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ON A THEOREM OF FEICHTINGER AND WEISZ

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Dedicated to the 70th birthday of Professor Karl-Heinz Indlekofer

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Abstract. The so-called θ-summation is well-known in the theory of approximation. A remarkable result gives a necessary and sufficient condition for uniformly or L^1-norm convergence of θ-means if θ has compact support. This condition is nothing else but the integrability of the (trigonometric) Fourier transform of θ. Later this theorem was improved by Feichtinger and Weisz showing the same result for θ's belonging to a suitable Wiener algebra $W(C,\ell_1)$. If θ is compactly supported then $\theta \in W(C,\ell_1)$ holds evidently but there are functions $\theta \in W(C,\ell_1)$ with unbounded support. In this work we extend the statement of Feichtinger and Weisz. To this end a new space $S(C,\ell_1)$ of functions will be constructed for which we prove the validity of the integrability condition. A simple consideration leads to the proper inclusion $W(C,\ell_1) \subset S(C,\ell_1)$.

1. Introduction

The so-called θ-summation, as a general method of summation generated by a single function θ is an intensively investigated area of approximation. (For this see e.g. [1], [5], [8] and references in [2], [6], [7] as illustration.) In

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this paper we consider θ-means of trigonometric Fourier series and investigate the question: for what functions θ do we have convergence result. To this end we summarize briefly the most important concepts, definitions and well-known facts about θ-summation (for historical background see also the references).

Next we denote by L^1 the set of the functions $f : \mathbb{R} \to \mathbb{R}$ integrable in the sense of Lebesgue. Furthermore, let $\|f\|_1 := \int |f(x)|\,dx := \int_{-\infty}^{+\infty} |f(x)|\,dx$. If $g \in L^1$ then

$$Pg(t) := \sum_{k=-\infty}^{+\infty} g(t+2k\pi) \qquad t \in ([-\pi,\pi])$$

is the so-called periodization of g. Since

$$\int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} |g(t+2k\pi)|dt = \int |g(t)|dt < +\infty,$$

the series $\sum_{k=-\infty}^{+\infty} g(t+2k\pi)$ is absolutely convergent a.e. $t \in \mathbb{R}$. It is clear that $\int_{-\pi}^{\pi} |Pg(t)|dt \le \|g\|_1$ and $\int_{-\pi}^{\pi} Pg(t)dt = \int g(t)dt$. Moreover, Pg is periodic by 2π.

Now let $f \in L^1[-\pi,\pi]$. We take it as $f : \mathbb{R} \to \mathbb{R}$ periodic by 2π and for $g \in L^1$ define $f \star g$ as the usual convolution of f and Pg :

$$f \star g(x) := f * Pg = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} f(x-t)g(t+2k\pi)dt \qquad (x \in \mathbb{R}).$$

Then $f \star g \in L^1[-\pi,\pi]$ and $\int_{-\pi}^{\pi} |f \star g(x)|dx \le \int_{-\pi}^{\pi} |f(x)|dx \cdot \|g\|_1 =: \|f\|_1 \cdot \|g\|_1$. For example if $f(t) := e_j(t) := e^{\imath jt} \quad (j \in \mathbb{Z}, t \in [-\pi,\pi])$ then

$$e_j \star g(x) = e^{\imath jx} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} e^{-\imath jt} g(t+2k\pi)dt =$$

$$= e^{\imath jx} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} e^{-\imath j(t+2k\pi)} g(t+2k\pi)dt = e^{\imath jx} \int g(t)e^{-\imath jt}dt \qquad (x \in \mathbb{R}),$$

i.e. $e_j \star g(x) = e^{\imath jx}\hat{g}(-j) \quad (x \in \mathbb{R})$. Here $\hat{g}$ stands for the (trigonometric) Fourier transform:

$$\hat{g}(x) := \int g(t)e^{\imath tx}dt \qquad (x \in \mathbb{R}).$$

We remember to the so-called inversion formula: if $g, \hat{g} \in L^1$ then

$$g(x) = \frac{1}{2\pi}\hat{\hat{g}}(-x) = \frac{1}{2\pi}\int \hat{g}(t)e^{-\imath tx}dt \qquad (\text{a.e. } x \in \mathbb{R}).$$

