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STRONG LAW OF LARGE NUMBERS FOR SCALAR-NORMED SUMS OF ELEMENTS OF REGRESSIVE SEQUENCES OF RANDOM VARIABLES

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Abstract. The necessary and sufficient conditions providing the strong law of large numbers for scalar-normed sums of elements of first-order regressive sequences of random variables with independent and symmetric disturbance are studied.

1. Introduction

Consider a linear regressive sequence of random variables $(\eta_k) = (\eta_k, k \geq 1)$ which obeys the system of following recurrence equations:

$$\eta_1 = \beta_1\theta_1, \quad \eta_k = \alpha_k\eta_{k-1} + \beta_k\theta_k, \quad k \geq 2, \tag{1}$$

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where (α_k) and (β_k) are nonrandom real sequences, and (θ_k) is a sequence of independent symmetric random variables such that

$$\mathbb{P}\{\theta_k = 0\} < 1, \quad k \geq 1. \tag{2}$$

Remind that the random variable θ is called *symmetric* if θ and $(-\theta)$ are identically distributed. Note that introduced recurrent scheme involves a wide class of sequences. For example, if (θ_k) is a standard Gaussian sequence then (η_k) is a Gaussian Markov sequence.

The partial case of sequence (1) is a sequence of independent random variables, when $\alpha_k = 0$, $k \geq 2$. It is well-known that in the classical works of P. Levy [13], A.Ya. Khintchine [11], A.N. Kolmogorov [12], Yu.V. Prokhorov [17] and M. Loeve [14] the theory of almost sure asymptotic behavior of increasing scalar-normed sums of independent random variables was constructed. This theory among other problems studies different versions of the strong law of large numbers. One of the most classical types of the strong laws of large numbers is the Prokhorov-Loeve type strong law for scalar-normed sums of independent random variables. Such type of the strong law of large numbers was intensively studied in the works of V.V. Buldygin [1], A.I. Martikainen and V.V. Petrov [15, 16], N.A. Volodin and S.V. Nagaev [22].

The main purpose of this paper is to obtain general conditions providing strong law of large numbers for scalar-normed sums of elements of sequence (1) similar to the Prokhorov-Loeve type. Namely, for the sequence (1) we will consider the sequence of partial sums (S_n), $S_n = \sum_{k=1}^{n} \eta_k$, $n \geq 1$, and study conditions providing

$$c_n S_n \underset{n\to\infty}{\longrightarrow} 0, \quad \text{almost surely},$$

where (c_n) is some nonrandom real sequence.

It is also worth noting that another partial case of sequence (1) is an autoregressive sequence of random variables, when $\alpha_k = q = const$, $k \geq 1$. In this case sequence (1) is a stationary sequence. For wide sense stationary sequences of random variables the strong law of large numbers was studied in the works of I.N. Verbitskaya [20, 21], and later V.F. Gaposhkin [8, 9, 10]. We consider autoregressive sequence as an example which illustrates main result of this paper.

The set problem is tightly connected with the problem on conditions for the almost sure convergence of series $\sum_{k=1}^{\infty} \eta_k$. Necessary and sufficient conditions providing almost sure convergence of such a series, in particular Gaussian Markov series, were obtained in papers of V.V. Buldygin and M.K. Runovska [3, 4, 5, 18, 19]. The technique applied by studying series $\sum_{k=1}^{\infty} \eta_k$ is based on the passage from the sequence of partial sums (S_n) of regressive sequence (1) to the sequence of sums of independent random elements in higher dimensional space. This technique is general and is effectively used also in this paper.

2. Strong law of large numbers for scalar-normed sums of elements of regressive sequences

Let $n \geq 1$ and

$$a(n,k) = \begin{cases} 0, & 1 \leq n < k; \\ 1, & n = k; \\ 1 + \sum\limits_{l=1}^{n-k} \Big(\prod\limits_{j=k+1}^{k+l} \alpha_j \Big), & n > k, \end{cases} \tag{3}$$

and $A(n,k) = \beta_k a(n,k)$, $n, k \geq 1$.

The following Theorem gives necessary and sufficient conditions providing the strong law of large numbers for sums whose terms are elements of regressive sequence (1). By $\mathfrak{R}^\infty$ denote the class of all sequences of positive integers increasing to infinity.

Theorem 2.1. *Let* $S_n = \sum_{k=1}^n \eta_k$, $n \geq 1$, *for the regressive sequence* (1), *and* (c_n) *be some nonrandom real sequence. In order for the relation*

$$c_n S_n \underset{n\to\infty}{\longrightarrow} 0, \quad \textit{almost surely,}$$

to hold it is necessary and sufficient that the following two conditions be satisfied:

1) $\lim\limits_{n\to\infty} c_n A(n,k) = 0$, *for any* $k \geq 1$;

2) *for all the sequences* (m_j) *of the class* $\mathfrak{R}^\infty$

$$\Big| c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} A(m_{j+1}, k)\theta_k \Big| \underset{j\to\infty}{\to} 0, \quad \textit{almost surely.}$$

Preparatory to proving the Theorem 2.1 we recall one auxiliary result which will be used to prove sufficiency of Theorem 2.1.

