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ON THE UNIFORMITY OF SOME SEQUENCES OF RATIONAL NUMBERS

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Abstract. Let $\mathcal{S}$ be a subset of rational numbers. For $x \geqslant 1$ we introduce the set $\mathcal{S}_x$, $\mathcal{S}_x \subset \mathcal{S}$, which consists of numbers $m/n \in \mathcal{S}, (m,n) = 1$, $n \leqslant x$. For $J = (\lambda_1; \lambda_2)$, $J \subset (0; +\infty)$, we denote $|J| = \lambda_2 - \lambda_1$, $J^u = (\lambda_1; \lambda_1 + u|J|)$, and $F_x(u) = \#(\mathcal{S}_x \cap J^u)/\#(\mathcal{S}_x \cap J)$, where $0 \leqslant u \leqslant 1$. The discrepancy $\sup_u |F_x(u) - u|$ is evaluated for some subsets $\mathcal{S}$, specified by arithmetical conditions.

1. Introduction

Let $\mathcal{F}_+$ be the set of positive rational numbers represented by fractions $\frac{m}{n}$, where m, n are coprime natural numbers. The coprimality of m, n will be denoted by $m \perp n$. Let us fix a number $x \geqslant 1$ and introduce the set

$$\mathcal{F}_x = \left\{ \frac{m}{n} : m \perp n, n \leqslant x \right\}.$$

For an interval $J \subset (0; +\infty)$ we set $\mathcal{F}_x^J = \mathcal{F}_x \cap J$. If $J = (0; 1]$, then the elements of $\mathcal{F}_x^J$ arranged in ascending order form the classical Farey sequence of positive rationals.

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It is well-known, that elements of $\mathcal{F}_x^{(0;1]}$ are asymptotically uniformly distributed in $(0;1]$, i.e. $\#\mathcal{F}_x^{(0;u]}/\#\mathcal{F}_x^{(0;1]} \to u$ for $0 \leqslant u \leqslant 1$ as $x \to \infty$. The proof is to be found in [9] (Chapter 4, Problem 189) and [6] (Chapter 2). Let us consider the discrepancy

$$D(\mathcal{F}_x^{(0;1]}) = \sup_{0\leqslant u\leqslant 1} \Big| \frac{\#\mathcal{F}_x^{(0;u]}}{\#\mathcal{F}_x^{(0;1]}} - u \Big|.$$

H. Niederreiter proved in [8] that

$$D(\mathcal{F}_x^{(0;1]}) \asymp \frac{1}{x}, \quad x \to \infty, \tag{1.1}$$

and F. Dress in [3] refined this to the final result: for all natural numbers x

$$D(\mathcal{F}_x^{(0;1]}) = \frac{1}{x}.$$

The purpose of this paper is to provide some examples of subsets of $\mathcal{F}_x$, specified by arithmetical conditions and asymptotically uniformly distributed in given intervals.

Let $\mathcal{S} \subset \mathcal{F}_+$ be some set of rational numbers. We introduce the notations:

$$\mathcal{S}_x = \mathcal{F}_x \cap \mathcal{S}, \quad \mathcal{S}_x^J = \mathcal{F}_x^J \cap \mathcal{S}.$$

For an interval $J = (\lambda_1; \lambda_2)$, $J \subset (0; +\infty)$, we denote $|J| = \lambda_2 - \lambda_1$, $J^u = (\lambda_1; \lambda_1 + u|J|)$ and define the discrepancy by

$$D(\mathcal{S}_x^J) = \sup_{0\leqslant u\leqslant 1} \Big| \frac{\#\mathcal{S}_x^{J^u}}{\#\mathcal{S}_x^J} - u \Big|. \tag{1.2}$$

The intervals J may depend on x, i.e. we suggest that $J = J_x$. If $D(\mathcal{S}_x^J) \to 0$, as $x \to \infty$, the elements of $\mathcal{S}_x$ are asymptotically uniformly distributed in the intervals J. Our approach to proving uniformity is straightforward: we establish the asymptotics

$$\#\mathcal{S}_x^J = G(x, \mathcal{S}) \cdot |J| \cdot (1 + O(\epsilon(x, J))), \quad x \to +\infty,$$

and using this derive the upper bound for (1.2).

