semi-automata

$$\mathcal{A}^{(i)} := (E, \{A_0^{(i)}, A_1^{(i)}\}, I_0) \quad (i \text{ equals } 0 \text{ or } 1),$$

with support $\mathbb{C}^2$, are built such that $\mathcal{A}^{(0)}$ recognizes μ and $\mathcal{A}^{(1)}$ recognizes ρ. A mixing of μ and ρ is obtained as follows. Let $\lambda : \mathbb{N}_0 \to B$ and $v_0 \in \mathbb{C}^2$, then the λ-mixing sequence of μ and ρ with base point v_0 is the sequence $\gamma : \mathbb{N}_0 \to \mathbb{C}^2$ defined by

$$\gamma(0) := A_0^{(\lambda(0))} v_0 \quad \text{and} \quad \gamma(n) := A_{e_0(n)}^{(\lambda(0))} \dots A_{e_{t_n}(n)}^{(\lambda(t_n))} v_0 \quad (n > 0)\,. \tag{1.2}$$

Notice that if the sequence λ is ultimately periodic, then γ is still automatic. In Section 3, the vector valued polynomial

$$P_\gamma(N,z) := \sum_{0\le n<N} z^n.\gamma(n) \quad (z\in\mathbb{C}) \tag{1.3}$$

is studied. Notice that $P_\gamma(N,z)$ belongs to $\mathbb{C}^2$ and by convention $z^0 = 1$, hence $P_\gamma(1,z) = \gamma(0)$ and also $P_\gamma(0,z) = 0$. The complex plane $\mathbb{C}^2$ is equipped with the standard hermitian product and its associated quadratic norm is noted $||\cdot||_2$. A summation formula involving the binary expansion of N is established. In Section 4, lower and upper bounds for $\sup_{|z|\le 1} ||P_\gamma(N,z)||_2$ are given. Finally, in the last section, additional results are derived from our method. This paper is a partial extension of a previous work of the first author [1]. Tools we use are in germ contained in [2, 4, 7, 8].

2. Automata recognizing PTM sequence and RS sequence

The classical automaton that recognizes the PTM sequence μ has the following linear version with support $\mathbb{R}$ (or $\mathbb{C}$):

$$\mathcal{A} = \big(\{+1,-1\}, \{A_0 : x\mapsto x,\ A_1 : x\mapsto -x\}, +1, \tau : x\mapsto x\big)\,. \tag{2.1}$$

Automata that recognize the RS sequence ρ are more complicated. A linear version $\mathcal{A}^{(1)}$ is depicted Figure 1, the output function being the first coordinate projection $p_0 : \left[\begin{smallmatrix} x_0\\ x_1\end{smallmatrix}\right] \mapsto x_0$. In order to use simultaneously automata computing both sequences μ and ρ, we introduce the linear automaton

$$\mathcal{A}^{(0)} = (E, \{A_0^{(0)}, A_1^{(0)}\}, \left[\begin{smallmatrix} +1\\ +1\end{smallmatrix}\right], p_0)$$

with support $\mathbb{C}^2$, where $E = \{\left[\begin{smallmatrix} +1\\ +1\end{smallmatrix}\right], \left[\begin{smallmatrix} +1\\ -1\end{smallmatrix}\right], \left[\begin{smallmatrix} -1\\ +1\end{smallmatrix}\right], \left[\begin{smallmatrix} -1\\ -1\end{smallmatrix}\right]\}$ (the space of states of $\mathcal{A}^{(1)}$), $A_0^{(0)} = \mathbf{I}_2$ (identity matrix of rank 2) and $A_1^{(0)} = -\mathbf{I}_2$. Notice that both vectors $\left[\begin{smallmatrix} +1\\ +1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} +1\\ -1\end{smallmatrix}\right]$ can be used as initial state to recognize μ.

