CONTINUATION OF THE LAUDATION
to Professor Imre Kátai

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Five years ago in Annales Volume 28 (2008) Professor Járai Antal wrote the Laudation to Professor Imre Kátai. Kátai retired in 2008 and works as professor emeritus of the Eötvös Loránd University (ELTE). The titles

Honorary doctor and professor of Eötvös Loránd University,

and

Honorary doctor and professor of the University of Vilnius

were awarded to him.

In the last five years he has been participating in different projects, e.g. TÁMOP, DFG, TÉT, visited several universities and wrote several papers mainly with coauthors.

We enlarge the categories to classify his new results as follows:

1. Construction of normal numbers

Given an integer $q \geq 2$, we say that an irrational number η is a *q-normal number*, or simply a *normal number*, if the q-ary expansion of η is such that any preassigned sequence, of length $k \geq 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$. The problem of determining if a given number is normal is unresolved. For instance, fundamental constants such as π, e, $\sqrt{2}$, $\log 2$ as well as the famous Apéry constant $\zeta(3)$, have not yet been proven to be normal numbers, although numerical evidence tends to indicate that they are. This is even more astounding if we recall that in 1909, Borel [E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo **27** (1909), 247–271] has shown that almost all numbers are normal in every base.

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In their 1995 paper, J.M. De Koninck and I. Kátai [230] introduced the notion of a *disjoint classification of primes*, that is a collection of $q+1$ disjoint sets of primes $\mathcal{R}, \wp_0, \wp_1, \dots, \wp_{q-1}$, whose union is $\wp$, the set of all primes, where $\mathcal{R}$ is a finite set (perhaps empty) and where the other q sets are of positive densities $\delta_0, \delta_1, \dots, \delta_{q-1}$ (with clearly $\sum_{i=0}^{q-1} \delta_i = 1$); setting $A_q = \{0, 1, \dots, q-1\}$, letting an expression of the form $i_1 \dots i_k$, where each $i_j \in A_q$, be a *word* of length k, writing A_q^* as the set of all words regardless of their length, and using the function $H : \mathbb{N} \to A_q^*$ defined by $H(n) = H(p_1^{a_1} \cdots p_r^{a_r}) = \ell_1 \dots \ell_r$, where each ℓ_j is such that $p_j \in \wp_{\ell_j}$, they investigated the size of the set of positive integers $n \le x$ for which $H(n) = \alpha$ for a given word $\alpha \in A_q^k$. By this approach, in [354], they could show the following result:

Let $q \ge 2$ be an integer and let $\mathcal{R}, \wp_0, \wp_1, \dots, \wp_{q-1}$ be a disjoint classification of primes. Assume that, for a certain constant $c_1 \ge 5$,

$$\pi([u, u+v] \cap \wp_i) = \frac{1}{q}\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right) \tag{1}$$

uniformly for $2 \le v \le u$, $i = 0, 1, \dots, q-1$, as $u \to \infty$. Furthermore, let $H : \wp \to A_q^*$ be defined by

$$H(p) = \begin{cases} \Lambda & \text{if } p \in \mathcal{R}, \\ \ell & \text{if } p \in \wp_\ell \text{ for some } \ell \in A_q \end{cases} \tag{2}$$

(here Λ stands for the empty word) and further let $T : \mathbb{N} \to A_q^*$ be defined by $T(1) = \Lambda$ and for $n \ge 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \dots H(p_r). \tag{3}$$

Then, the number $0.T(1)T(2)T(3)T(4)\dots$ is a q-normal number.

In a subsequent paper [380], they weakened condition (1) to allow for the construction of even larger families of normal numbers. For instance, they showed the following result:

Assume that $\mathcal{R}, \wp_0, \dots, \wp_{q-1}$ are disjoint sets of primes, whose union is $\wp$, and assume that there exists a positive number $\delta < 1$ and a real number $c_1 \ge 5$ such that

$$\pi([u, u+v] \cap \wp_i) = \delta\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right)$$

holds uniformly for $2 \le v \le u$, $i = 0, 1, \dots, q-1$, and similarly

$$\pi([u, u+v] \cap \mathcal{R}) = (1 - q\delta)\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right).$$

Let H and T be defined as in (2) and (3). Then, the numbers $0.T(1)T(2)T(3)\ldots$ and $0.T(1)T(2)T(4)T(6)T(10)\ldots T(p-1)\ldots$ (where p runs through the sequence of primes) are q-normal numbers.

This type of result motivated Igor Shparlinski to ask if the number

$$0.P(2)P(3)P(4)\ldots,$$

where $P(n)$ stands for the largest prime factor of n, is a normal number in base 10. He further asked if the number $0.P(2+1)P(3+1)P(5+1)P(7+1)P(11+1)\ldots P(p+1)\ldots$ is also a normal number in base 10.