(It is well-known that $\hat{g}$ is continuous, so in this case it can be assumed that g is also continuous. Hence the above equality holds for all $x \in \mathbb{R}$).) Let $\theta \in L^1$ be given such that $\hat{\theta} \in L^1$ and with a natural number $m = 1, 2, ...$ we take

$$\theta_m(t) := \frac{m}{2\pi} \cdot \hat{\theta}(mt) \qquad (t \in \mathbb{R}).$$

By means of θ_m let us considered the operators

$$T_m^\theta f := f \star \theta_m \qquad (f \in L^1[-\pi, \pi]).$$

Then for all $f \in L^1[-\pi, \pi]$ we have

$$\|T_m^\theta f\|_1 = \|f \star \theta_m\|_1 \leq \|f\|_1 \cdot \|\theta_m\|_1 = \frac{m\|f\|_1}{2\pi} \int |\hat{\theta}(mt)| dt =$$

$$= \frac{\|f\|_1}{2\pi} \int |\hat{\theta}(t)| dt = \frac{\|\hat{\theta}\|_1}{2\pi} \|f\|_1.$$

In other words the sequence of (obviously linear) operators

$$T_m^\theta : L^1[-\pi, \pi] \to L^1[-\pi, \pi] \qquad (0 < m \in \mathbb{N})$$

are uniformly bounded with respect to the norm $\| \cdot \|_1$ of the Banach space $L^1[-\pi, \pi]$. Special (see above) $T_m^\theta e_j = e_j \widehat{\theta_m}(-j) \ \ (j \in \mathbb{Z})$, where by the inversion formula

$$\widehat{\theta_m}(-j) = \frac{m}{2\pi} \int \hat{\theta}(mt) e^{-\imath jt} dt = \frac{1}{2\pi} \int \hat{\theta}(t) e^{-\imath jt/m} dt = \theta(j/m) \qquad (x \in \mathbb{R}).$$

Therefore $T_m^\theta e_j = \theta(j/m) e_j \quad (j \in \mathbb{Z})$.

Further we assume that the function θ is also continuous and satisfies the condition

$$C_m := \sum_{k=-\infty}^{+\infty} |\theta(k/m)| < +\infty \qquad (0 < m \in \mathbb{N}).$$

Under these assumptions we consider the mappings $\sigma_m^\theta \ \ (0 < m \in \mathbb{N})$ as follows:

$$\sigma_m^\theta f := \sum_{k=-\infty}^{+\infty} \theta(k/m) c_k(f) e_k \qquad (f \in L^1[-\pi, \pi]),$$

where $c_k(f) := (2\pi)^{-1} \int_{-\pi}^{\pi} f(t) e^{-\imath kt} dt$ is the usual k-th Fourier coefficient of f. Since $|c_k(f)| \leq \|f\|_1/(2\pi) \ (k \in \mathbb{Z})$, i.e. $\sum_{k=-\infty}^{+\infty} |\theta(k/m) c_k(f) e_k| \leq$

$\leq C_m\|f\|_1/(2\pi)$, thus the series in question converges uniformly and for every $f \in L^1[-\pi,\pi]$ we get

$$\|\sigma_m^\theta f\|_1 \leq \sum_{k=-\infty}^{+\infty} |\theta(k/m)|\cdot|c_k(f)|\cdot\|e_k\|_1 \leq C_m\|f\|_1.$$

This means that for all $m = 1, 2, ...$ the (linear) operator $\sigma_m^\theta : L^1[-\pi,\pi] \to$ $\to L^1[-\pi,\pi]$ is also bounded. Furthermore,

$$c_k(e_j) = \delta_{kj} = \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases} \qquad (k, j \in \mathbb{Z}),$$

which involves (see above) $\sigma_m^\theta e_j = \theta(j/m)e_j = T_m^\theta e_j$ $(j \in \mathbb{Z}, 0 < m \in \mathbb{N})$. From this it follows immediately the analogous equality for all trigonometric polynomials. They form a dense set in $L^1[-\pi,\pi]$ with respect to $\|\cdot\|_1$, therefore

$$\sigma_m^\theta f = T_m^\theta f \qquad (f \in L^1[-\pi,\pi],\ 0 < m \in \mathbb{N}).$$