From now on the sequence γ is given by (1.2) with the above automata $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(1)}$. The set of possible values of γ are mainly related to v_0 and by homogeneity it is enough to consider these values up to the multiplication by a non zero complex number, more precisely:

Theorem 2.1. *The sequence γ takes a finite number of vector values. There is only one for $v_0 = \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]$ (namely the null vector) and there all are in E if $v_0 \in E$. There are 10 possible values if $v_0 = \left[\begin{smallmatrix} x\\ 1\end{smallmatrix}\right]$ or $v_0 = \left[\begin{smallmatrix} 1\\ x\end{smallmatrix}\right]$ with $x \in \mathbb{C}\setminus\{0\}$ and only 7 possible values if $x = 0$.*

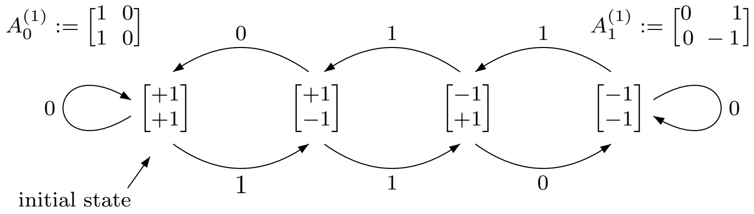

Figure 1. The classical linear Rudin-Shapiro automaton $\mathcal{A}^{(1)}$. Arrows labeled by 0 (resp. 1) correspond to the instruction matrix $A_0^{(1)}$ (resp. $A_1^{(1)}$)

Proof. The cases $v_0 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $v_0 \in E$ are clear and if $\lambda(n) = 0$ for all $n \geq 0$, then γ takes only 2 values. Assume that $v_0 = \left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$. By applying the instruction matrices, one sees that the set of possible values of γ is exactly

(2.2)
$$E_x := \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix}, \begin{bmatrix} -1 \\ -x \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} x \\ -x \end{bmatrix}, \begin{bmatrix} -x \\ x \end{bmatrix}, \begin{bmatrix} -x \\ -x \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix} \right\}$$

The case $v_0 = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ is analogous. ■

3. Summation formulae

We expand $P_\gamma(N, z)$ according to the binary expansion $N = \sum_{m=0}^{t_N} e_m(N) 2^m$, with $N \geq 1$ and $e_{t_N}(N) \neq 0$. In the sequel, t is currently written for t_N. For all integers $k \geq 0$ the shifted sequence $\lambda_k(\cdot)$ is defined by $\lambda_k(n) := \lambda(n+k)$ and γ_k will be the sequence defined by (1.2) but with λ_k in place of λ. Define also the matrices $A^{(i)}(z) := A_0^{(i)} + z.A_1^{(i)}$.

3.1. The regular case. We assume here that γ is *regular* that is to say $A_0^{(\lambda(0))} v_0 = v_0$, hence $\gamma(0) = v_0$ and for any binary word $w_k \ldots w_0$, $\gamma(\dot{w}) = = A_{w_0}^{(\lambda(0))} \ldots A_{w_k}^{(\lambda(k))} v_0$. Moreover

$$\begin{aligned} z^j.A_j^{(\lambda(0))}(z^{2n}.\gamma_1(n)) &= z^{2n+j}.A_j^{(\lambda(0))} A_{e_n(n)}^{(\lambda(1))} \ldots A_{e_{t_n}(n)}^{(\lambda(t_n))} v_0 = \\ &= z^{2n+j}.A_j^{(\lambda(0))} A_{e_n(n)}^{(\lambda(1))} \ldots A_{e_{t_N}(n)}^{(\lambda(t_N))} v_0 = \\ &= z^{2n+j}.\gamma(2n+j) \end{aligned}$$

for all $j \in B$. After summation

$$\begin{aligned} A^{\lambda(0)}(z)\Big(\sum_{0\le n<N} z^{2n}.\gamma_1(n)\Big) &= \sum_{0\le n<N} \big(z^{2n}.\gamma(2n)+z^{2n+1}.\gamma(2n+1)\big) = \\ &= \sum_{0\le \nu<2N} z^{\nu}.\gamma(\nu)\,, \end{aligned}$$

hence, one gets

$$P_\gamma(2N,z) = A^{(\lambda(0))}(z)P_{\gamma_1}(N,z^2)\,. \tag{3.1}$$

Set $\Pi(0,z)=\mathbf{I}_2$ and for integer $m\ge 1$:

$$\Pi(m,z) := A^{(\lambda(0))}(z)A^{(\lambda(1))}(z^2)\dots A^{(\lambda(m-1))}(z^{2^{m-1}})\,. \tag{3.2}$$

