APPROXIMATELY CONVEX FUNCTIONS

Zoltán Boros (Debrecen, Hungary)

Noémi Nagy (Budapest, Hungary)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

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Abstract. A Rolewicz type theorem concerning the superstability of approximate convexity is established. Namely, it is proved that any real valued function f, defined on an open, convex subset D of a linear normed space, which satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + c\,(\lambda(1-\lambda)\|x-y\|)^p$$

for every $x, y \in D$ and $\lambda \in [0,1]$, with a fixed non-negative real number c, and a fixed exponent $p > 1$, has to be convex, i.e., satisfies the above inequality with $c = 0$ as well.

1. Introduction

Investigations of approximate convexity, in various cases, usually involves the study of functions f satisfying an inequality of the form

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C\Phi\,(t\,,\,1-t)\,\psi\,(\|x-y\|)\,, \tag{1.1}$$

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where $f : D \to \mathbb{R}$ is defined on a convex, open subset D of a normed space X, $\|u\|$ denotes the norm of $u \in X$, C is a (usually non-negative) fixed real number, $\Phi : [0,1] \times [0,1] \to \mathbb{R}$ and $\psi : [0,+\infty[\to \mathbb{R}$ are given functions, while inequality (1.1) is supposed to hold for all $t \in [0,1]$ and x, $y \in D$. In several papers, investigations are restricted to the case $X = \mathbb{R}$, when f is defined on an open interval and $\|u\|$ has to be replaced by the absolute value $|u|$ of the real number u.

In case $C = 0$, the inequality (1.1) describes the concept of convex functions. If $C \geq 0$ and $\Phi(t, 1-t) = \psi(\|x-y\|) = 1$ for all $t \in [0,1]$, $x, y \in D$, a function $f : D \to \mathbb{R}$ satisfying (1.1) is called C-convex. The first investigations of C-convex functions are due by Hyers and Ulam [6]. According to their result, if the underlying space X is of finite dimension n and the function f is C-convex, then there exists a convex function $g : D \to \mathbb{R}$ such that

$$|f(x) - g(x)| \leq k_n C$$

for all $x \in D$. Concerning the constant k_n, they established the inequality

$$k_n \leq \frac{n(n+3)}{4(n+1)}.$$

C-convex functions were studied by Green [3] as well. He obtained better estimations. On the other hand, Laczkovich [7] proved that $k_n \geq \frac{1}{4}\log_2(n/2)$. This estimation shows that the statement cannot be extended to infinite dimensional spaces. A counterexample in this direction was earlier constructed by Casini and Papini [2].

Luc, Ngai and Théra [8] investigated the solutions f of the inequality (1.1) when $\Phi(t,s) = ts$ and $\psi(h) = h$, X is a Banach space. They assumed, in addition, that f is lower semicontinuous.

In a series of papers, Rolewicz introduced and investigated the concepts of ψ-paraconvex and strongly ψ-paraconvex functions, corresponding to the choices $\Phi(t,s) = 1$ and $\Phi(t,s) = \min\{t,s\}$, respectively, in the inequality (1.1). He obtained various results according to the assumptions on X and the local behaviour of the function ψ around the origin. When $X = \mathbb{R}$, $\psi(h) = h^p$ with some fixed $p > 2$, $C \geq 0$ and $\Phi(t,s) = 1$, he proved [13] that every solution $f : D \to \mathbb{R}$ of (1.1) is convex. Later he extended this result [14] to the more general case when X is a Banach space and $\psi : [0,+\infty[\to \mathbb{R}$ fulfils the assumption $\lim_{h\to 0} \psi(h)/h^2 = 0$. His further results show that the assumption on ψ is essential. For instance, one can easily verify that the real function $f(x) = -Cx^2$ $(x \in \mathbb{R})$ is strongly ψ-paraconvex with $\psi(h) = h^2$ but f is not convex when $C > 0$. Via similar calculations one can prove the following statement: if $X = \mathbb{R}$, $\Phi(t,s) = ts$, $\psi(h) = h^2$, and f satisfies (1.1) for all $t \in [0,1]$, $x, y \in D$, then the function $g(x) = f(x) + Cx^2$ $(x \in D)$ is convex. The statement is valid for negative C as well, when f is called strongly convex (cf. [5, Prop. 1.1.2], [10]). We note that the choices $\Phi(t,s)=\min\{t,s\}$ and $\Phi(t,s)=ts$ in (1.1) are essentially equivalent as $\frac{1}{2}\min\{\,t\,,\,1-t\,\}\le t(1-t)\le\min\{\,t\,,\,1-t\,\}$ for every $t\in[0,1]$.