2. Rationals with the congruence constraints

The subsequences of Farey fractions with the conditions on denominators were studied by many authors (see [4], [5], [2]). Let b, B be some natural numbers and

$$\mathcal{S} = \Big\{\frac{m}{n} \in \mathcal{F}_+ : n \equiv b \ (\mathrm{mod}\ B)\Big\}.$$

The discrepancies of subsequences of $\mathcal{S}$ in short intervals were investigated by Ledoan in [7]. It follows from his work, that

$$D(\mathcal{S}_x^{(0;1]}) \asymp \frac{1}{x}, \quad x \to \infty,$$

cf. (1.1).

We consider the rationals with the nominators and denominators in some arithmetical progressions. With the natural numbers $a, b, A, B, a \perp A, b \perp B$ let us define

$$\mathcal{S} = \Big\{\frac{m}{n} \in \mathcal{F}_+ : m \equiv a \ (\mathrm{mod}\ A), n \equiv b \ (\mathrm{mod}\ B)\Big\} \tag{2.1}$$

and consider the discrepancies (1.2).

Theorem 2.1. *For the sets in* (2.1) *and arbitrary intervals J we have*

$$D(\mathcal{S}_x^J) \ll \frac{\log x}{x} + \frac{1}{|J| \cdot \log x}.$$

Corollary 2.1. *If $|J| \cdot \log x \to \infty$ as $x \to \infty$, then elements of $\mathcal{S}_x$ are asymptotically uniformly distributed in J.*

We precede the proof of the Theorem with two lemmas.

Lemma 2.1. *For and arbitrary function $f : \mathcal{F}_+ \to \mathbb{R}$ and an interval $J \subset (0; +\infty)$ denote*

$$S(f, \mathcal{F}_x^J) = \sum_{r \in \mathcal{F}_x^J} f(r).$$

Then

$$S(f, \mathcal{F}_x^J) = \sum_{n \leqslant x} M\Big(\frac{x}{n}\Big) T(n), \tag{2.2}$$

where $M(u)$ is the summatory function of the Möbius function $\mu(n)$, and

$$T(n) = \sum_{\lambda_1 n < m < \lambda_2 n} f\big(\frac{m}{n}\big).$$

Note, that in the definition of $T(n)$ the coprimality of m and n is not required. The proof of (2.2) is straightforward: start with the equality

$$S(f,\mathcal{F}_x^J) = \sum_{\substack{n\leqslant x\\ \lambda_1 n<m<\lambda_2 n}} f\Big(\frac{m}{n}\Big) \sum_{d|(m,n)} \mu(d)$$

and proceed by interchanging the order of summation, see also this Lemma in [12].

We shall now derive the asymptotics for $\#\mathcal{S}_x^J$.

Lemma 2.2. *Let $\mathcal{S}$ be the set defined in (2.1) and $J=(\lambda_1;\lambda_2), J\subset (0;+\infty)$. Then*

$$\#\mathcal{S}_x^J = \frac{3}{\pi^2}\cdot\frac{|J|}{AB}\cdot x^2\Big\{\prod_{p|AB}\Big(1-\frac{1}{p^2}\Big)^{-1} + O\Big(\frac{B\log x}{x}+\frac{AB}{\varphi(B)}\cdot\frac{1}{|J|\cdot\log x}\Big)\Big\} \tag{2.3}$$

holds as $x\to\infty$ with the constants in O-sign not depending on a,b,A,B,J.