By expanding the product and using the regularity of γ, one obtains directly $\Pi(m,z)v_0 = P_\gamma(2^m,z)$, a formula which is also a consequence of (3.1). For m such that $0\le m\le t=t_N$ set $s_m := \sum_{m\le r\le t} e_r(N)2^r$ and $s_{t+1}=0$. Then, if $0\le n<2^m$ and $e_m(N)=1$, the definition of $\gamma(\cdot)$ leads to

$$\begin{aligned} \gamma(s_{m+1}+0.2^m+n) &= A^{(\lambda(0))}_{e_0(n)}\dots A^{(\lambda(m-1))}_{e_{m-1}(n)}A^{(\lambda(m))}_0 A^{(\lambda(m+1))}_{e_{m+1}(N)}\dots A^{(\lambda(t))}_{e_t(N)}v_0 = \\ &= A^{(\lambda(0))}_{e_0(n)}\dots A^{(\lambda(m-1))}_{e_{m-1}(n)}A^{(\lambda(m))}_0\gamma_{m+1}\Big(\Big\lfloor\frac{N}{2^{m+1}}\Big\rfloor\Big)\,. \end{aligned}$$

Consequently,

$$\sum_{s_{m+1}\le s<s_m} z^s.\gamma(s) = z^{s_{m+1}}.\Pi(m,z)A^{(\lambda(m))}_0\gamma_{m+1}\Big(\Big\lfloor\frac{N}{2^{m+1}}\Big\rfloor\Big) \qquad (e_m(N)=1) \tag{3.3}$$

and

$$P_\gamma(N,z) = \sum_{m=0}^{t_N} e_m(N)z^{s_{m+1}}.\Pi(m,z)A^{(\lambda(m))}_0\gamma_{m+1}\Big(\Big\lfloor\frac{N}{2^{m+1}}\Big\rfloor\Big)\,. \tag{3.4}$$

3.2. The non-regular case. Suppose that γ is not necessarily regular. In that case, Formula (3.4) is generally wrong. But (3.3) remains unchanged as long as $m\ne t$ since in that case, either $e_m(N)=0$ and the sum on the left side is null anyway or $e_m(N)=1$, but $\gamma_{m+1}\big(\lfloor\frac{N}{q^{m+1}}\rfloor\big) = A^{(\lambda(m+1))}_{e_{m+1}(N)}\dots A^{(\lambda(t))}_{e_t(N)}v_0$ with $e_t(N)=1$, so that the fact that γ is not regular has no effect. For $m=t$ $(=t_N)$, the sum in (3.3) equals $P_\gamma(2^t,z)$. But, in all generality, for $\ell\ge 1$,

$$\begin{aligned} P_\gamma(2^\ell,z) &= z^{2^{\ell-1}}.\Pi(\ell-1,z)A^{(\lambda(\ell-1))}_1 v_0 + P_\gamma(2^{\ell-1},z) = \\ &= \big(z^{2^{\ell-1}}.\Pi(\ell-1,z)A^{(\lambda(\ell-1))}_1 + \dots + z.\Pi(0,z)A^{\lambda(0))}_1 + A^{\lambda(0))}_0\big)v_0\,. \end{aligned}$$

Collecting all these calculations leads to

$$
\begin{aligned}
P_\gamma(N,z) &= \sum_{m=0}^{t_N-1} e_m(N) z^{s_{m+1}}.\Pi(m,z) A_0^{(\lambda(m))}\gamma_{m+1}\Big(\Big\lfloor \frac{N}{2^{m+1}}\Big\rfloor\Big) + \\
&\qquad + \sum_{\ell=0}^{t_N-1} z^{2^\ell}.\Pi(\ell,z) A_1^{(\lambda(\ell))} v_0 + A_0^{(\lambda(0))} v_0\,. \qquad (3.5)
\end{aligned}
$$