Motivated by results on C-convex functions and investigations in the spirit of Luc, Ngai and Théra, Páles [12] proved the following theorem: Let I denote an open interval in $\mathbb{R}$ and ε, δ be nonnegative real numbers. A function $f:I\to\mathbb{R}$ satisfies the inequality

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+\varepsilon t(1-t)|x-y|+\delta$$

for all $x,y\in I$ and $t\in[0,1]$ if, and only if, f can be represented in the form $f=g+\alpha+\beta$, where $g:I\to\mathbb{R}$ is convex, $\alpha:I\to\mathbb{R}$ is a Lipschitz function and $\beta:I\to\mathbb{R}$ is a bounded function.

The notion of midconvex (or Jensen convex) functions concerns functions $f:D\to\mathbb{R}$ that satisfy (1.1) for all $x,y\in D$ with $t=1/2$ and $C=0$. According to the celebrated Bernstein–Doetsch theorem [1], if f is midconvex and locally bounded above, then f is convex. Analogously, if f satisfies (1.1) with $t=1/2$, $C\ge 0$ and $\Phi(1/2\,,\,1/2)=\psi(\|x-y\|)=1$ for all $x,y\in D$ and f is locally bounded above, then f is $2C$-convex [11]. Házy and Páles [4], considering an exponent $p\in[0,1]$, investigated the relations among the solutions of inequality (1.1) with $\Phi(t,s)=(ts)^p$, $\psi(h)=h^p$, and those of the special case $t=1/2$, obtaining similar results. Their results were generalized to more general choices of Φ and ψ by Makó and Páles [9]. A comparison of these results with those of Rolewicz is elaborated by Jacek Tabor and Józef Tabor [15].

2. Results

We consider approximate convexity of the form (1.1) in case $\Phi(t,s)=(ts)^p$, $\psi(h)=h^p$, under the assumption that $p>1$. We begin the investigation and reformulation of the problem in case of real variables.

2.1. Approximate convexity on intervals

Proposition 2.1. *Let $I\subset\mathbb{R}$ be an open interval, $c\ge 0$, $p>1$. A function $f:I\to\mathbb{R}$ fulfils the inequality*

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)+c\,(\lambda(1-\lambda)\,|x-y|)^p \tag{2.1}$$

for every $x, y \in I$ *and* $\lambda \in [0,1]$ *if, and only if,* f *satisfies the inequality*

$$f(y) \leq \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) + c\left(\frac{(z-y)(y-x)}{z-x}\right)^p \tag{2.2}$$

for every $x, y, z \in I$ *fulfilling* $x < y < z$ *.*

Proof. Let us assume that (2.1) holds for all $x, y \in I$ and $\lambda \in [0,1]$. Let us consider $x, y, z \in I$ such that $x < y < z$. Let $\lambda = \frac{z-y}{z-x}$. Then $0 < \lambda < 1$, $1-\lambda = \frac{y-x}{z-x}$, and $y = \lambda x + (1-\lambda)z$. Thus, applying the inequality (2.1) with z in place of y, we obtain (2.2).

Conversely, suppose that f satisfies (2.2) for all $x, y, z \in I$ fulfilling $x < < y < z$, and let $0 < \lambda < 1$, x, $z \in I$ such that $x < z$. Introducing $y = \lambda x + (1-\lambda)z$, we obtain $x < y < z$ and all the above listed expressions for λ and $1-\lambda$. Therefore (2.2) yields (2.1) with z in place of y. In other words, (2.1) is verified if $x < y$ and $0 < \lambda < 1$. Since λ can be replaced with $1-\lambda$ (as both are between 0 and 1), the inequality (2.1) is symmetric with respect to x and y. So we obtained (2.1) from (2.2) for $x \neq y$ and $0 < \lambda < 1$. In the remaining cases (i.e., when $x = y$ or $\lambda \in \{0, 1\}$) (2.1) obviously holds with equality.

