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NORMALIZATION OF BEURLING GENERALIZED PRIMES WITH RIEMANN HYPOTHESIS

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Dedicated to Professor K.-H. Indlekofer on his seventieth birthday

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Abstract. Two Beurling generalized number systems are given such that the primes of both systems up to any assigned large number are exactly all the rational primes not exceeding the number and both consist of only rational numbers of the form $n+\ell/2^n$. The counting functions of g-integers of both satisfy $N(x) = x + x^{1/2}O(\exp\{c(\log x)^{2/3}\})$. The first realizes the RH and the second realizes exactly the classical zero-free region of the Riemann zeta function and the de la Vallée Poussin error term.

1. Introduction

Let $\mathcal{P} = \{p_i\}$ be an unbounded sequence of real numbers satisfying $1 < p_1 \leq \leq p_2 \leq \cdots$. We call $\mathcal{P}$ a sequence of Beurling generalized primes (henceforth, g-primes) and the free multiplicative semigroup $\mathcal{N}$ generated by $\mathcal{P}$ a system of Beurling generalized integers (g-integers). Note that it is not assumed that either $\mathcal{P}$ or $\mathcal{N}$ lies in the positive integers (whole numbers) $\mathbb{N}$ nor that the unique factorization is in force in $\mathcal{N}$.

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Let

$$\pi(x) = \pi_{\mathcal{P}}(x) := \#\{\mathcal{P} \cap [1, x]\} \quad \text{and} \quad N(x) = N_{\mathcal{P}}(x) := \#\{\mathcal{N} \cap [1, x]\}$$

be the counting functions of g-primes in $\mathcal{P}$ and g-integers in $\mathcal{N}$, respectively. The general question is how hypotheses on one of $N(x)$ and $\pi(x)$ imply conclusions on the another. Also, let

$$\zeta(s) = \zeta_{\mathcal{P}}(s) := \int_{1-}^{\infty} x^{-s}\, dN(x) = \exp\left\{\int_{1}^{\infty} \log\,(1 - x^{-s})^{-1}\, d\pi(x)\right\},$$
$$\sigma := \Re s > 1$$

be the zeta function associated with the g-integer system $\mathcal{N}$, which is the analytic term combining $N(x)$ and $\pi(x)$.

A simple example of g-primes is the sequence $\mathcal{P}$ of all odd primes in $\mathbb{N}$. Here $\mathcal{N}$ is just the sequence of odd whole numbers. Hence $N_{\mathcal{P}} = (1/2)x + O(1)$ and $\pi_{\mathcal{P}}(x)$ is the counting function of odd primes. The omission of the only even prime 2 cuts the density of whole numbers in half.

In case $\mathcal{N} = \mathbb{N}$, we have the classical counting functions of rational primes and whole numbers and the Riemann zeta function. In 1899 de la Vallée Poussin [5] proved that

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - c/\log\,(|t| + 4) \tag{1.1}$$

with some constant $c > 0$, where $s = \sigma + it$. From this so-called "classical" zero-free region, he could deduce the prime number theorem (PNT) with an error term as

$$\pi(x) = \mathrm{li}(x) + O(x \exp\{-c(\log\, x)^{1/2}\}), \tag{1.2}$$

where c is also a positive constant (but need not be the same c in (1.1)). Later it was proved that if

$$\zeta(s) \neq 0, \quad \text{for} \quad \sigma > 1/2 \quad \text{(the Riemann Hypothesis)}$$

then

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2} \log\, x).$$

The determination of the truth of RH is one of most important problems in today's mathematics.

In general case, in 1937, Beurling [2] showed that if

$$N(x) = Ax + O(x \log^{-\gamma} x) \tag{1.3}$$

with constants $A > 0$ and $\gamma > 3/2$ then

$$\pi(x) \sim x/\log\, x, \tag{1.4}$$

i.e., the analogue of the PNT holds for this g-number system.

Better PNT error terms than (1.2) for whole numbers are known these days, with any exponent less than 3/5 in place of 1/2. All these improvements are made in using the well-spacing of whole numbers, i.e., the additive structure of positive integers.

Beurling's result can be viewed as an abstraction of an earlier proof by E. Landau [4] of the prime ideal theorem for algebraic number fields; but with the error term as (1.2). Landau's argument analyzes the analytic connection between the counting functions of integral ideals and prime ideals according to the norms of ideals. The norm function on algebraic number fields is multiplicative but not additive. From this view-point, it has long been conjectured [1] that (1.2) may be optimal for g-numbers.