De Koninck and Kátai [350] answered in the affirmative to both these questions and proved even more. To understand their breakthrough, we must first introduce some notation. Let $q \geq 2$ be a fixed integer and let A_q be as above. Given a positive integer n, write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where $\varepsilon_i(n) \in A_q$ for $0 \leq i \leq t$ and $\varepsilon_t(n) \neq 0$. To this representation, associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \in A_q^{t+1}.$$

Let $F \in \mathbb{Z}[x]$ be a polynomial with positive leading coefficient and of positive degree r. Then the numbers

$$0.\overline{F(P(2))}\,\overline{F(P(3))}\,\overline{F(P(4))}\ldots$$

and

$$0.\overline{F(P(2+1))}\,\overline{F(P(3+1))}\ldots\overline{F(P(p+1))}\ldots$$

are normal numbers in base q.

In [363], they used polynomials to further construct various families of normal numbers. Let $Q_1, Q_2, \ldots, Q_h \in \mathbb{Z}[x]$ be distinct irreducible primitive monic polynomials each of degree no larger than 3. For each $\nu = 0, 1, 2, \ldots, q-1$, let $c_1^{(\nu)}, c_2^{(\nu)}, \ldots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^{h} Q_j(x + c_j^{(\nu)})$, with $F_\nu(0) \neq 0$ for each ν. Moreover, assume that the integers $c_i^{(\nu)}$ are chosen in such a way that $F_\nu(x)$ are squarefree polynomials and $\gcd(F_\nu(x), F_\mu(x)) = 1$ when $\nu \neq \mu$. Let $\wp_0$ be the finite set of prime numbers p for which there exist $\mu \neq \nu$ and $m \in \mathbb{N}$ such that $p|\gcd(F_\nu(m), F_\mu(m))$. Now let

$$U(n) = F_0(n)F_1(n)\cdots F_{D-1}(n) = \vartheta\, p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r},$$

where $\vartheta \in \mathcal{N}(\wp_0)$ and $p_1 < p_2 < \cdots < p_r$ are primes not belonging to $\mathcal{N}(\wp_0)$, while the a_i's are positive integers. Then, let h_n be defined on the prime divisors p^a of $U(n)$ by

$$h_n(p^a) = h_n(p) = \begin{cases} \Lambda & \text{if } p|\vartheta, \\ \ell & \text{if } p|F_\ell(n),\ p \notin \wp_0 \end{cases}$$

and further define $\alpha_n := h_n(p_1^{a_1})h_n(p_2^{a_2})\dots h_n(p_r^{a_r})$, where on the right hand side we omit Λ, the empty word, when $h_n(p_i^{a_i}) = \Lambda$ for some i. They considered the real number η whose q-ary expansion is given by $\eta = 0.\alpha_1\alpha_2\alpha_3\dots$ and showed that η is a q-normal number. Moreover, assuming that $\deg(Q_j) \le 2$ for $j = 1, 2, \dots, h$, they proved that the number $0.\alpha_2\alpha_3\alpha_5\dots\alpha_p\dots$ (where the subscripts run over primes p) is a normal number.

Recently, in the paper [376] dedicated to Karl-Heinz Indlekofer on the occasion of his 70^{th} anniversary, they used the large prime divisors of integers to construct other families of normal numbers. Indeed, let $\eta(x)$ be a slowly varying function, that is a function satisfying $\lim_{x\to\infty} \dfrac{\eta(cx)}{\eta(x)} = 1$ for any fixed constant $c > 0$, and assume also that $\eta(x)$ does not tend to infinity too fast in the sense that it satisfies the additional condition $\dfrac{\log \eta(x)}{\log x} \to 0$ as $x \to \infty$. Then, let $Q(n)$ be the smallest prime divisor of n which is larger than $\eta(n)$, while setting $Q(n) = 1$ if $P(n) > \eta(n)$. Then, they showed that the real number $0.\overline{Q(1)}\,\overline{Q(2)}\,\overline{Q(3)}\dots$ is a q-normal number. With various similar constructions, they created large families of normal numbers in any given base $q \ge 2$. For instance, consider the product function $F(n) = n(n+1)\cdots(n+q-1)$. Observe that if for some positive integer n, we have $p = Q(F(n)) > q$, then $p|n+\ell$ only for one $\ell \in \{0, 1, \dots, q-1\}$, implying that ℓ is uniquely determined for all positive integers n such that $Q(F(n)) > q$. Thus we may define the function

$$\tau(n) = \begin{cases} \ell & \text{if } p = Q(F(n)) > q \text{ and } p|n+\ell, \\ \Lambda & \text{otherwise.} \end{cases}$$

Using this notation, they proved that the number $0.\tau(q+1)\tau(q+2)\tau(q+3)\dots$ is a q-normal number.