4. Lower and upper bounds

In what follows the quadratic norm $||M||$ of a matrix application $M : \mathbb{C}^2 \to \to \mathbb{C}^2$ is defined by $||M|| = \sup_{||X||_2\leq 1} ||M(X)||_2$. It is well known that $||M||$ is equal to the square root of the largest modulus of the eigenvalues of M^*M, where M^* is the adjoint of M with respect to the standard hermitian product. Easy calculations show $||A_0^{(0)}|| = ||A_0^{(0)}|| = 1$, $||A_0^{(1)}|| = ||A_0^{(1)}|| = \sqrt{2}$ and $||A^{(0)}(z)|| = |1-z|$, $||A^{(1)}(z)|| = \sqrt{2}$. The following lemma will be useful. It is proved with more general assumptions in [4, Lemma 2.2].

Lemma 4.1. *For all $\beta \in]0,1]$ and all integers $N \geq 1$,*

$$\sum_{m=0}^{k} e_m(N) 2^{\beta m} \leq \frac{1}{2^\beta - 1} N^\beta\,.$$

4.1. Lower bound for $||P_\gamma(N,z)||_2$. The following theorem is classical.

Theorem 4.1. *Let $x \in \mathbb{C}\setminus\{0\}$ and $v_0 \in E_x$ then, for all integers $N \geq 1$,*

$$\sup_{|z|\leq 1} ||P_\gamma(N,z)||_2 \geq c_x\sqrt{N} \qquad (4.1)$$

with $c_x := \min_{e\in E_x} ||e||_2$.

The proof is based on the standard computation

$$\int_0^1\int_0^1 ||P_\gamma(N, te^{2i\pi\theta})||_2^2 dt\, d\theta = \sum_{0\leq n<N} ||v(n)||_2^2\,.$$

By definition of E_x and independently of λ, $\gamma(\mathbb{N}) \subset E_x$, hence

$$\sup_{|z|\leq 1} ||P_v(N,z)||_2^2 \geq c_x^2 N$$

proving (4.1).

Remark 4.1. The constant c_x is equal to $\min\{\sqrt{2}, \sqrt{2}|x|, \sqrt{1+|x|^2}\}$. But, if $v_0 \in E_1$ then $\gamma(E_1) \subset E_1$, hence $\sup_{|z|\le 1} ||P_v(N,z)||_2 \ge \sqrt{2N}$.

4.2. Upper bound for $||P_\gamma(N,z)||_2$, regular case. In this part we assume that $v_0 = \left[\begin{smallmatrix}+1\\+1\end{smallmatrix}\right]$ or $v_0 = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$ and to emphasize this fact, the sequence γ defined in (1.2) will be denoted by η.

Theorem 4.2. *For all N:*

$$\sup_{|z|\le 1} ||P_\eta(N,z)||_2 \le \sqrt{2}\sum_{m=0}^{t_N} e_m(N) 2^{m-\frac{1}{2}(\lambda(0)+\cdots+\eta(m-1))} . \tag{4.2}$$

Proof. By construction, the sequence η is regular and takes its values in E_1. Therefore (4.2) is a straightforward consequence of (3.4) and the inequality

$$||\Pi(m,z)||_2 \le 2^{m-\frac{1}{2}(\lambda(0)+\cdots+\lambda(m-1))} \tag{4.3}$$

is valid for all complex numbers z such that $|z| \le 1$. ■

4.3. General case. It is interesting to consider the general situation where v_0 belongs to $\mathbb{C}^2 \setminus \left\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right\}$ with a summation running from b to $N+b$ in order to obtain a reasonable bound uniformly in b. Going back to the notation γ, the homogeneity allows to reduce the study for v_0 in E_x with $0 < |x| \le 1$. Notice that now $\max_{e\in E_x} ||e||_2 = \sqrt{2}$.