Proof. Let f be the indicator function of the set $\mathcal{S}$. With the notations of Lemma 2.1 we have $S(f,\mathcal{F}_x^J)=\#\mathcal{S}_x^J$,

$$T(n) = \sum_{\substack{d|n\\ n/d\equiv b \pmod B}} \#\Big\{m : d|m, \frac{m}{d}\equiv a \pmod A, \frac{m}{d}\perp\frac{n}{d}, \lambda_1\frac{n}{d}<\frac{m}{d}<\lambda_2\frac{n}{d}\Big\}.$$

We denote the summand in $T(n)$ by $T(n,d)$, i.e.

$$T(n) = \sum_{\substack{d|n\\ n/d\equiv b \pmod B}} T(n,d).$$

Using this in (2.2) we get

$$S(f,\mathcal{F}_x^J) = \sum_{n\leqslant x} M\Big(\frac{x}{n}\Big)\sum_{d|n} T(n,d) = \sum_{\substack{d,n\\ dn\leqslant x\\ n\equiv b \pmod B}} M\Big(\frac{x}{dn}\Big)T_n^*,$$

where

$$T_n^* = \#\{m : \lambda_1 n<m<\lambda_2 n, m\equiv a \pmod A, m\perp n\}.$$

Let us compute T_n^* :

$$T_n^* = \sum_{\substack{\lambda_1 n<m<\lambda_2 n\\ m\equiv a \pmod A}} \sum_{\delta|(m,n)} \mu(\delta) = \sum_{\delta|n}\mu(\delta) \sum_{\substack{\lambda_1 n<\delta m<\lambda_2 n\\ \delta m\equiv a \pmod A}} 1.$$

Note that the sum corresponding to $\delta, (\delta, A) > 1$, is empty. If $\delta \perp A$, we replace the condition $\delta m \equiv a \pmod{A}$ by $m \equiv a_\delta \pmod{A}$, where a_δ is some natural number. Hence,

$$T_n^* = \sum_{\substack{\delta | n \\ \delta \perp A}} \mu(\delta) \sum_{\substack{\lambda_1 n/\delta < m < \lambda_2 n/\delta \\ m \equiv a_\delta \pmod{A}}} 1 = \sum_{\substack{\delta | n \\ \delta \perp A}} \mu(\delta) \Big\{ (\lambda_2 - \lambda_1) \cdot \frac{n}{\delta A} + \theta_{n,\delta} \Big\}.$$

Using this in the expression of $S(f, \mathcal{F}_x^J)$ we have

$$S(f, \mathcal{F}_x^J) = \frac{\lambda_2 - \lambda_1}{A} \cdot S + O(E), \tag{2.4}$$

where

$$\begin{aligned} S &= \sum_{\substack{d,n \\ dn \leqslant x \\ n \equiv b \pmod{B}}} M\Big(\frac{x}{dn}\Big) n \sum_{\substack{\delta | n \\ \delta \perp A}} \frac{\mu(\delta)}{\delta}, \\ E &= \sum_{\substack{d,n \\ dn \leqslant x \\ n \equiv b \pmod{B}}} \Big| M\Big(\frac{x}{dn}\Big) \Big| \sum_{\substack{\delta | n \\ \delta \perp A}} |\mu(\delta)|. \end{aligned}$$

Let $\tau(n)$ stand for the number of different divisors of n. Then

$$E \ll \sum_{\substack{n \leqslant x \\ n \equiv b \pmod{B}}} \tau(n) \sum_{d \leqslant x/n} \Big| M\Big(\frac{x}{dn}\Big) \Big| \ll x \sum_{\substack{n \leqslant x \\ n \equiv b \pmod{B}}} \frac{\tau(n)}{n},$$

here we used the bound

$$\sum_{m \leqslant v} \Big| M\Big(\frac{v}{m}\Big) \Big| \ll v,$$

which follows from the estimate $M(u) \ll u \exp\{-c\sqrt{\log u}\}, u \geqslant 2$, (see also [12]). Using the Shiu's result for the sums of non-negative multiplicative function on arithmetic progression for the multiplicative function $f(n) = \tau(n)/n$ (see ([10], Theorem 1) we get

$$E \ll \frac{1}{\varphi(B)} \cdot \frac{x^2}{\log x}. \tag{2.5}$$

We proceed with the evaluation of the sum S. With the notation

$$g(n, A) = \sum_{\substack{\delta | n \\ \delta \perp A}} \frac{\mu(\delta)}{\delta}$$

we have

$$S=\sum_{\substack{d,n\\ dn\leqslant x\\ n\equiv b \pmod{B}}} M\Big(\frac{x}{n}\Big)g(n,A)=\sum_{\substack{n\leqslant x\\ n\equiv b \pmod{B}}} ng(n,A)\sum_{d\leqslant x/n} M\Big(\frac{x}{dn}\Big).$$