In previous works [3] and [6], examples are given in which the classical zero-free region (1.1) and the PNT error term (1.2) are exactly realized for Beurling g-number systems. Also, in the latter, examples are given in which the Riemann Hypothesis are realized for Beurling g-number systems.

It is then of interest to view how close these examples could be to the natural numbers $\mathbb{N}$. We shall show that, on the basis of g-primes $\mathcal{P}_R$ and $\mathcal{P}_B$ given in [6], g-primes with the same properties and more features imitating the primes in $\mathbb{N}$ (see (i) and (ii) below) can be constructed further.

Theorem 1. *Given any whole number n_R, there is g-primes $\mathcal{P}_R$ such that*

(i) *$\mathcal{P}_R$ consists of numbers in the sequence*

$$\left\{ v_k = n + \frac{\ell}{2^n} \quad for \quad k = 2^n + \ell, n = 0, 1, 2, \cdots, 0 \leq \ell < 2^n \right\} \tag{1.5}$$

and $\mathcal{P}_R \cap [1, n_R]$ consists exactly of primes in $\mathbb{N}$ not exceeding n_R;

(ii) *the counting function $N_R(x)$ of the resulting g-integers satisfies*

$$N_R(x) = x + O(x^{1/2} \exp\{c(\log x)^{2/3}\});$$

(iii) *the associated zeta function $\zeta_R(s)$ is analytic for $\sigma > 1/2$ except a simple pole at $s = 1$ with residue 1;*

(iv) *the function $\zeta_R(s)$ has no zeros on the half plane $\sigma > 1/2$ (the Riemann Hypothesis);*

(v) *the g-prime counting function $\pi_R(x)$ satisfies*

$$\pi_R(x) = li(x) + O(x^{1/2}).$$

Theorem 2. *Given any whole number n_B, there is g-primes $\mathcal{P}_B$ such that the functions $N_B(x)$, $\pi_B(x)$ and $\zeta_B(s)$ have the properties* (i)–(iii) *of Theorem* 1 *and*

(iv) *the function $\zeta_B(s)$ has infinitely many zeros on the curve $\sigma = 1 - -1/\log|t|$, $|t| \geq e^2$ and no zeros to its right;*

(v) *the Chebyshev function $\psi_B(x)$ satisfies*

$$\limsup_{x\to\infty} \frac{\psi_B(x) - x}{x\exp\{-2\sqrt{\log x}\}} = 2$$

$$\liminf_{x\to\infty} \frac{\psi_B(x) - x}{x\exp\{-2\sqrt{\log x}\}} = -2.$$

Remark. The g-primes $\mathcal{P}_R$ and $\mathcal{P}_B$ consist of rational numbers of the form $n+\ell/2^n$. We are unable to construct $\mathcal{P}_R$ or $\mathcal{P}_B$ consisting of only positive integers.

2. Normalization of g-primes

We shall give only the proof of Theorem 1 because Theorem 2 can be proved in the same way. Without loss of generality, we may assume that $n_R > 4$.

The proof is a construction procedure consisting of a series of successively adding chosen g-primes to a system and deleting other chosen g-primes from a system. Hence, during the procedure, $\mathcal{P}, \zeta, \pi$ and N with subscripts $1, 2, 3, \ldots$ will denote a series of particular g-prime systems, the associated zeta functions, the counting functions of g-primes, and the counting functions of the g-integers, respectively.

We first insert all rational primes up to n_R that are not already in $\mathcal{P}_R$ given in [6] and delete all g-primes up to n_R that are not rational primes. In this way the new system $\mathcal{P}_1$ satisfies properties (i), (iv), (v) of Theorem 1 and

(ii)$_1$ the counting function $N_1(x)$ of the resulting g-integers satisfies

$$N_1(x) = \kappa_1 x + O(x^{1/2}\exp\{c(\log x)^{2/3}\})$$

with $\kappa_1 > 0$;

(iii)$_1$ the associated zeta function $\zeta_1(s)$ is analytic for $\sigma > 1/2$ except a simple pole at $s = 1$ with residue κ_1.

The desired estimates of $(\mathrm{ii})_1$ and $(\mathrm{iii})_1$ can be shown by using the inclusion-exclusion principle since only a finite number of g-primes are involved.