A simple calculation shows that

$$\sigma_m^\theta f(x) = \int_{-\pi}^{\pi} f(t)K_m^\theta(x-t)\,dt \qquad (x \in [-\pi,\pi]),$$

where the (2π periodic) kernel K_m^θ is defined as follows:

$$K_m^\theta := \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \theta(k/m)e_k \qquad (0 < m \in \mathbb{N}).$$

The assumption $C_m < +\infty$ guaranties that K_m^θ is continuous. On the other hand

$$\|\sigma_m^\theta f\|_\infty := \max_{x\in[-\pi,\pi]} |\sigma_m^\theta f(x)| = \|T_m^\theta f\|_\infty \leq$$

$$\leq \|f\|_\infty\cdot\|\theta_m\|_1 = \frac{1}{2\pi}\|\hat{\theta}\|_1\|f\|_\infty \qquad (f \in C[-\pi,\pi]).$$

This leads by standard argument to the inequality $\|K_m^\theta\|_1 \leq \|\hat{\theta}\|_1/(2\pi)$ $(m =$ $= 1, 2, \ldots)$, i.e.

$$\sup_{0<m\in\mathbb{N}} \|K_m^\theta\|_1 \leq \frac{1}{2\pi}\|\hat{\theta}\|_1.$$

A remarkable result in the theory of approximation (see e.g. [8]) says that in the above estimation we can write equality:

$$\sup_{0<m\in\mathbb{N}} \|K_m^\theta\|_1 = \frac{1}{2\pi}\|\hat{\theta}\|_1.$$

Moreover, the "reverse" implication was also investigated. Namely (see e.g. [3], [4], [8]), if a continuous and integrable function $\theta : \mathbb{R} \to \mathbb{R}$ has compact support, then the next implication is true:

$$(*) \qquad \sup_{0<m\in\mathbb{N}} \|K_m^\theta\|_1 < +\infty \implies \hat{\theta} \in L^1.$$

Here the assumption on the compactness of the support of θ can be weakened. To this end let for a function $f : \mathbb{R} \to \mathbb{R}$

$$\|f\|_W := \sum_{k=-\infty}^{\infty} \sup_{x\in[0,1)} |f(k+x)|$$

and denote $W(C,\ell_1)$ the Wiener algebra of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $\|f\|_W < +\infty$. Then the following statement holds (Feichtinger and Weisz [2]): the assumption $\theta \in W(C,\ell_1)$ is enough to the implication $(*)$. It is clear that every continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support belongs to $W(C,\ell_1)$. Furthermore, a simple example can be constructed to show that there are functions in $W(C,\ell_1)$ with unbounded supports.

2. The space $(S(C,\ell_1), \|\cdot\|_S)$

Next we prove that the just mentioned result of Feichtinger and Weisz can be improved. In other words the space $W(C,\ell_1)$ can be so enlarged that the implication $(*)$ remains true. For this purpose we introduce for a continuous function $f : \mathbb{R} \to \mathbb{R}$ a new norm $\|f\|_S$ as follows:

$$\|f\|_S := \sum_{j=-\infty}^{+\infty} \sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{l=0}^{m-1} |f(j+l/m)|$$

and the space

$$S(C,\ell_1) := \{f \in C : \|f\|_S < +\infty\}.$$

Since $m^{-1}\sum_{l=0}^{m-1} |f(j+l/m)|$ $(f \in S(C,\ell_1), 0 < m \in \mathbb{N}, j \in \mathbb{Z})$ is a Riemann sum of the integral $\int_j^{j+1} |f(t)|\,dt$ thus

$$\lim_{m\to\infty} \frac{1}{m} \sum_{l=0}^{m-1} |f(j+l/m)| = \int_j^{j+1} |f(t)|\,dt.$$