Theorem 4.3. *For all positive integers N,*

$$\sup_{b\ge 0}\sup_{|z|\le 1} \Big\| \sum_{b\le n<N+b} z^n.\gamma(n)\Big\|_2 \le 2\sqrt{2} \sum_{0\le m\le t_N} 2^{m-\frac{1}{2}(\lambda(0)+\cdots+\lambda(m-1))} .$$

Proof. For non-negative integers r, s and a, set

$$S(r,s,a,z) := \sum_{r2^s\le n<r2^s+a} z^n.\gamma(n) . \tag{4.4}$$

We distinguish several cases.

Case 1: $r = 0$. By definition $S(0,s,a,z) = P_\gamma(a,z)$ (independent of s) and using (3.5)

$$\begin{aligned} ||S(0,s,a,z)||_2 &\le \sqrt{2}\Big(\sum_{m=0}^{t_a-1} e_m(a)||\Pi(m,z)||_2 + \sum_{\ell=0}^{t_a-1} ||\Pi(\ell,z)||_2 + 1\Big) \le \\ &\le \sqrt{2}\Big(1 + \sum_{m=0}^{k_a-1} (e_m(a)+1)2^{\sigma(m)}\Big) \end{aligned} \tag{4.5}$$

with $\sigma(m) = m - \frac{1}{2}(\lambda(0) + \cdots + \lambda(m-1))$ if $m \geq 1$ and $\sigma(0) = 0$.

Case 2: $r \neq 0$ ***and*** $a = 2^s$**.** The sum (4.4) can be written successively

$$\begin{aligned} S(r,s,2^s,z) &= \sum_{r2^s \leq n < (r+1)2^s} z^n.\gamma(n) = \\ &= z^{r2^s}. \sum_{0 \leq \ell < 2^s} z^\ell.\gamma(r2^s + \ell) = \\ &= z^{r2^s}.\Pi(s,z) A^{(\lambda(s))}_{e_0(r)} \ldots A^{(\lambda(s+t_r))}_{e_{t_r}(r)} v_0 \end{aligned}$$

and (4.3) leads to

$$||S(r,s,2^s,z)||_2 \leq \sqrt{2}\, 2^{\sigma(s)} . \tag{4.6}$$

Case 3: $a \neq 0$**,** $r \neq 0$**,** r ***odd and*** $s > t_a$**.** Set $s_m := \sum_{m \leq j \leq t_a} e_j(a) 2^j$ and cut the sum $S(r,s,a,z)$ into a sum of partial summations:

$$S(r,s,a,z) := z^{r2^s}. \sum_{0 \leq m \leq t_a} \sum_{s_{m+1} \leq \ell < s_m} z^\ell.\gamma(r2^s + \ell) .$$

For m running from 0 to t_a,

$$\begin{aligned} \sum_{s_{m+1} \leq \ell < s_m} z^\ell.\gamma(r2^s + \ell) &= z^{s_{m+1}}. \sum_{0 \leq \nu < e_m(a) 2^m} z^\nu.\gamma(r2^s + s_{m+1} + \nu) = \\ &= \Big(e_m(a) z^{s_{m+1}}. \sum_{0 \leq \nu < e_m(a)2^m} z^\nu.A(\nu,m)\Big) A_0^{(\lambda(m))} B(a,r,s,m) v_0 \end{aligned}$$

with

$$B(a,r,s,m) := A^{(\lambda(m+1))}_{e_{m+1}(a)} \ldots A^{(\lambda(t_a))}_{e_{t_a}(a)} A^{(\lambda(t_a+1))}_{e_{t_a+1}(r2^s)} \ldots A^{(\lambda(s+t_r))}_{e_{s+t_r}(r2^s)}$$

if $m < k_a$,

$$B(r,s,t_a) := A^{(\lambda(t_a+1))}_{e_{t_a+1}(r2^s)} \ldots A^{(\lambda(s+t_r))}_{e_{s+t_r}(r2^s)}$$

and $A(\nu,m) := A^{(\lambda(0))}_{e_0(\nu)} \ldots A^{(\lambda(m-1))}_{e_{m-1}(\nu)}$. Now $S(r,s,a,z)$ can be written in the form

$$z^{r2^s}. \textstyle\sum_{0 \leq m \leq t_a} e_m(a).\Pi(m,z) A_0^{(\lambda(m))} B(r,s,m) v_0$$

so that, using (4.3),

$$||S(r,s,a,z)||_2 \leq \sqrt{2} \sum_{0 \leq m \leq t_a} e_m(a) 2^{\sigma(m)} . \tag{4.7}$$