If $\kappa_1 = 1$ then the construction is done. Otherwise, if $\kappa_1 > 1$, we may move a finite number of g-primes exceeding n_R from $\mathcal{P}_1$ so that the counting function $N_2(x)$ of the g-integers of the resulting system satisfies

$$N_2(x) = \kappa_2 x + O(x^{1/2}\exp\{c(\log x)^{2/3}\})$$

with $\kappa_2 < 1$ since

$$\prod_{\substack{p_j\in\mathcal{P}_1\\ p_j>n_R}} (1-p_j^{-1}) = 0.$$

Hence we may further assume that

$$|N_2(x) - \kappa_2 x| \le C_2 x^{1/2}\exp\{c(\log x)^{2/3}\}$$

with

$$1 - A^{-1} < \kappa_2 < 1,$$

where A is an integer satisfying $A > n_R$. Otherwise,

$$0 < \kappa_2 \le 1 - (n_R+1)^{-1}.$$

Then there is a number $m \in \mathbb{N}$ satisfying $m \ge n_R + 1$ such that

$$1 - \frac{1}{m+1} < \kappa_2 \prod_{n=n_R+1}^{m} (1-n^{-1})^{-1} \le 1.$$

Then new g-primes $n_R+1, \cdots, m$ are added to $\mathcal{P}_2$. If the right-hand side is an equality then the construction is done. Otherwise,

$$1 - \frac{1}{m+1} < \kappa_3 := \kappa_2 \prod_{n=n_R+1}^{m} (1-n^{-1})^{-1} < 1$$

and the counting function $N_3(x)$ of the new system $\mathcal{P}_3$ satisfies

$$|N_3(x) - \kappa_3 x| \le C_3 x^{1/2}\exp\{c(\log x)^{2/3}\}$$

with $C_3 = C_2\prod_{n=n_R+1}^{m}(1-n^{-1/2})^{-1}$.

To complete the construction, we appeal to the following lemma and leave its proof to the next section.

Lemma 1. *Given κ satisfying*

$$1 - A^{-1} < \kappa < 1$$

with an integer $A > n_R$, there is a finite or infinite sequence $\{w_n\}$ of $w_n \in \mathbb{N}$ such that

$$w_n > n_R, \quad \prod_n (1 - w_n^{-1}) = \kappa, \quad \sum_n w_n^{-1/2} < \infty, \quad \sum_{w_n \le x} 1 = O(\log x).$$

Thus we set $\kappa = \kappa_3$ and apply Lemma 1. Without loss of generality, we may assume that the sequence $\{w_n\}$ is increasing, i.e., $w_n \le w_{n+1}$. Note that $\prod(1 - w_n^{-s})^{-1}$ is analytic for $\sigma > 1/2$ and has no zeros there. We enlarge $\mathcal{P}_3$ to contain $\{w_n\}$. Then the resulting g-prime system $\mathcal{P}_R$ consists of numbers in $\{v_k\}$ of (1.5) (but may not be a subsequence of $\{v_k\}$) and $\mathcal{P}_{\mathcal{R}} \cap [1, n_R]$ consists exactly of primes in $\mathbb{N}$ not exceeding n_R since all new added $w_n > n_R$. Also, the associated zeta function is given by

$$\zeta_R(s) = \zeta_3(s) \prod_n (1 - w_n^{-s})^{-1},$$

which is analytic for $\sigma > 1/2$ except a simple pole at $s = 1$ with residue

$$\kappa_3 \prod_n (1 - w_n^{-1})^{-1} = 1$$

and has no zeros on $\sigma > 1/2$. Moreover, the counting function $\pi_R(x)$ of g-primes in $\mathcal{P}_R$ satisfies

$$\pi_R(x) = \pi_3(x) + \sum_{w_n \le x} 1 = \mathrm{li}(x) + O(x^{1/2}).$$

Finally, the counting function $N_R(x)$ of g-integers satisfies

$$|N_R(x) - x| \le C x^{1/2} \exp\{c(\log x)^{2/3}\}. \tag{2.1}$$

with

$$C = C_3 \prod_n (1 - w_n^{-1/2})^{-1}.$$

Actually, if $\{w_n\}$ is a finite sequence then (2.1) is plainly true. Otherwise, $\{w_n\}$ is infinite and hence $w_n \to \infty$. For any given $x \ge 1$, on the one hand, if $w_{n_1} > x$ then,

$$\begin{aligned} N_R(x) &\le \kappa_3 \prod_{n=1}^{n_1} (1 - w_n^{-1})^{-1} x + C_3 \prod_{n=1}^{n_1} (1 - w_n^{-1/2})^{-1} x^{1/2} \exp\{c(\log x)^{2/3}\} \le \\ &\le x + C_3 \prod_{n=1}^{\infty} (1 - w_n^{-1/2})^{-1} x^{1/2} \exp\{c(\log x)^{2/3}\} \end{aligned}$$

since g-integers in $\mathcal{N}_R$ not exceeding x have no g-prime divisors exceeding w_{n_1}. On the other hand, choosing n_1 sufficiently large so that $\sum_{n>n_1} w_n^{-1} \le x^{-1}/2$, we have

$$\prod_{n>n_1} (1 - w_n^{-1}) \ge \exp\left\{-2 \sum_{n>n_1} w_n^{-1}\right\} \ge \exp\{-x^{-1}\}$$

and hence

$$x\left(1 - \prod_{n>n_1} (1 - w_n^{-1})\right) \le x(1 - \exp\{-x^{-1}\}) \le 1.$$