The proof of the following lemma consists of straightforward calculations, so it is left to the reader.

Lemma 2.2. *Let us suppose that* x *,* y *,* $z \in I$ *satisfy* $x < y < z$ *. Then* (2.2) *is equivalent to each of the following three inequalities:*

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} + c\left(\frac{z-y}{z-x}\right)^p (y-x)^{p-1}, \tag{2.3}$$

$$\frac{f(z)-f(x)}{z-x} - c\left(\frac{y-x}{z-x}\right)^p (z-y)^{p-1} \leq \frac{f(z)-f(y)}{z-y}, \tag{2.4}$$

and

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y} + c\left(\frac{(z-y)(y-x)}{z-x}\right)^{p-1}. \tag{2.5}$$

Theorem 2.3. *Let* $I \subset \mathbb{R}$ *be an open interval,* $c \geq 0$ *,* $p > 1$ *and* $f : I \to \mathbb{R}$ *such that, for every* $x, y \in I$ *and* $\lambda \in [0,1]$*,* f *satisfies* (2.1)*. Then, for every* $a \in I$ *, there exist the limits*

$$f'_-(a) := \lim_{s\to a-} \frac{f(s)-f(a)}{s-a} = \sup\left\{ \frac{f(s)-f(a)}{s-a} \,:\, a > s \in I \right\} \in \mathbb{R} \quad \text{and}$$

$$f'_+(a) := \lim_{t\to a+} \frac{f(t)-f(a)}{t-a} = \inf\left\{ \frac{f(t)-f(a)}{t-a} \,:\, a < t \in I \right\} \in \mathbb{R}\,.$$

Moreover, $f'_-(a) \leq f'_+(a)$ *.*

Proof. First we show that $f'_+(a)$ exists, it is real and it coincides with the greatest lower bound of the given set of difference ratios. Let $s, t \in I$ such that $s < a < t$. Then from (2.5) we get

$$\begin{aligned} \frac{f(t)-f(a)}{t-a} &\geq \frac{f(a)-f(s)}{a-s} - c\left(\frac{(t-a)(a-s)}{t-s}\right)^{p-1} \\ &\geq \frac{f(a)-f(s)}{a-s} - c\,(a-s)^{p-1}\,. \end{aligned}$$

Thus the set

$$S_a^+ = \left\{ \frac{f(t)-f(a)}{t-a} \,\middle|\, t \in I,\ a < t \right\}$$

is bounded below, therefore

$$\varphi(a) := \inf S_a^+ \in \mathbb{R}.$$

Let $\varepsilon_1 > 0$. Since $\lim\limits_{d \to 0+} c d^{p-1} = 0$, it follows that there exists $\delta_0 > 0$ such that $c \cdot \delta_0^{p-1} < \frac{\varepsilon_1}{2}$. Moreover, there exists $u \in I$ such that $u > a$ and $\frac{f(u)-f(a)}{u-a} < < \varphi(a) + \frac{\varepsilon_1}{2}$. Let $\delta = \min\{\delta_0, u-a\}$. Obviously, $\delta > 0$. If $a < t < a + \delta$, then $a + \delta \leq a + (u-a) = u$ and from (2.3) we get

$$\begin{aligned} \varphi(a) \leq \frac{f(t)-f(a)}{t-a} &\leq \frac{f(u)-f(a)}{u-a} + c\left(\frac{u-t}{u-a}\right)^p (t-a)^{p-1} \\ &< \varphi(a) + \frac{\varepsilon_1}{2} + c\delta_0^{p-1} < \varphi(a) + \varepsilon_1\,. \end{aligned}$$

Hence, we have $\varphi(a) = \lim\limits_{t \to a+} \frac{f(t)-f(a)}{t-a} = f'_+(a)\,.$

We can apply an analogous argument, based on the inequalities (2.5) and (2.4), to show that $f'_-(a)$ exists, it is real and it coincides with the least upper bound of the given set of difference ratios.