Recently, in [383], De Koninck and Kátai used a totally different approach to create normal numbers. Their idea is based on the behaviour of the size of the gap between the prime factors of integers. It goes as follows. Let $q \ge 2$ be a fixed integer. Given a positive integer $n = p_1^{e_1}\cdots p_{k+1}^{e_{k+1}}$ with primes $p_1 < \cdots < p_{k+1}$ and positive exponents $e_1, \dots, e_{k+1}$, consider the numbers $c_1(n), \dots, c_k(n)$ defined by

$$c_j(n) := \left\lfloor \frac{q \log p_j}{\log p_{j+1}} \right\rfloor \in A_q \qquad (j = 1, \dots, k)$$

and introduce the arithmetic function

$$H(n) = \begin{cases} c_1(n)\dots c_k(n) & \text{if } \omega(n) \ge 2, \\ \Lambda & \text{if } \omega(n) \le 1, \end{cases}$$

where $\omega(n)$ stands for the number of distinct prime factors of the integer $n \ge 2$, with $\omega(1) = 0$. They showed that the number $0.H(1)H(2)H(3)\dots$ is a q-normal number.

Reduced residue classes can also yield a way of producing normal numbers. Indeed, letting φ stand for the Euler function, set $B_{\varphi(q)} = \{\ell_1, \dots, \ell_{\varphi(q)}\}$ as the set of reduced residues modulo q. Let ε_n be a real function which tends monotonically to 0 as $n \to \infty$ but in such a way that $(\log\log n)\varepsilon_n \to \infty$ as $n \to \infty$. Letting $p(n)$ stand for the smallest prime factor of n, consider the set

$$\mathcal{N}^{(\varepsilon_n)} := \{n \in \mathbb{N} : p(n) > n^{\varepsilon_n}\} = \{n_1, n_2, \dots\}.$$

De Koninck and Kátai proved in [384] that the infinite word $\mathrm{res}_q(n_1)\mathrm{res}_q(n_2)\dots$, where $\mathrm{res}_q(n) = \ell$ if $n \equiv \ell \pmod q$, contains every finite word whose digits belong to $B_{\varphi(q)}$ infinitely often.

Is it possible to generate normal numbers using the k-th largest prime factor of an integer? It is! Indeed, given an integer $k \geq 1$, for each integer $n \geq 2$, let $P_k(n)$ stand for the k-th largest prime factor of n if $\omega(n) \geq k$, while setting $P_k(n) = 1$ if $\omega(n) \leq k-1$. Thus, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ stands for the prime factorization of n, where $p_1 < p_2 < \cdots < p_s$, then

$$P_1(n) = P(n) = p_s, \qquad P_2(n) = p_{s-1}, \qquad P_3(n) = p_{s-2}, \dots$$

Let $F \in \mathbb{Z}[x]$ be a polynomial of positive degree satisfying $F(x) > 0$ for $x > 0$. Also, let $T \in \mathbb{Z}[x]$ be such that $T(x) \to \infty$ as $x \to \infty$ and assume that $\ell_0 = \deg T$. Fix an integer $k \geq \ell_0$. Then, De Koninck and Kátai showed in [385] that the numbers

$$0.\overline{F(P_k(T(2)))}\,\overline{F(P_k(T(3)))}\dots\overline{F(P_k(T(n)))}\dots$$

and (assuming that $k \geq \ell_0 + 1$)

$$0.\overline{F(P_k(T(2+1)))}\,\overline{F(P_k(T(3+1)))}\dots\overline{F(P_k(T(p+1)))}\dots$$

are q-normal numbers.

2. Exponential sums and uniform distribution mod 1

It is known, since Vinogradov, that, given any irrational number α, the sequence αp_n, $n = 1, 2, \dots$, where p_n stands for the n-th prime, is uniformly distributed in $[0, 1]$. In 2005, Banks, Harman and Shparlinski proved that for every irrational number α, the sequence $\alpha P(n)$, $n = 1, 2, \dots$, where $P(n)$ stands for the largest prime factor of n, is uniformly distributed mod 1. They did so by establishing that $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} e(\alpha P(n)) = 0$, where $e(z) := \exp\{2\pi i z\}$. In [347], De Koninck and Kátai generalized this result by proving that, given any complex valued multiplicative function f such that $|f(n)| = 1$ for all positive

integers n, then $\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n)e(\alpha P(n)) = 0$. They also showed that this general result further holds if one replaces $e(\alpha P(n))$ by $T(P(n))$, where T is any function defined on primes satisfying $|T(p)| = 1$ for all primes p and such that $\sum_{p\le x} T(p) = o(\pi(x))$, where $\pi(x)$ stands for the number of primes $\le x$. They also proved that if f is a complex-valued multiplicative function satisfying $|f(n)| = 1$ for all positive integers n and if the series $\displaystyle\sum_p \frac{1 - \Re\left(f(p)p^{-it}\right)}{p}$ converges for some $t \in \mathbb{R}$, then, given any irrational number α,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)\, e(\alpha P(n-1)) = 0.$$

Now, let $Q(n) = \alpha_k n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_1 n$ be a polynomial with real coefficients and such that at least one of the coefficients $\alpha_k, \ldots, \alpha_1$ is an irrational number. Let f be a complex-valued multiplicative function such that $|f(n)| \le 1$ for all positive integers n. Let $F_1(x), \ldots, F_s(x)$ be polynomials with integer coefficients which take only positive values at positive arguments. For $j = 1, \ldots, s$, let $\rho_j(d)$ stand for the number of solutions of $F_j(n) \equiv 0 \pmod d$. Moreover, let $\rho(d_1, \ldots, d_s)$ be the number of solutions of the congruence system $F_j(n) \equiv 0 \pmod{d_j}$, $j = 1, \ldots, s$. Further let $g_1, \ldots, g_s$ be complex valued multiplicative functions each satisfying the following four conditions:

(i) $|g_j(n)| = 1$ for all $n \in \mathbb{N}$;

(ii) g_j is strongly multiplicative;

(iii) $\lim_{p\to\infty} g_j(p) = 1$;

(iv) $\displaystyle\sum_p \frac{\Re(1 - g_j(p))\rho_j(p)}{p} < \infty$.

Consider the arithmetic function $\ell(n) := g_1(F_1(n)) \cdots g_s(F_s(n))$ and define $S_f(x) := \displaystyle\sum_{n\le x} f(n)\ell(n)e(Q(n))$. In [362], De Koninck and Kátai showed that

$$\sup_{f\in\mathcal{M}_1} \frac{|S_f(x)|}{x} \to 0 \qquad \text{as } x \to \infty$$

and

$$\left| \frac{1}{\mathrm{li}(x)} \sum_{q\le x} \ell(q)e(Q(q)) \right| \to 0 \qquad \text{as } x \to \infty.$$

De Koninck and Kátai then further studied exponential sums using their notion of disjoint classification of primes, in the following manner. Let β be an

arbitrary irrational number. Assume that

$$\wp = \wp_0 \cup \wp_1 \ldots \cup \wp_{q-1} \cup \mathcal{R},$$

where $\wp_0, \wp_1, \ldots, \wp_{q-1}, \mathcal{R}$ are disjoint subsets of primes, with $\mathcal{R}$ containing at most finitely many elements (and in fact maybe none). Let $\pi([a,b])$ be the number of primes belonging to the interval $[a,b]$. Let $\pi(I|\wp_i) = \#\{p \in \wp_i \cap I\}$. In what follows assume that

$$\pi([u,u+v]|\wp_i) = \delta_i \pi([u,u+v]) + O\left(\frac{u}{(\log u)^{c_1}}\right)$$

holds uniformly for $2 \le v \le u$, $i = 0, \ldots, q-1$, where c_1 is a positive constant and $\delta_0, \ldots, \delta_{q-1}$ are positive constants such that $\sum_{i=0}^{q-1} \delta_i = 1$. Given an integer $t \ge 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each i_j is one of the numbers $0, 1, \ldots, q-1$, is called a *word* of length t. Let $\mathcal{A}_t$ be the set of all words of length t. Define $\mathcal{A}_0$ to be the set containing the *empty word* Λ and set $\mathcal{A}^* := \bigcup_{i=0}^{\infty} \mathcal{A}_i$. Define the arithmetic functions H, u and v as follows. First let $H(1) = \Lambda$ and then, for an arbitrary prime number p and positive integer a, define

$$H(p^a) = \begin{cases} \Lambda & \text{if} \quad p \in \mathcal{R}, \\ j & \text{if} \quad p \in \wp_j. \end{cases}$$

Then, for integers $n = b p_1^{a_1} \cdots p_r^{a_r}$, where all the prime factors of b belong to $\mathcal{R}$ and where $p_1 < \cdots < p_r$ are primes and each $a_i \in \mathbb{N}$, let

$$\begin{aligned} H(n) &:= H(p_1^{a_1}) \ldots H(p_r^{a_r}), \\ u(n) &:= H(p_1^{a_1}) + H(p_2^{a_2}) q + \cdots + H(p_r^{a_r}) q^{r-1}, \\ v(n) &:= H(p_r^{a_r}) + H(p_{r-1}^{a_{r-1}}) q + \cdots + H(p_1^{a_1}) q^{r-1}. \end{aligned}$$

With this setup, in [379], De Koninck and Kátai showed various results concerning exponential sums and in particular that if

$$S_1(x) = \sum_{n \le x} e(u(n)\beta) \qquad \text{and} \qquad S_2(x) = \sum_{n \le x} e(v(n)\beta),$$

then, for $j = 1, 2$,

$$\lim_{x \to \infty} \frac{S_j(x)}{x} = 0.$$

In a paper dedicated to Jean-Paul Allouche on the occasion of his 60^{th} birthday, De Koninck and Kátai [386] proved that, if $P_k(n)$ stands for the k-th largest prime factor of $n \ge 2$ and if α is an irrational number, while f is a

multiplicative function such that $|f(n)| = 1$ for all positive integers n, then $\sum_{n\le x} f(n)\exp\{2\pi i\alpha P_k(n)\} = o(x)$ as $x \to \infty$.