Finally, consider the sum $S(r,s,-a,z) := \sum_{r2^s - a \leq n < r2^s} z^n.\gamma(n)$. One has

$$S(r,s,-a,z) = \sum_{0 \leq m \leq t_a} \sum_{r2^s - s_m \leq n < r2^s - s_{m+1}} z^n.\gamma(n). \tag{4.8}$$

Notice that $r2^s - s_m = \frac{r-1}{2}2^{s+1} + 2^{s-1} + \cdots + 2^{t_a+1} + +(1-e_{t_a}(a))2^{t_a} + \cdots + (1-e_{m+1}(a))2^{m+1} + (2-e_m(a))2^m$ if $m < t_a$ and $r2^s - s_m = r2^s - 2^{t_a} = \frac{r-1}{2}2^{s+1} + 2^{s-1} + \cdots + 2^{t_a+1} + 2^{t_a}$ for $m = t_a$. If $e_m(a) = 0$, the partial sum in (4.8) corresponding to m is empty, its value is the null vector. If $e_m(a) = 1$, then $s_n = s_{n+1} + 2^m$ and the previous binary expansions justify the equalities

$$\begin{aligned}\sum_{r2^s - s_m \le n < r2^s - s_{m+1}} z^n.\gamma(n) &= e_m(a) z^{r2^s - s_m}. \sum_{0 \le \ell < 2^m} z^\ell.\gamma(r2^s - s_m + \ell) = \\ &= e_m(a) z^{r2^s - s_m}.\Pi(m,z) A_0^{(\lambda(m))} B'(r,s,m) v_0\end{aligned}$$

with $B'(r,s,m) := A_{e_{m+1}(r2^s - s_m)}^{(\lambda(m+1))} \ldots A_{e_{s+t'_r}(r2^s - s_m)}^{(\lambda(s+t'_r))}$ where $t'_r = t_r$ if $r \ge 3$ and $t'_r = -1$ if $r = 1$. This equality is still valid when $e_m(a) = 0$. Summarizing,

$$S(r,s,-a,z) = \sum_{0 \le m \le t_a} e_m(a) z^{r2^s - s_m}.\Pi(m,z) A_0^{(\lambda(m))} B'(r,t,m) v_0$$

and consequently

$$||S(r,s,-a,z)||_2 \le \sqrt{2} \sum_{0 \le m \le t_a} e_m(a) 2^{\sigma(m)}\,. \tag{4.9}$$

4.4. Final step. We are ready to bound the norm of the sum of vectors $z^n.\gamma(n)$ on the interval $[b, b+N)$ with $N \ge 3$. If $b \ne 0$, take the largest integer $s \ge 1$ such that there exists an integer $r \ge 1$ verifying $r2^s \in [b, b+N)$. Necessarily r is odd. The couple (r,s) obtained in this way is unique, otherwise $(r+1)2^s$ or $(r-1)2^s$ belongs to $[b, b+N[$ and the choice of s will be not optimal. In particular $N \le 2^s$. Introduce the integers $a = N + b - r2^s$ and $a' = r2^s - b$. Notice that $a \ne 0$. We cut the summation on $[b, b+N)$ into summations on $[b, r2^s)$ and $[r2^s, b+N)$.

If $r = 0$, an upper bound of $||S(0,s,a,z)||_2$ comes from Case 1.

If $a = 2^s$, then $N = 2^s$ and $b = r2^s$ ($a' = 0$), and calculation of an upper bound of $||S(r,s,a,z)||_2$ relies on Case 2.

If $a = N$, then $b = r2^s$ and $a \le 2^s$ but the case $a = 2^s$ has been already considered. For $a < 2^s$ (hence $s > t_a$) an upper bound of $||S(r,s,a,z)||_2$ is given by the first part of Case 3.

It remains to work on $S(r,s,-a',z)$. The above analysis allows to eliminate $a' = 0$ and $a' = 2^t$ and the case $0 < a' < 2^t$ is related to the second part of Case 3 where a' replaces a.