Therefore

$$\sup_{0<m\in\mathbb{N}} \frac{1}{m}\sum_{l=0}^{m-1} |f(j+l/m)| \geq \int\limits_j^{j+1} |f(t)|\,dt$$

and we get

$$\int |f(t)|\,dt = \sum_{j=-\infty}^{+\infty} \int\limits_j^{j+1} |f(t)|\,dt \leq$$

$$\leq \sum_{j=-\infty}^{+\infty} \sup_{0<m\in\mathbb{N}} \frac{1}{m}\sum_{l=0}^{m-1} |f(j+l/m)| = \|f\|_S < +\infty.$$

Hence $S(C,\ell_1)\subset L^1$. Furthermore, if $f\in S(C,\ell_1)$ and $m=1,2,\ldots$ then

$$\sum_{k=-\infty}^{+\infty} |f(k/m)| \leq m\cdot \sum_{j=-\infty}^{+\infty} \frac{1}{m}\sum_{l=0}^{m-1} |f(j+l/m)| \leq m\cdot \|f\|_S < +\infty.$$

Next we list some basic properties of $(S(C,\ell_1),\|\cdot\|_S)$.

1^o First of all, a simple consideration shows that $(S(C,\ell_1),\|\cdot\|_S)$ is a normed space. Indeed, $\|0\|_S=0$ is trivial. If $f\in S(C,\ell_1)$ and $\|f\|_S=0$ then for every $j\in\mathbb{Z}$, $0<m\in\mathbb{N}$ we have $f(j+l/m)=0 \quad (l=0,...,m-1)$. Let $x\in\mathbb{R}$, $\varepsilon>0$. By the continuity of f there are $j\in\mathbb{Z}$, $0<m\in\mathbb{N}$, $l=0,\ldots,m-1$ such that with $y:=j+l/m$ the inequality $|f(x)-f(y)|=|f(x)|<\varepsilon$ holds. Hence $f(x)=0$, i.e. $f\equiv 0$. Furthermore, the equality $\|\lambda f\|_S=|\lambda|\cdot\|f\|_S$ $(f\in S(C,\ell_1),\lambda\in\mathbb{R})$ and the inequality $\|f+g\|_S\leq\|f\|_S+\|g\|_S \quad (f,g\in \in S(C,\ell_1))$ are obvious.

2^o Now, we take a sequence $f_n\in S(C,\ell_1) \quad (n\in\mathbb{N})$ of functions which is convergent in $S(C,\ell_1)$. In other words there exists $f\in S(C,\ell_1)$ such that

$$\|f_n-f\|_S\to 0 \qquad (n\to\infty).$$

This means that

$$\sum_{j=-\infty}^{+\infty} \sup_{0<m\in\mathbb{N}} \frac{1}{m}\sum_{l=0}^{m-1} |f_n(j+l/m)-f(j+l/m)| \to 0 \qquad (n\to\infty).$$

If $0\neq r\in\mathbb{R}$ is a rational number then with suitable $j_0\in\mathbb{Z}, 0<m_0\in\mathbb{N}$, $l_0=0,\ldots,m_0-1$ the equality $r=j_0+l_0/m_0$ holds. It is clear that

$$|f_n(r)-f(r)|\leq \sum_{l=0}^{m_0-1} |f_n(j_0+l/m_0)-f(j_0+l/m_0)| \leq m_0\|f_n-f\|_S \qquad (n\in\mathbb{N}),$$

i.e. $f(r)=\lim_{n\to\infty} f_n(r)$.