In [388], they obtained various results involving exponential sums and shifted primes. In particular they proved the following result:

Let f be a complex-valued multiplicative function with $|f(n)| = 1$ for $n = 1, 2, 3, \ldots$, and let T be a function defined on prime numbers and satisfying $|T(p)| = 1$ for each prime p and such that $\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{\substack{p\le x \\ p\equiv \ell \pmod d}} T(p) = 0$ *for every fixed integers $d > 0$ and $\ell \neq 0$ satisfying $(\ell, d) = 1$. Then, assuming that the series* $\sum_p \frac{1 - \Re(f(p)p^{-i\tau})}{p}$ *is convergent, we have that*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} f(p+1)T(p) = 0.$$

3. Arithmetic functions defined on special sets

Let $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be a multiplicative function such that $f(p^a)$ depends only on a for all prime powers p^a. Let $Q_1, Q_2, \ldots, Q_h$ be distinct irreducible primitive monic polynomials each of degree no larger than 3. For each $\nu = 1, 2, \ldots, t$, let $c_1^{(\nu)}, c_2^{(\nu)}, \ldots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^h Q_j(x + c_j^{(\nu)})$ $(\nu = 1, 2, \ldots, t)$, with $F_\nu(0) \neq 0$ for each ν. Let us assume that $(F_\nu(x), F_\mu(x)) = 1$ if $\nu \neq \mu$. In the paper [358] dedicated to János Galambos on his 70^{th} birthday, De Koninck, Doyon and Kátai showed that there exists a non negative constant d_0 such that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : f(F_\ell(n)) \text{ divides } f(F_{\ell+1}(n)) \text{ for } \ell = 1, 2, \ldots, t-1\} = d_0.$$

Interesting arithmetic functions to which one can apply this result are $\tau(n)$ (the number of divisors of n), $\tau_k(n)$ (the number of ways one can write n as the product of k positive integers taking into account the order in which they are written), $\beta(n)$ (the product of the exponents in the prime factorization of n) and $a(n)$ (the number of finite non isomorphic abelian groups with n elements).

In the paper [368] dedicated to Dr. Bui Minh Phong on his 60^{th} anniversary, De Koninck and Kátai established short interval estimates for a given strongly additive function satisfying certain conditions and restricted to the set of shifted primes and they also considered similar sums, but running on sets of integers $m+1$, where each integer m has a fixed number of prime factors. For instance, they proved the following:

Let $\varepsilon > 0$ be a fixed small number. Let $I_{x,y} = [x, x+y]$, where $x^{\frac{7}{12}+\varepsilon} \le y \le x$, and let $\pi(I_{x,y}) := \sum_{p \in I_{x,y}} 1$. If g is a a strongly multiplicative function such that $|g(p)| \le 1$ and $g(p) \to 1$ as $p \to \infty$ and if the infinite sum $\sum_p \frac{1-g(p)}{p}$ converges, then,

$$\max_{x^{7/12+\varepsilon} \le y \le x} \left| \frac{1}{\pi(I_{x,y})} \sum_{p \in I_{x,y}} g(p+1) - M(g) \right| \to 0 \quad \text{as } x \to \infty,$$

where $M(g) := \prod_p \left(1 + \frac{g(p)-1}{p-1}\right)$.

Another important typical result proved in this paper can be stated as follows:

Let f be a strongly additive function such that $f(p) \neq 0$ for all primes p and such that $f(p) \to 0$ as $p \to \infty$. Let $A(x) = \sum_{p \le x} \frac{f(p)}{p-1}$ and assume that $\sum_p \frac{f^2(p)}{p} < \infty$. Moreover, let

$$\varphi(\tau) := \prod_p \left(1 + \frac{e^{i\tau f(p)} - 1}{p-1}\right) e^{-i\tau f(p)/(p-1)}$$

and let $F(u)$ be the distribution function whose characteristic function is $\varphi(\tau)$. Let also

$$F^{(k)}_{I_{x,y}}(u) := \frac{1}{\Pi_k(I_{x,y})} \#\{m \in I_{x,y} \cap \wp_k : f(m+1) - A(x) < u\},$$

where $\wp_k = \{n \in \mathbb{N} : \omega(n) = k\}$ and $\Pi_k(I_{x,y}) = \#\{n \in I_{x,y} : \omega(n) = k\}$. Then,

$$\lim_{x \to \infty} \sup_{k \le k_x} \max_{x^{7/12+\varepsilon} \le y \le x} \max_{u \in \mathbb{R}} \left| F^{(k)}_{I_{x,y}}(u) - F(u) \right| = 0.$$

In [378], De Koninck and Kátai studied the distribution of the values of certain additive functions restricted to those integers with a fixed number of prime divisors. Indeed, given an additive function f for which there exists a real number $C > 0$ such that $|f(p^a)| < C$ for all prime powers p^a, let

$$A_x = \sum_{p \le x} \frac{f(p)}{p}$$

and let f^* be the additive function (which depends on x) defined on prime powers by $f^*(p^a) = f(p^a) - \dfrac{a}{x_2}A_x$, where $x_2 = \log\log x$. Further let

$$B_x = \sqrt{\sum_{p\le x} \frac{(f^*(p))^2}{p}}$$

and assume that $B_x \to \infty$. Then, for each integer $k \ge 1$, let

$$\xi_{k,x} := \frac{k}{x_2}, \qquad \wp_k := \{n \in \mathbb{N} : \omega(n) = k\}, \qquad \pi_k(x) := \#\{n \le x : n \in \wp_k\}.$$