Let us group all the bounds obtained above. Case 1 gives from (4.5) the upper bound

$$\Big\| \sum_{0 \le n \le N} z^n.\gamma(n) \Big\|_2 \le \sqrt{2} \sum_{m=0}^{t_N} 2^{\sigma(m)}\,.$$

Case 2 furnishes the inequality (4.6) which is better that the one considered in Case 1. Case 3 requires to make run a in the open interval $(0, N)$. Finally, inequalities (4.7) and (4.9) conduct easily to the following upper bound of $||\sum_{b\le n<b+N} z^n.\gamma(n)||_2$:

$$\max_{1\le a<N} \sqrt{2}\Big(\sum_{0\le m\le t_a} e_m(a)2^{\sigma(m)} + \sum_{0\le m\le t_{N-a}} e_m(N-a)2^{\sigma(m)}\Big)$$

which is less than $2\sqrt{2}\sum_{m=0}^{t_N} 2^{\sigma(m)}$. This value works well for $N = 1$ and $N = 2$. Summarizing, one gets the uniform upper bound exhibited in the theorem. ■

Corollary 4.1. *Assume that $\lambda^{-1}(1)$ is infinite, then*

$$\limsup_{N} \sup_{b\ge 0}\Big(\sup_{|z|\le 1} \Big\|\frac{1}{N}\sum_{b\le n\le b+N} z^n.\gamma(n)\Big\|_2\Big) = 0.$$

Proof. Let $K \ge 1$. There exists s_K such that $\sum_{0\le m<s_K} \lambda(m) = K$. Hence, for all integers $N \ge 1$ with $t_N + 1 > s_K$:

$$\begin{aligned}\Big\|\sum_{b\le n<N+b} z^n.\gamma(n)\Big\|_2 &\le 2\sqrt{2}\Big(\sum_{m=0}^{s_K-1} 2^m + 2^{-\frac{1}{2}K}\sum_{m=s_K}^{t_N} 2^m\Big) \\ &\le 2\sqrt{2}\Big(2^{s_K} + \frac{2N}{2^{K/2}}\Big).\end{aligned}$$

Since the choice of K is arbitrary, this equality proves the corollary. In fact for any $\varepsilon > 0$, we can choose a suitable K such that the above bound is less than εN for all integers N large enough. ■

Corollary 4.2. *Suppose there exist a real number $\alpha \in (0, 1]$ and an integer M such that*

$$\mathrm{card}(\lambda^{-1}(1) \cap [0, N)) \ge \alpha N$$

for all integers $N \ge M$. Then there exists an explicit constant $C(\alpha, M)$ (depending on α and M) such

$$\sup_{b\ge 0}\sup_{|z|\le 1} \Big\|\sum_{b\le n<b+N} z^n.\gamma(n)\Big\|_2 \le C(\alpha, M)N^{1-\frac{\alpha}{2}}.$$

Proof. Assume $N \ge M$. Theorem 4.3 implies

$$\Big\|\sum_{b\le n<N+b} z^n.\gamma(n)\Big\|_2 \le 2\sqrt{2}\Big(\sum_{m=0}^{M-1} 2^m + \sum_{m=0}^{t_N} 2^{m(1-\frac{1}{2}\alpha)}\Big).$$

Now by applying Lemma 4.1, the inequality

$$\sup_{|z|\leq 1} ||P_\eta(N,z)||_2 \leq 2\sqrt{2}\Big(2^M + \frac{1}{2^{1-\frac{1}{2}\alpha}-1}N^{1-\frac{1}{2}\alpha}\Big)$$

holds and Corollary 4.2 follows easily. ■

5. Applications

5.1. Well-distribution. Let $\psi(\cdot)$ be a sequence like $\gamma(\cdot)$ defined by (1.2) but with initial value v_0 in E_1. Then, $\psi(\cdot)$ takes its values in the multiplicative Abelian group $G = \{+1,-1\} \times \{+1,-1\}$. The non-trivial characters of G are the first coordinate map $p_0(\cdot)$, the second coordinate map $p_1(\cdot)$ and the product $\chi(\cdot) = p_0(\cdot)p_1(\cdot)$.