3^o The space $(S(C,\ell_1), \|\cdot\|_S)$ does not form a Banach space. Indeed, if $n = 1, 2, \dots$ and

$$f_n(x) := \begin{cases} \sin(\pi/x) & (1/n \le x \le 1) \\ 0 & (x \in \mathbb{R} \setminus (1/n, 1)), \end{cases}$$

then $f_n \in C$ and

$$\|f_n\|_S = \sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{l=0}^{m-1} |f_n(l/m)| \le 1$$

implies $f_n \in S(C,\ell_1)$. Furthermore, if $n, k, m = 1, 2, ...$ and $k > n$, then

$$\sum_{l=0}^{m-1} |f_n(l/m) - f_k(l/m)| = \sum_{l=0, 1/k<l/m<1/n}^{m-1} |f_n(l/m) - f_k(l/m)| =$$

$$= \sum_{l=0, m/k<l<m/n}^{m-1} |f_k(l/m)| \le \frac{m}{n} - \frac{m}{k}.$$

From this it follows that

$$\|f_n - f_k\|_S = \sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{l=0}^{m-1} |f_n(l/m) - f_k(l/m)| \le$$
$$\le \frac{1}{n} - \frac{1}{k} \to 0 \qquad (n, k \to \infty).$$

Hence the sequence (f_n) is a Cauchy sequence with respect to $\|\cdot\|_S$. Assume the existence $f \in S(C,\ell_1)$ such that $\|f_n - f\|_S \to 0 \quad (n \to \infty)$. Then it would be true by 2^o for all rational $r \in (0, 1)$ that

$$f(r) = \lim_{n\to\infty} f_n(r) = \sin(\pi/r).$$

However, such a continuous function $f : \mathbb{R} \to \mathbb{R}$ does not exist.

4^o Let $f \in C[0, 1]$ and

$$s_m(f) := \frac{1}{m} \sum_{l=0}^{m-1} |f(l/m)| \qquad (m = 1, 2, \dots).$$

If

$$f_{ml} := \max\{|f(t)| : l/m \le t \le (l + 1)/m\} \qquad (l = 0, \dots, m - 1)$$

and

$$S_m(f) := \frac{1}{m}\sum_{l=0}^{m-1} f_{ml} \qquad (m = 1, 2, \ldots),$$

then

$$s(f) := \sup_m s_m(f) \leq S(f) := \sup_m S_m(f).$$

Now let $n = 1, 2, ...$ be given and let us consider the function $f_n \in C[0,1]$ in the following way: $f_n(0) := f_n(t) := 0 \quad (1/n \leq t \leq 1)$, $f_n(1/(2n)) := 1$ and the graph of f_n over $[0, 1/n]$ is a triangle. Then $S_1(f_n) = 1$ which implies $S(f_n) \geq 1$. On the other hand for $m = 1, ..., n$ it follows $s_m(f_n) = 0$ but

$$s_m(f_n) = \frac{1}{m}\sum_{l=1}^{[m/n]} f_n(l/m) \leq \frac{1}{m}\sum_{l=1}^{[m/n]} 1 \leq \frac{1}{n} \qquad (m = n+1, n+2, \ldots).$$

Therefore $s(f_n) \leq 1/n \ \ (0 < n \in \mathbb{N})$, i.e. it does not exist constant $q \geq 0$ such that $S(f) \leq q \cdot s(f) \ \ (f \in C[0,1])$.

5^o Define for $f \in S(C, \ell_1)$ the symbol $\|f\|_{SW}$ as follows:

$$\|f\|_{SW} := \sum_{j=-\infty}^{+\infty} \sup_{0<m\in\mathbb{N}} \frac{1}{m}\sum_{l=0}^{m-1} \|f\chi_{[j+l/m, j+(l+1)/m]}\|_\infty .$$

It is not hard to see that $\|\cdot\|_{SW}$ is norm and $\|\cdot\|_S \leq \|\cdot\|_{SW}$. However, the functions $f_n \quad (n = 1, 2, ...)$ from 4^o show that $\|\cdot\|_S$, $\|\cdot\|_{SW}$ are not equivalent.