Finally, let $\delta < \frac{1}{2}$ be a fixed positive number. Then, they proved that

$$\lim_{x\to\infty} \max_{\substack{k \\ \xi_{k,x}\in[\delta,2-\delta]}} \max_{y\in\mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \le x : n \in \wp_k,\ \frac{f^*(n)}{B_x\sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0,$$

where $\Phi(y) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{y} e^{-u^2/2}\, du.$

In the paper [381] dedicated to Professor Jonas Kubilius on the occasion of his 90^{th} anniversary, De Koninck and Kátai studied the normality of the distribution of consecutive digits in the q-ary expansion of integers belonging to particular subsequences of the positive integers.

In [382], they studied the distribution of arithmetic functions restricted to particular subsets of integers. Let $Q_1, \dots, Q_t \in \mathbb{R}[x]$ be polynomials with no constant term for which each linear combination $m_1Q_1(x)+\cdots+m_tQ_t(x)$, with $m_1, \dots, m_t \in \mathbb{Z}$ and not all 0, always has an irrational coefficient. Let $I_1, \dots, I_t$ be sets included in the interval $[0, 1)$, each of which being a union of finitely many subintervals of $[0, 1)$. Furthermore, let $\mathcal{T}$ be the set of those positive integers n for which $\{Q_1(n)\} \in I_1, \dots, \{Q_t(n)\} \in I_t$ holds simultaneously, where $\{y\}$ stands for the fractional part of y. Let $t_1, t_2, \dots$ be a sequence of complex numbers uniformly summable and set $T(x) = \sum_{n\le x} t_n$ and $T(x|\mathcal{T}) = \sum_{\substack{n\le x \\ n\in\mathcal{T}}} t_n$. They proved that, as $x \to \infty$, $T(x)/x \sim T(x|\mathcal{T})/(\lambda(I_1)\cdots\lambda(I_t)x)$, where $\lambda(I)$ stands for the Lebesgue measure of the set I.

Finally, in [387], they studied the distribution modulo 1 of sequences involving the largest prime factor function $P(n)$. More precisely, they proved the following result:

Let $g : [1, \infty) \to \mathbb{R}$ be a differentiable function and let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(u) = g(\log u)$. Assume that the function $vg'(v)$ is increasing and tends to infinity. For $x \ge 2$, let $R(x) := \pi(x) - li(x)$ be the error term in the

Prime Number Theorem and further assume that, for any given real number $d > 0$,

$$\lim_{y\to\infty} \int_y^{y^{1+d}} \frac{|R(u)|}{u} |f'(u)|\, du = 0.$$

Then the sequence $(f(P(n)))_{n\geq 1}$ *is uniformly distributed modulo 1.*

They also showed that, given an arbitrary real number $\tau \neq 0$ and an arbitrary integer $k \geq 1$, the sequence $(\tau \log P_k(n))_{n\geq 2}$ is uniformly distributed modulo 1.

4. The Theorem of H. Daboussi

According to a 1974 result of Daboussi and Delange, if we denote by $\mathcal{M}_1$ the set of all complex-valued multiplicative functions whose modulus is bounded by 1, then for every irrational number α, $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n)e(n\alpha) = 0$, uniformly for $f \in \mathcal{M}_1$. Their proof is based on the large sieve inequality. In [330], Kátai used a Turán-Kubilius type inequality to prove a more general result, namely the following:

Let t *be an arbitrary real-valued arithmetic function and assume that for every positive constant* K, *there exist primes* $p_1, \ldots, p_R$ *such that* $\sum_{j=1}^{R} \frac{1}{p_j} > K$ *and such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(t(p_i n) - t(p_j n)) = 0$$

for any distinct numbers $i, j \in \{1, \ldots, R\}$. *Then there exists a sequence of positive real numbers* (ρ_N) *tending to 0 as* $N \to \infty$ *and such that*

$$\sup_{f\in\mathcal{M}_1} \left| \frac{1}{N} \sum_{n\leq N} f(n)e(t(n)) \right| \leq \rho_N.$$

Let G^* be the multiplicative group of Gaussian integers, W be the union of finitely many convex domains in $\mathbb{C}$, and set

$$E = \{z \in \mathbb{C} : 0 \leq \Re(z) < 1,\ 0 \leq \Im(z) < 1\}.$$

In [344], continuing the work done in a 2003 paper by Bassily and Kátai, De Koninck and Kátai explored the asymptotic distribution mod E of functions

defined on G^*. For instance, they showed that, if $P(z)$ is a polynomial of degree $k \geq 1$ with leading coefficient a, if $F : G* \to \mathbb{C}$ is an arbitrary additive function and if we consider the distribution function

$$V_x(u,v) := \frac{1}{N(x)} \#\{\alpha \in xW \cap G^* : \Re(P(\alpha)+F(\alpha)) < u,\ \Im(P(\alpha)+F(\alpha)) < v\},$$

where $u + iv \in E$ and $N(x) := \#\{\alpha \in xW \cap G^*\}$, then

$$V_x(u,v) = uv + o(1) \qquad \text{as } x \to \infty$$

uniformly for $0 \leq u, v \leq 1$ provided that the numbers $1, \Re(a), \Im(a)$ are rationally independent.