Thus, we arrive at

$$\begin{aligned} N_R(x) \ge \kappa_3 \prod_{n=1}^{n_1} (1 - w_n^{-1})^{-1} x - C_3 \prod_{n=1}^{n_1} (1 - w_n^{-1/2})^{-1} x^{1/2} \exp\{c(\log x)^{2/3}\} = \\ = x - x\left(1 - \prod_{n>n_1} (1 - w_n^{-1})\right) - \\ - C_3 \prod_{n=1}^{n_1} (1 - w_n^{-1/2})^{-1} x^{1/2} \exp\{c(\log x)^{2/3}\} \ge \\ \ge x - C_3 \prod_{n=1}^{\infty} (1 - w_n^{-1/2})^{-1} x^{1/2} \exp\{c(\log x)^{2/3}\}. \end{aligned}$$

Therefore the system $\mathcal{P}_R$ has all expected properties and the construction is finished.

3. Proof of Lemma 1

Let a function f be defined by

$$f(\alpha) = \prod_{n\ge 1} (1 - \alpha^{-n})^{-1}, \quad 1 < \alpha < \infty.$$

Then $f(\alpha)$ is continuous and strictly decreasing,

$$\lim_{\alpha\to 1+} f(\alpha) = \infty, \quad \text{and} \quad \lim_{\alpha\to\infty} f(\alpha) = 1.$$

Hence $f(\alpha) = 1/\kappa$ has a unique solution α. Note that $\alpha > A > n_R$ since

$$\prod_{n} (1 - \alpha^{-n})^{-1} = \frac{1}{\kappa} < \left(1 - \frac{1}{A}\right)^{-1}.$$

If $\alpha \in \mathbb{N}$, put $w_n = \alpha^n$ and the lemma is proved. Otherwise, $\alpha \notin \mathbb{N}$ and we set

$$a_n := \lfloor \alpha^n \rfloor \le \alpha^n \le \lceil \alpha^n \rceil =: b_n.$$

Note that $a_1^n \le a_n \le b_n \le b_1^n$ and $a_1 \ge A$. Hence

$$\prod_n \left(1 - \frac{1}{a_n}\right)^{-1} > f(\alpha) = \frac{1}{\kappa} > \prod_n \left(1 - \frac{1}{b_n}\right)^{-1}.$$

It follows that

$$\prod_n \left(1 - \frac{1}{a_n}\right)\left(1 - \frac{1}{b_n}\right)^{-1} < \kappa \prod_n \left(1 - \frac{1}{b_n}\right)^{-1} < 1.$$

Then it can be shown that

$$\prod_n \left(1 - \frac{1}{a_n}\right)\left(1 - \frac{1}{b_n}\right)^{-1} > 1 - \frac{2}{A^2}. \tag{3.1}$$

Actually, the left-hand side of (3.1) equals

$$\exp\left\{ \sum_{n=1}^{\infty} \frac{a_n - b_n}{a_n b_n} + \sum_{n=1}^{\infty} \sum_{k \ge 2} \frac{1}{k} \frac{a_n^k - b_n^k}{a_n^k b_n^k} \right\}.$$

Note that $a_n - b_n \ge -1$. The second sum in the exponent is at least

$$\sum_n \sum_{k \ge 2} \frac{1}{k} \frac{(-1)(a_n^{k-1} + a_n^{k-2} b_n + \cdots + a_n b_n^{k-2} + b_n^{k-1})}{a_n^k b_n^k} >$$
$$> -\sum_n \sum_{k \ge 2} \frac{1}{a_n^k b_n} = -\sum_n \frac{1}{a_n b_n} \frac{1}{a_n - 1}.$$