6^o It is clear that for all continuous functions $\theta : \mathbb{R} \to \mathbb{R}$

$$\frac{1}{m}\sum_{l=0}^{m-1} |\theta(j + l/m)| \leq \sup_{0\leq x<1} |\theta(j + x)| \qquad (j \in \mathbb{Z}),$$

which means that $W(C, \ell_1) \subset S(C, \ell_1)$. A simple example proves that this inclusion is proper, i.e. $W(C, \ell_1) \neq S(C, \ell_1)$. Indeed, let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\|\theta\chi_{(j,j+1/j)}\|_\infty = 1/j \quad (j = 1, 2, ...)$ and $\theta(t) = 0$ $(t \in \mathbb{R} \setminus A)$, where $A := \bigcup_{j=1}^{\infty}(j, j + 1/j)$ (see also Feichtinger and Weisz [2]). Then $\theta \in L^1$ and for all $m = 1, 2, ...$ and $j \in \mathbb{Z}$

$$\frac{1}{m}\sum_{l=0}^{m-1} |\theta(j + l/m)| \leq \begin{cases} 0 & (j \leq 0) \\ m^{-1}\sum\limits_{l=1}^{[m/j]} 1/j \leq j^{-2} & (j > 0). \end{cases}$$

Hence $\theta \in S(C, \ell_1)$. However,

$$\sum_{j=-\infty}^{+\infty} \sup_{0\leq x<1} |\theta(j + x)| = \sum_{j=1}^{+\infty} \frac{1}{j} = +\infty,$$

in other words $\theta \notin (W, \ell_1)$.

3. Main result

Now, we prove the main result of this work.

Theorem 3.1. *For all functions $\theta \in S(C, \ell_1)$ the implication $(*)$ is true.*

Proof. Let m, M, N be positive natural numbers and assume $M \leq m\pi$. Then $\|K_m^\theta\|_1$ can be considered as follows (the basic idea in the first steps is due to [8]):

$$2\pi\|K_m^\theta\|_1 = \int_{-\pi}^{\pi} \left| \sum_{k=-\infty}^{+\infty} \theta(k/m)e^{\imath kt} \right| dt = \int_{-m\pi}^{m\pi} \left| \frac{1}{m} \sum_{k=-\infty}^{+\infty} \theta(k/m)e^{\imath kt/m} \right| dt \geq$$

$$\geq \int_{-M}^{M} \left| \frac{1}{m} \sum_{k=-\infty}^{+\infty} \theta(k/m)e^{\imath kt/m} \right| dt \geq$$

$$\geq \int_{-M}^{M} \left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| dt - \int_{-M}^{M} \left| \frac{1}{m} \sum_{|k|>mN} \theta(k/m)e^{\imath kt/m} \right| dt \geq$$

$$\geq \int_{-M}^{M} \left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| dt - \frac{2M}{m} \sum_{|k|>mN} |\theta(k/m)|.$$

Here the sum $m^{-1}\sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m}$ $(-M \leq t \leq M)$ is nothing else but a Rieman sum of the continuous function $[-N, N] \ni x \mapsto \theta(x)e^{\imath tx}$. Therefore

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} = \int_{-N}^{N} \theta(x)e^{\imath tx}dx \qquad (-M \leq t \leq M).$$

However, if $-M \leq t \leq M$ then

$$\left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| \leq (2N+1) \max_{|x|\leq N} |\theta(x)|,$$

i.e. by the Lebesgue's dominated convergence theorem

$$\lim_{m\to\infty} \int_{-M}^{M} \left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| dt = \int_{-M}^{M} \left| \int_{-N}^{N} \theta(x)e^{\imath tx}dx \right| dt.$$

Furthermore,

$$\frac{1}{m}\sum_{|k|>mN}|\theta(k/m)|\leq\frac{1}{m}\sum_{|j|\geq N}\sum_{l=0}^{m-1}|\theta(j+l/m)|\leq\sum_{|j|\geq N}\gamma_j,$$

where

$$\gamma_j:=\sup_{0<n\in\mathbb{N}}\frac{1}{n}\sum_{l=0}^{n-1}|\theta(j+l/n)|.$$

We remark that $\theta\in S(C,\ell_1)$ implies $\sum_{j=-\infty}^{+\infty}\gamma_j=\|\theta\|_S<+\infty$, i.e. $\sum_{|j|\geq N}\gamma_j\to$ $\to 0\quad(N\to\infty)$.