In [348], Kátai considered an analogue of Daboussi's theorem on some semigroups of integers generated by particular sets of primes. More precisely, let $1, \beta_1, \ldots, \beta_t$ be real numbers which are rationally independent; let $I_1, I_2, \ldots, I_t \subset [0,1)$ be sets, each of which is a finite union of intervals; let $\wp^*$ be the set of prime numbers p for which $\{\beta_j p\} \in I_j$ for $j = 1, \ldots, t$ (here $\{y\}$ stands for the fractional part of y); finally, let $\mathcal{B}^*$ be the semigroup generated by $\wp^*$. Then, in this context, Kátai proved that if α is an irrational number satisfying certain conditions, then, given an arbitrary additive function F, the values $F(n) + \alpha n$, as n runs through $\mathcal{B}^*$, are uniformly distributed.

In [353], Bassily and Kátai further expanded Daboussi's theorem over Gaussian integers. But first observe that in a 2003 paper, Bassily, De Koninck and Kátai had established that, if W stands for the union of finitely many convex bounded domains in $\mathbb{C}$ and if $\mathcal{A}$ stands for the set of those additive characters χ such that $\chi(1) = e^{2\pi i A}$ and $\chi(i) = e^{2\pi i B}$, where at least one of A and B is irrational, and if $\chi \in \mathcal{A}$, then for every multiplicative function $g : \mathbb{Z}[i] \setminus \{0\} \to \mathbb{C}$ such that $|g(\alpha)| \leq 1$, $\lim_{x\to\infty} \frac{1}{|xW|} \sum_{\beta \in xW} g(\beta)\chi(\beta) = 0$, where the convergence is uniform in g. Recently, in a joint paper, Bassily and Kátai [353] proved analogous results for the summation domain $\beta \in xW \cap \mathcal{J}$, where $\mathcal{B} = \{\beta \in \mathbb{Z}[i] \setminus \{0\} : \{\gamma_j \beta\} \in S,\ j = 1, \ldots, k\}$, where γ_j are complex numbers satisfying certain conditions (here the fractional part of a complex number z is $\{z\} = \{\Re(z)\} + i\{\Im(z)\}$) and S is the union of domains whose boundaries are given by rectifiable continuous curves.

5. Distribution of q-additive functions

Recall the notion of q-additive and q-multiplicative functions. Let $q \geq 2$ be an integer and let $A_q = \{0, 1, \ldots, q-1\}$. Given a nonnegative integer n, write

its q-ary expansion as $n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)$, where each $\varepsilon_i(n) \in A_q$ and $\varepsilon_t(n) \neq 0$. A function f defined on the nonnegative integers is called a q-additive (resp. multiplicative) function if it satisfies $f(0) = 0$ (resp. $f(0) = 1$) and $f(n) = \sum_{i=0}^{\infty} f(\varepsilon_j(n)q^j)$ (resp. $f(n) = \prod_{i=0}^{\infty} f(\varepsilon_j(n)q^j)$) for all integers $n \geq 0$.

In [323], Kátai and Subbarao obtained results on the distribution of q-additive functions on some subsets of integers. Given a q-additive function f, consider the corresponding function

$$F(y) = \lim_{N\to\infty} \frac{1}{q^N} \#\{n < q^N : f(n) < y\}.$$

It is a consequence of the 3-series theorem of Kolmogorov that for any q-additive function f, the corresponding function F exists and it is a distribution function if and only if the series $\sum_{j=0}^{\infty}\sum_{b=1}^{q-1} f(bq^j)$ and $\sum_{j=0}^{\infty}\sum_{b=1}^{q-1} f^2(bq^j)$ are both convergent. In [323] investigate the distribution of q-additive functions over the set of integers having a fixed number of prime factors.

In a joint paper with N. L. Bassily [333], the following analogue of the theorem of H. Daboussi is proved:

Let $q \geq 2$, $q \in \mathbb{N}$, $t : \mathbb{N}_0 \to \mathbb{R}$. Suppose that for all $\nu \in \mathbb{N}$ and $0 \leq a_1, a_2 < q^\nu, a_1 \neq a_2$, the sequence

$$\eta_{a_1,a_2}(b) := t(\alpha_1 + bq^\nu) - t(\alpha_2 + bq^\nu)$$

satisfies the relation

$$\frac{1}{x}\sum_{b<x} e(\eta_{a_1,a_2}(b)) \to 0 \quad (x \to \infty).$$

Then

$$\sup_{g\in\overline{\mathcal{M}}_q} |\frac{1}{x}\sum_{n\leq x} g(n)e(t(n))| \to 0 \quad (x \to \infty),$$

where $\overline{\mathcal{M}}_q$ is the set of q-multiplicative function g such that $|g(n)| \leq 1$ $(n \in \mathbb{N})$.