Hence we see that

$$\begin{aligned}
\prod_n \left(1 - \frac{1}{a_n}\right)\left(1 - \frac{1}{b_n}\right)^{-1} &\ge \exp\left\{ -\sum_n \frac{1}{a_n b_n} - \sum_n \frac{1}{a_n b_n} \frac{1}{a_n - 1} \right\} \ge \\
&\ge \exp\left\{ -\frac{a_1}{a_1 - 1} \sum_n \frac{1}{a_n^2} \right\} \ge \\
&\ge \exp\left\{ -\frac{a_1}{a_1 - 1} \sum_n \frac{1}{a_1^{2n}} \right\} = \\
&= \exp\left\{ -\frac{a_1}{(a_1 - 1)(a_1^2 - 1)} \right\} \ge \exp\left\{ -\frac{2}{A^2} \right\} > \\
&> 1 - \frac{2}{A^2}.
\end{aligned}$$

We now recursively define a sequence, finite or infinite, of triples

$$\{(\kappa_m, A_m, S_m)\}$$

as follows. First, let

$$\kappa_1 := \kappa, \quad A_1 := A, \quad \text{and} \quad S_1 := \{b_n\}.$$

Then define $A_2 := A_1^2/2$ or $(A_1 - 1)^2/2$ according to A_1 is even or odd so that A_2 is even. Note that A_2 satisfies

$$1 - \frac{1}{A_2} < \kappa_1 \prod_{w \in S_1} (1 - w^{-1})^{-1} \leq 1.$$

In general, if (κ_m, A_m, S_m) and A_{m+1} have been defined and if

$$1 - \frac{1}{A_{m+1}} < \kappa_m \prod_{w \in S_m} (1 - w^{-1})^{-1} = 1$$

then (k_m, A_m, S_m) is the last triple of the sequence. In this case, we set $\{w_n\} = = \cup_{j=1}^m S_j$ and the lemma is proved. Otherwise,

$$1 - \frac{1}{A_{m+1}} < \kappa_m \prod_{w \in S_m} (1 - w^{-1})^{-1} < 1$$

then a repeat of the same argument given above with

$$\kappa = \kappa_{m+1} := \kappa_m \prod_{w \in S_m} (1 - w^{-1})^{-1}$$

and $A = A_{m+1}$ yields a subset S_{m+1} of $\mathbb{N}$ such that

$$1 - \frac{2}{A_{m+1}^2} < \kappa_{m+1} \prod_{w \in S_{m+1}} (1 - w^{-1})^{-1} \leq 1.$$

Hence $(\kappa_{m+1}, A_{m+1}, S_{m+1})$ and $A_{m+2} := A_{m+1}^2/2$ are defined. This procedure yields a finite or infinite sequence $\{(\kappa_m, A_m, S_m)\}$.

Finally let $\{w_n\} := \cup_m S_m$. Note that

$$A_2 > A_1 > n_R > 4, \quad \text{and} \quad A_3 = A_2^2/2 > A_2^{3/2}$$

and by induction

$$A_m \geq A_2^{(3/2)^{m-2}}, \quad m \geq 2.$$

Therefore
$$\lim_{m\to\infty} \kappa_m \prod_{w\in S_m} (1-w^{-1})^{-1} = 1,$$
i.e.,
$$\kappa \prod_{w\in S_1} (1-w^{-1})^{-1} \prod_{w\in S_2} (1-w^{-1})^{-1} \cdots = 1$$
Moreover, from the definition of S_m, we have
$$\begin{aligned}
&\sum_{w\in S_1} w^{-1/2} + \sum_{w\in S_2} w^{-1/2} + \cdots \le \\
&\le 2(A_1^{-1/2} + A_2^{-1/2} + A_3^{-1/2} + \cdots) \ll \\
&\ll A_1^{-1/2} + A_2^{-1/2} + (A_2^{-1/2})^{3/2} + (A_2^{-1/2})^{(3/2)^2} + \cdots \le \\
&\le A_1^{-1/2} + A_2^{-1/2}/(1 - A_2^{-1/4})
\end{aligned}$$
and
$$\begin{aligned}
&\sum_{\substack{w\in S_1\\ w\le x}} 1 + \sum_{\substack{w\in S_2\\ w\le x}} 1 + \sum_{\substack{w\in S_3\\ w\le x}} 1 + \cdots \le \\
&\le \frac{\log\, x}{\log\, A_1} + \frac{\log\, x}{\log\, A_2} + \frac{\log\, x}{(3/2)\log\, A_2} + \frac{\log\, x}{(3/2)^2 \log\, A_2} + \cdots = \\
&= \frac{\log\, x}{\log\, A_1} + \frac{3\log\, x}{\log\, A_2}.
\end{aligned}$$
This completes the proof of Lemma 1. ■

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