Summarizing the above facts we get

$$2\pi\|K_m^\theta\|_1\geq\int_{-M}^{M}\left|\frac{1}{m}\sum_{k=-mN}^{mN}\theta(k/m)e^{\imath kt/m}\right|dt-2M\sum_{|j|\geq N}\gamma_j,$$

from which it follows that

$$2\pi\sup_{0<m\in\mathbb{N}}\|K_m^\theta\|_1\geq\lim_{m\to\infty}\int_{-M}^{M}\left|\frac{1}{m}\sum_{k=-mN}^{mN}\theta(k/m)e^{\imath kt/m}\right|dt-2M\sum_{|j|\geq N}\gamma_j=$$

$$=\int_{-M}^{M}\left|\int_{-N}^{N}\theta(x)e^{\imath tx}dx\right|dt-2M\sum_{|j|\geq N}\gamma_j.$$

Taking into account $\left|\int_{-N}^{N}\theta(x)e^{\imath tx}dx\right|\leq\|\theta\|_1\quad(|t|\leq M)$ the above mentioned theorem of Lebesgue guaranties that

$$\lim_{N\to\infty}\int_{-M}^{M}\left|\int_{-N}^{N}\theta(x)e^{\imath tx}dx\right|dt=\int_{-M}^{M}\left|\lim_{N\to\infty}\int_{-N}^{N}\theta(x)e^{\imath tx}dx\right|dt=$$

$$=\int_{-M}^{M}\left|\int_{-\infty}^{+\infty}\theta(x)e^{\imath tx}dx\right|dt=\int_{-M}^{M}\left|\hat{\theta}(t)\right|dt.$$

Thus

$$\sup_{0<m\in\mathbb{N}}\|K_m^\theta\|_1\geq\lim_{N\to\infty}\int_{-M}^{M}\left|\int_{-N}^{N}\theta(x)e^{\imath tx}dx\right|dt-2M\lim_{N\to\infty}\sum_{|j|\geq N}\gamma_j=\int_{-M}^{M}\left|\hat{\theta}(t)\right|dt,$$

from which

$$\|\hat{\theta}\|_1 = \lim_{M\to\infty} \int\limits_{-M}^{M} \left|\hat{\theta}(t)\right| dt \le 2\pi \sup_{0<m\in\mathbb{N}} \|K_m^\theta\|_1 < +\infty,$$

i.e., $\hat{\theta} \in L^1$ follows.

4. Remarks

1° It is clear that for compactly supported θ the proof can be simplified (see e.g. [8]). Namely, in this case $\operatorname{supp}\theta \subset [-N, N]$ can be supposed in the above proof. Then $\sum_{|k|>mN} \theta(k/m)e^{\imath kt/m} = 0 \quad (|t| \le M)$ holds trivially, hence

$$2\pi\|K_m^\theta\|_1 \ge \int\limits_{-M}^{M} \left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| dt.$$

Therefore

$$2\pi \sup_{0<m\in\mathbb{N}} \|K_m^\theta\|_1 \ge \lim_{m\to\infty} \int\limits_{-M}^{M} \left| \frac{1}{m} \sum_{k=-mN}^{mN} \theta(k/m)e^{\imath kt/m} \right| dt =$$

$$= \int\limits_{-M}^{M} \left| \int\limits_{-N}^{N} \theta(x)e^{\imath tx} dx \right| dt = \int\limits_{-M}^{M} \left|\hat{\theta}(x)\right| dt$$

and the proof can be finished as above.