In [360], Kátai continued his work done with M.V. Subbarao on the distribution of the values of q-additive functions. Again, using the representation $n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t$, let $\alpha(n) := \varepsilon_0(n) + \cdots + \varepsilon_t(n)$ and, for each $h \in A_q$, set $\beta_h(n) := \#\{j \geq 0 : \varepsilon_j(n) = h\}$. Kátai then examines the asymptotic mean values of a complex-valued q-multiplicative function g satisfying $|g(n)| = 1$ for all integers $n \geq 0$ and the distribution of the

values of a real-valued q-additive function on subsets $S_N(\overline{r}) := \{0 \leq n < q^N : \beta_0(n) = r_0, \ldots, \beta_{q-1}(n) = r_{q-1}\}$, where $\overline{r} = (r_0, \ldots, r_{q-1}) \in \mathbb{N}_0^q$, or $\{0 \leq n < q^N : \alpha(n) = k\}$, as $N \to \infty$.

In papers [369] and [370] written jointly with L. Germán, Kátai studied the distribution of q-additive functions on multiplicative semigroups. More precisely, in [369], they investigated the existence of limit distribution of q-additive functions over the set of integers characterized by the sum of its digits, while in [370], they examined polynomial sequences of semigroups whose subsets contain integers with a given number of prime divisors.

6. On random arithmetical functions. Renewal theorems

In a series of joint papers ([341], [359] and [374]) written with K.-H. Indlekofer and O. I. Klesov, Kátai studied random arithmetic functions.

More precisely, let $\xi_p \ \ (p \in \mathcal{P} = \text{set of primes})$ be independent random variables distributed uniformly on $A_Q = \{\xi | \xi^Q = 1\}$. Let f be a completely multiplicative function defined on $\mathcal{P}$ by $f(p) = \xi_p$. They investigated the density of those n for which $f(n+j) = \kappa_j \ \ (j = 0, \ldots, t), \ \kappa_j \in A_Q$.

In [374], they examine the mean values of random multiplicative functions over polynomial values and the mean values of random multiplicative functions defined on the set of gaussian integers.

In [359], they obtain some renewal theorems with weighted renewal functions.

7. Arithmetical functions satisfying some relations

Kátai conjectured that if $f_0, \ f_1, \ \ldots, f_k$ are real-valued completely additive functions and

$$f_0(n) + f_1(n+1) + \cdots + f_k(n+k) \equiv 0 \pmod 1$$

for all $n \in \mathbb{N}$, then

$$f_0(n) \equiv f_1(n) \equiv \cdots \equiv f_k(n) \equiv 0 \pmod 1$$

are satisfied for all $n \in \mathbb{N}$. This conjecture has been proved for $k = 2, 3$ by I. Kátai [142, 150]. Recently, in a joint paper [361] with K. Chakraborty and B. M. Phong, the case $k = 4$ is proved by assuming the fulfilment of relation

$$f_0(n) + f_1(n+1) + \cdots + f_4(n+4) \equiv 0 \pmod 1$$

for every $n \in \mathbb{Z}$.

In another joint paper [372] with K. Chakraborty and B. M. Phong, the following result is proved:

Assume that G is an Abelian group with identity element 0 and $f_0,\ f_1,\ f_2$ are G-valued completely additive functions. We have

(a) *If* $f_0(n) + f_1(2n+1) + f_2(n+2) = 0 \quad (\forall n \in \mathbb{N})$, *then*

$$f_0(n) = f_1(2n+1) = f_2(n) = 0 \quad (\forall n \in \mathbb{N}).$$

(b) *If* $f_0(n) + f_1(2n-1) + f_2(n+2) = 0 \quad (\forall n \in \mathbb{N})$, *then*

$$f_0(n) = f_1(2n-1) = f_2(n) = 0 \quad (\forall n \in \mathbb{N}).$$

In a joint paper of K. Chakraborty and B. M. Phong [373] the values of arithmetic functions in short intervals are investigated. The following result is proved:

For positive integer d and $c > 0$, let $J_c(n) = [n, n + c\sqrt{n}]$ and $\mathcal{K}_d = \{n \in \in \mathbb{N} \mid (n,d) = 1\}$. Let $1 < N_1 < N_2 < \cdots$ be an infinite sequence of integers and assume that $\ell_1, \ell_2, \ldots$ are integers coprime to d. Assume that f and g are completely additive functions defined on $\mathcal{K}_d$ such that $f(n) = g(n)$ if $n \equiv \ell_j \pmod d$, $n \in J_c(N_j)$ $(j = 1, 2, \ldots)$. If $c > 2d$, then $f(n) = g(n)$ identically on $\mathcal{K}_d$.

Dear Kátai Imre, we wish you a happy 75-th birthday and we also wish you and your family very good health and spirits.