Note that for all $u\geqslant 1$

$$\sum_{m\leqslant u} M\Big(\frac{u}{m}\Big)=\sum_{\delta\leqslant u}\mu(\delta)\Big\lfloor\frac{u}{\delta}\Big\rfloor=1.$$

Hence,

$$\begin{aligned} S &= \sum_{\substack{n\leqslant x\\ n\equiv b \pmod{B}}} ng(n,A)=\sum_{\substack{\delta\leqslant x\\ \delta\perp A}}\frac{\mu(\delta)}{\delta}\sum_{\substack{m\leqslant x/\delta\\ \delta m\equiv b \pmod{B}}}\delta m\\ &= \sum_{\substack{\delta\leqslant x\\ \delta\perp AB}}\mu(\delta)\sum_{\substack{m\leqslant x/\delta\\ m\equiv b_\delta \pmod{B}}} m=\sum_{\substack{\delta\leqslant x\\ \delta\perp AB}}\mu(\delta)\Big\{\frac{1}{2B}\frac{x^2}{\delta^2}+O\Big(\frac{x}{\delta}\Big)\Big\}. \end{aligned}$$

Here as in the computation of T_n^* we replaced the condition $\delta m\equiv b \pmod{B}$ by $m\equiv b_\delta \pmod{B}$.

It follows now by the standard arguments that

$$S=\frac{3}{\pi^2}\cdot\frac{x^2}{B}\prod_{p|AB}\Big(1-\frac{1}{p^2}\Big)^{-1}+O(x\log x). \tag{2.6}$$

Finally from (2.4), (2.5) and (2.6) we get

$$S(f,\mathcal{F}_x^J)=\frac{3}{\pi^2}\cdot\frac{|J|}{AB}\cdot x^2\Big\{\prod_{p|AB}\Big(1-\frac{1}{p^2}\Big)^{-1}+O\Big(\frac{B\log x}{x}+\frac{AB}{\varphi(B)}\cdot\frac{1}{|J|\cdot\log x}\Big)\Big\}.$$

■

Proof of the Theorem. The statement of the Theorem is trivial if the quantity $|J|\cdot\log x$ is bounded. Let $|J|\cdot\log x\to\infty$ as $x\to\infty$. We use the asymptotics (2.3) in the form

$$\#\mathcal{S}_x^J=C(A,B)\cdot|J|\cdot x^2\Big\{1+O\Big(\frac{\log x}{x}+\frac{1}{|J|\cdot\log x}\Big)\Big\}.$$

Then

$$\frac{\#\mathcal{S}_x^{J^u}}{\#\mathcal{S}_x^J}=\frac{|J^u|\cdot\Big\{1+O\Big(\frac{\log x}{x}+\frac{1}{|J^u|\cdot\log x}\Big)\Big\}}{|J|\cdot\Big\{1+O\Big(\frac{\log x}{x}+\frac{1}{|J|\cdot\log x}\Big)\Big\}},$$

and because of $|J^u| = u|J|$ we get

$$\frac{\#\mathcal{S}_x^{J^u}}{\#\mathcal{S}_x^J} = u + O\Big(\frac{\log x}{x} + \frac{1}{|J| \cdot \log x}\Big)\Big\},$$

which yields the statement of the Theorem. ■

3. Divisibility, multiples, additive functions

Let B and b_i $(i \geqslant 1)$ be some natural numbers, such that

$$\sum_i \frac{1}{b_i} < \infty.$$

In [1] the authors consider the set of rational numbers

$$\mathcal{S} = \Big\{\frac{m}{n} \in \mathcal{F}_+ : n \equiv b \ (\mathrm{mod}\ B), n \perp b_i, i \geqslant 1\Big\}$$

and prove that

$$D(\mathcal{S}_x^{(0;1]}) \asymp \frac{1}{x}, \quad x \to \infty.$$

The results on the asymptotical uniformity of the sets of rationals, satisfying some divisibility constraints can be derived from the following Lemma, see [12].