In order to verify the inequality $f'_-(a) \leq f'_+(a)$, let us consider $x, z \in I$ such that $x < a < z$. Writing a in the place of y in (2.5) we get

$$\begin{aligned} \frac{f(a)-f(x)}{a-x} &\leq \frac{f(z)-f(a)}{z-a} + c\left[\frac{(z-a)(a-x)}{z-x}\right]^{p-1} \leq \\ &\leq \frac{f(z)-f(a)}{z-a} + c\left[\frac{(z-a)(z-x)}{z-x}\right]^{p-1} = \frac{f(z)-f(a)}{z-a} + c(z-a)^{p-1}. \end{aligned}$$

Hence we have

$$f'_-(a) = \lim_{x \to a-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a-} \frac{f(a)-f(x)}{a-x} \leq \frac{f(z)-f(a)}{z-a} + c(z-a)^{p-1},$$

and thus

$$f'_-(a) \le \lim_{z\to a+} \left(\frac{f(z)-f(a)}{z-a} + c(z-a)^{p-1} \right) = f'_+(a). \qquad \blacksquare$$

Theorem 2.4. *Let $I \subset \mathbb{R}$ be an open interval, $c \ge 0$, $p > 1$ and $f : I \to \mathbb{R}$ such that f satisfies* (2.1) *for every $x, y \in I$ and $\lambda \in [0,1]$. Then f satisfies* (2.1) *with $c = 0$ as well, so f is convex.*

Proof. Suppose that $x, y, z \in I$ satisfy $x < y < z$. According to Theorem 2.3, we have the inequalities

$$\frac{f(y)-f(x)}{y-x} \le f'_-(y) \le f'_+(y) \le \frac{f(z)-f(y)}{z-y}\,.$$

Therefore inequality (2.5) is satisfied with $c = 0$ as well. Thus inequalities (2.2) and (2.1) are also valid with $c = 0$. Hence, f is convex by definition. $\blacksquare$

Remark 2.1. Let us consider the example $f(x) = -\frac{c}{4}x^2$ ($x \in \mathbb{R}$), which was mentioned in the introduction as well. Clearly, f is continuous, bounded above, and it fulfils (2.1) with $p = 2$ for $\lambda = 1/2$ and for all x, $y \in \mathbb{R}$. However, it is not convex (when $c > 0$), hence, due to Theorem 2.4, it cannot satisfy (2.1) with $p = 2$ (and any constant in place of c) for all $\lambda \in [0,1]$. Therefore the Bernstein–Doetsch theorem cannot be extended to this type of approximately convex functions.

2.2. Approximate convexity in normed spaces

Theorem 2.5. *Let $(X, \|\cdot\|)$ denote a linear normed space, $D \subset X$ be open and convex, $c \ge 0$, $p > 1$ and let us suppose that $f : D \to \mathbb{R}$ satisfies*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) + c\,(\lambda(1-\lambda)\|x-y\|)^p$$

for every $x, y \in D$ and $\lambda \in [0,1]$. Then f is convex.

Proof. Fix $x, y \in X$ and let $u = \frac{x+y}{2}$, $w = \frac{y-x}{2}$. Note that $u - w = x \in D$ and $u + w = y \in D$, hence there exists an open interval I such that $\pm 1 \in I$ and $u + sw \in D$ for all $s \in I$. Let

$$g(s) = f(u+sw) \qquad (s \in I).$$

Then, for every $s, t \in I$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} g(\lambda s + (1-\lambda)t) &= f(\lambda(u+sw) + (1-\lambda)(u+tw)) \leq \\ &\leq \lambda f(u+sw) + (1-\lambda) f(u+tw) + \\ &\qquad + c\,(\lambda(1-\lambda)\|(u+sw)-(u+tw)\|)^p = \\ &= \lambda g(s) + (1-\lambda)g(t) + c\|w\|^p\,(\lambda(1-\lambda)|s-t|)^p\ . \end{aligned}$$

Thus g satisfies the assumptions of Theorem 2.4 (with the constant $c\|w\|^p$ in place of c), hence it is convex. In particular, we have, for every $\lambda \in [0,1]$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= g(\lambda(-1) + (1-\lambda)\cdot 1) \\ &\leq \lambda g(-1) + (1-\lambda)g(1) = \lambda f(x) + (1-\lambda) f(y)\,. \end{aligned}$$

As x and y were arbitrarily fixed, this completes the proof. ■

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Z. Boros
Institute of Mathematics
University of Debrecen
Debrecen
Hungary
zboros@science.unideb.hu

N. Nagy
Tomori Pál College
Budapest
Hungary
nagy.noemi@tpfk.hu