2° If $\theta \in S(C, \ell_1)$ then

$$\sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{k=-\infty}^{+\infty} |\theta(k/m)| \le \sum_{j=-\infty}^{+\infty} \sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{l=0}^{m-1} |\theta(j + l/m)| = \|\theta\|_S < +\infty.$$

So the next estimation holds:

$$(**) \qquad \sup_{0<m\in\mathbb{N}} \frac{1}{m} \sum_{k=-\infty}^{+\infty} |\theta(k/m)| < +\infty.$$

On the other hand a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ can be constructed such that

$$\sum_{l=0}^{m-1} |\theta(j + l/m)| \sim |j|^{-1-1/m} \qquad (0 < m, |j| \in \mathbb{N}).$$

For this θ it follows that $\sum_{k=-\infty}^{+\infty} |\theta(k/m)| \sim m$. This means that $(**)$ holds but θ does not belong to $S(C, \ell_1)$). Indeed, then $\gamma_j \sim 1/(|j| \ln |j|) \quad (1 < |j| \in \mathbb{N})$, i.e. $\|\theta\|_S = +\infty$.

3° The following question remains open: is the assumption $(**)$ on a continuous and integrable function $\theta : \mathbb{R} \to \mathbb{R}$ enough to the implication $(*)$?

4° Let $\theta \in S(C, \ell_1)$ and $(X, \|.\|_*)$ be defined as follows:

$$(X, \|\cdot\|_*) := \begin{cases} (C[-\pi, \pi], \|\cdot\|_\infty) \\ \text{or} \\ \big(L^1[-\pi, \pi], \|\cdot\|_1\big)\,. \end{cases}$$

It is well-known that the norm of the operator $\sigma_m^\theta : X \to X$ is nothing else but $\|K_m^\theta\|_1 \quad (m = 1, 2, ...)$. Assume that the function θ satisfies also $\theta(0) = 1$. Then a simple calculation shows that $\|\sigma_m^\theta e_j - e_j\|_* \to 0 \quad (m \to \infty)$, from which the same convergence follows for all trigonometric polynomials. Since the set of the trigonometric polynomials is dense in X, the theorem of Banach and Steinhaus (taking into account also our theorem) implies the following corollary:

$$\lim_{m\to\infty} \|\sigma_m^\theta f - f\|_* = 0 \quad (f \in X) \quad \Longleftrightarrow \quad \hat{\theta} \in L^1.$$

5° We remark that (see Feichtinger and Weisz [2]) if $\theta \in W(C, \ell_1), \theta(0) = 1$, then

$$\lim_{m\to\infty} \|\sigma_m^\theta f - f\|_2 = 0 \qquad (f \in L^2[-\pi, \pi]).$$

However, let $\theta : \mathbb{R} \to \mathbb{R}$ be continuous such that $\|\theta\chi_{(j,j+1/j^2)}\|_\infty = \sqrt{j} \quad (j = = 1, 2, \ldots)$ and $\theta(t) = 0 \quad (t \in \mathbb{R} \setminus B)$, where $B := \bigcup_{j=1}^\infty (j, j + 1/j^2)$ and $\theta(j + 1/(2j^2)) = \sqrt{j} \;\; (0 < j \in \mathbb{N})$. For this function the relation $\theta \in S(C, \ell_1)$ follows with $j^{-3/2}$ instead of $j^{-2} \quad (0 < j \in \mathbb{N})$ analogously as above in similar situation (see our example for the illustration of $W(C, \ell_1) \neq S(C, \ell_1)$). Furthermore for $m := 2j^2$, $k := jm + 1 \quad (0 < j \in \mathbb{N})$ we get

$$\frac{k}{m} = j + \frac{1}{2j^2},$$

hence $|\theta(k/m)| = \sqrt{j}$. This implies

$$\sup_{0<m\in\mathbb{N}} \sup_{k\in\mathbb{Z}} |\theta(k/m)| = +\infty.$$

The norm of the operator $\sigma_m^\theta : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ $(0 < m \in \mathbb{N})$ is $\|\sigma_m^\theta\| = = \sup_{k\in\mathbb{Z}} |\theta(k/m)|$ (see [2]). Hence $\sup_{0<m\in\mathbb{N}} \|\sigma_m^\theta\| = +\infty$ and the theorem of Banach and Steinhaus gives $f \in L^2$ such that the sequence $(\sigma_m^\theta f)$ diverges in $\|\cdot\|_2$ norm. In other words the last mentioned Feichtinger and Weisz's theorem on L^2-convergence cannot be extended to the space $S(C,\ell_1)$.

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