Lemma 3.1. *Let Q_0, Q_1, Q_2 be some coprime natural numbers and*

$$S = S(Q_0, Q_1, Q_2) = \Big\{\frac{m}{n} : m \perp Q_0 Q_1, n \perp Q_0 Q_2\Big\}. \tag{3.1}$$

Then uniformly over Q_0, Q_1, Q_2 and intervals $J \subset (0; +\infty)$

$$\#\mathcal{S}_x^J = C(Q_0, Q_1, Q_2) \cdot |J| \cdot x^2 \{1 + O(R(x, Q_0, Q_1, Q_2)\},$$

where

$$\begin{aligned} C(Q_0, Q_1, Q_2) &= \frac{3}{\pi^2} \prod_{p|Q_0} \Big(1 - \frac{2}{p+1}\Big) \prod_{p|Q_1 Q_2} \Big(1 - \frac{1}{p+1}\Big), \\ R(x, Q_0, Q_1, Q_2) &= 3^{\omega(Q_0 Q_1 Q_2)} \Big(\frac{\log x}{x} + \frac{1}{|J| \cdot x}\Big), \end{aligned}$$

and $\omega(n)$ denotes the number of distinct prime divisors of n.

The bound for discrepancy $D(\mathcal{S}_x^J)$ follows as in the proof of previous Theorem.

Theorem 3.1. *For the sets in (3.1) and the intervals* $J \subset (0; +\infty)$ *we have*

$$D(\mathcal{S}_x^J) \ll \frac{\log x}{x} + \frac{1}{|J| \cdot x}.$$

The asymptotics of the Lemma can be used for proving uniformity results for the sets specified by various arithmetical conditions. We give two examples.

For a subset of natural numbers A let $\mathcal{M}(A)$ denote the set of multiples of $a \in A$, i.e. the set of natural numbers divisible by at least one $a \in A$.

For two sets $A, B \subset \mathbb{N}$, such that there exist at least one pair of numbers $a \in A, b \in B, a \perp b$, we set

$$\mathcal{S} = \Big\{ \frac{m}{n} \in \mathcal{F}_+ : m \in \mathcal{M}(A), n \in \mathcal{M}(B) \Big\}. \tag{3.2}$$

The density questions for the sets (3.2) are considered in [11]. The case of finite sets A, B is easy. Using the combinatorial including-excluding principles we can prove, that the Theorem 3.1 is true for the sets defined in (3.2).

Let now $f : \mathcal{F}_+ \to \mathcal{G}$ be some additive arithmetical function taking the values in an Abelian group $\mathcal{G}$, i.e. for all $m_1/n_1, m_2/n_2 \in \mathcal{F}_+, m_1 n_1 \perp m_2 n_2$ satisfying

$$f\Big(\frac{m_1}{n_1} \cdot \frac{m_2}{n_2}\Big) = f\Big(\frac{m_1}{n_1}\Big) + f\Big(\frac{m_2}{n_2}\Big).$$

Let for some value $g \in \mathcal{G}$

$$\mathcal{S} = \Big\{ \frac{m}{n} \in \mathcal{F}_+ : f\Big(\frac{m}{n}\Big) = g \Big\}. \tag{3.3}$$

In the simplest case, when the set of powers of primes $\{p^\alpha : \alpha \in \mathbb{Z}, f(p^\alpha) \neq 0\}$ is finite, we derive that the Theorem 3.1 is true for the set (3.3), supposed that it is not empty.

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