Theorem 5.1. *Assume that $\lambda^{-1}(1)$ is infinite. The corresponding above sequence $\psi(\cdot)$ is well-distributed in the group G.*

Proof. Theorem 4.3 implies

$$\lim_N \sup_{b\geq 0} \Big|\frac{1}{N}\sum_{b\leq n<b+N} p_i(\psi(n))\Big| = 0$$

for $i = 0, 1$. Now, let k be the smallest integer ℓ such that $\lambda(\ell) = 1$, then $\chi \circ \psi(n) = (-1)^{e_k(n)}$ if $n \geq 2^k$ due to $\chi(A_i^{(0)}e) = \chi(e)$ and $\chi(A_i^{(1)}e) = (-1)^i$ for all $e \in E_1$ and $i = 0, 1$. It follows

$$\lim_N \sup_{b\geq 0} \Big|\frac{1}{N}\sum_{b\leq n<b+N} \chi(\psi(n))\Big| = 0\,.$$

The well distribution of $n \mapsto \psi(n)$ in the group G derives from a Weyl's criterion concerning well distribution in compact metrizable Abelian groups. ■

5.2. Generalizations. Mixing automata can be realized in a more general context but always with linear automata having the same space of states and the same number of instructions. Lower and upper bounds obtained in our particular but significative case can be generalized as well by using similar methods. For example, in the non mixing case (see [1]), let $\mathcal{A} = (E, (A_j)_{0\leq j<q}, I_0)$ be a left-regular complete and linear semi-automaton in base q, with support a

real or complex vector space, generated by E and equipped with a norm $||\cdot||$. Define

$$\theta_{\mathcal{A}} := \sup_{|z|\le 1} |||A_0 + z.A_1 + \cdots + z^{q-1}.A_{q-1}|||$$

where $|||\cdot|||$ is the supremum norm for endomorphisms of V. Then,

Theorem 5.2. *Let γ be a sequence defined as in (1.2) with $\lambda(\cdot)$ constant equal to 0, $\mathcal{A}^{(0)} = \mathcal{A}$ and let $P_\gamma(N,z)$ as in (1.3). If $\mathcal{A}$ is contractive, i.e. $\theta_{\mathcal{A}} < q$ then there exists a constant C such that*

$$\sup_{|z|\le 1} ||P_\gamma(N,z)|| \le CN^{\log_q \theta_{\mathcal{A}}}\,.$$

Another generalization can be done in the spirit of [5]. It consists in replacing the sum (1.3) by

$$P_\gamma(N,f) := \sum_{0\le n<N} f(n).\gamma(n)\,,$$

where f is any unimodular 2-multiplicative sequence. Notice that $n \mapsto z^n$ is 2-multiplicative and this property is used in the key product formula (3.2).

We end with an explicit example issued from [7]. The sequence $g_0 : n \mapsto$ $\mapsto (-1)^{s_1(n)+s_1(3n)}$ is obviously 2-automatic. A linear automaton can be constructed from the sequence $g : \mathbb{N}_0 \to \mathbb{R}^3$ given by its coordinates $g_i(n) =$ $= (-1)^{s_1(n)+s_1(3n+i)}$. The space of states is $g(\mathbb{N}_0)$ and the instruction matrices give

$$A(z) = \begin{bmatrix} 1 & z & 0 \\ -1 & 0 & -z \\ 0 & 1 & z \end{bmatrix}.$$

One gets $\theta_{\mathcal{A}} = 2$ for the quadratic norm, that is not so surprising to compare with the case of PTM sequence which is also not contractive. The supremum giving $\theta_{\mathcal{A}}$ is reached with $z = \pm 1$. But $A(1)$ has eigenvalues 1 and $(1 \pm i\sqrt{7})/2$ (of modulus $\sqrt{2}$). Then, a standard computation involving Lemma 4.1 with $z = 1$ gives $||P_\gamma(N,1)||_2 \in \mathcal{O}(\sqrt{N})$. Further investigations in these directions are studied in [3].

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