We interpret the space $\mathbb{R}^d$, $d \geq 1$, as d-dimensional real Euclidean space of column vectors with the inner product $\langle X, Y \rangle$ and Euclidean norm $\|X\| = \sqrt{\langle X, X \rangle}$, $X, Y \in \mathbb{R}^d$. Let $c_0(\mathbb{R}^d)$ be the space of all convergent to zero sequences of elements of the space $\mathbb{R}^d$; (X_n) be the sequence of independent symmetric random vectors in $\mathbb{R}^d$; $\Xi_n = \sum_{k=1}^n X_k$, $n \geq 1$; $(\mathcal{A}_n)$ be the sequence of continuous linear operators mapping $\mathbb{R}^d$ into $\mathbb{R}^d$.

The following Proposition is due to V.V. Buldygin and S.A. Solntsev (see [6, 7]) and provides a criterion for the almost sure convergence to zero of operator-normed sums of independent symmetric random vectors in the space $\mathbb{R}^d$.

Proposition 2.1. *In order for the relation $(\mathcal{A}_n \Xi_n) \in c_0(\mathbb{R}^d)$, almost surely, to hold, it is necessary and sufficient that the following two conditions be satisfied*:

A) $(\mathcal{A}_n X_k) \in c_0(\mathbb{R}^d)$, *almost surely, for any* $k \geq 1$;

B) *for all the sequences* (m_j) *of the class* $\mathfrak{R}^\infty$

$$\|\mathcal{A}_{m_{j+1}}(\Xi_{m_{j+1}} - \Xi_{m_j})\| \underset{j\to\infty}{\to} 0, \quad \textit{almost surely.}$$

Proof of Theorem 2.1. First let us prove the necessity of Theorem 2.1. Consider the sequence $(c_n S_n)$ and represent it in the form of a series:

$$\begin{pmatrix} c_1 S_1 \\ c_2 S_2 \\ \vdots \\ c_n S_n \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1 A(1,1) \\ c_2 A(2,1) \\ \vdots \\ c_n A(n,1) \\ \vdots \end{pmatrix} \theta_1 + \begin{pmatrix} 0 \\ c_2 A(2,2) \\ \vdots \\ c_n A(n,2) \\ \vdots \end{pmatrix} \theta_2 + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_n A(n,n) \\ \vdots \end{pmatrix} \theta_n + \cdots ,$$

whose terms are independent symmetric random elements in the sequence space $\mathbb{R}^\infty$. Emphasize that this series is coordinate-wise convergent. Let us rewrite the latter series as follows

$$\overrightarrow{S} = \sum_{k=1}^{\infty} (\overrightarrow{A}_k \theta_k),$$

where

$$\overrightarrow{S} = \begin{pmatrix} c_1 S_1 \\ c_2 S_2 \\ \vdots \\ c_n S_n \\ \vdots \end{pmatrix}, \quad \overrightarrow{A}_k = \begin{pmatrix} c_1 A(1,k) \\ c_2 A(2,k) \\ \vdots \\ c_n A(n,k) \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_k A(k,k) \\ c_{k+1} A(k+1,k) \\ \vdots \end{pmatrix}, \qquad k \geq 1.$$

Since $(c_n S_n) \in c_0(\mathbb{R})$, almost surely ,then $\overrightarrow{A}_k \theta_k \in c_0(\mathbb{R})$ almost surely, for any $k \geq 1$. Indeed, let us represent the $\mathbb{R}^\infty$-valued random element $\overrightarrow{S}$ in the form of the sum $\overrightarrow{S} = \overrightarrow{S}_1 + \overrightarrow{S}_2$, where $\overrightarrow{S}_1 = \overrightarrow{A}_1 \theta_1$, $\overrightarrow{S}_2 = \sum_{k=2}^{\infty} \overrightarrow{A}_k \theta_k$, and show that $\overrightarrow{A}_1 \theta_1 \in c_0(\mathbb{R})$, almost surely. Since (θ_k) is a sequence of independent symmetric random variables, then elements $\overrightarrow{S}_1$ and $\overrightarrow{S}_2$ are independent symmetric elements in the space $\mathbb{R}^\infty$. Hence elements $(\overrightarrow{S}_1 + \overrightarrow{S}_2)$ and $(\overrightarrow{S}_1 - \overrightarrow{S}_2)$ are identically distributed. Since $\overrightarrow{S} \in c_0(\mathbb{R})$, almost surely, i.e. $(\overrightarrow{S}_1 + \overrightarrow{S}_2) \in c_0(\mathbb{R})$, almost surely, then one has that also $(\overrightarrow{S}_1 - \overrightarrow{S}_2) \in c_0(\mathbb{R})$, almost surely. Taking into account the linearity of the space $c_0(\mathbb{R})$, we get that $\overrightarrow{S}_1 \in c_0(\mathbb{R})$, almost surely, and $\overrightarrow{S}_2 \in c_0(\mathbb{R})$, almost surely, that is $\overrightarrow{A}_1\theta_1 \in c_0(\mathbb{R})$, almost surely, and $\sum_{k=2}^{\infty} \overrightarrow{A}_k\theta_k \in c_0(\mathbb{R})$, almost surely. By analogue, one can see that $\overrightarrow{A}_k\theta_k \in c_0(\mathbb{R})$, almost surely, for any $k \geq 1$. The latter relation together with (2) implies that $\overrightarrow{A}_k \in c_0(\mathbb{R})$, for any $k \geq 1$, i.e. $\lim_{n\to\infty} c_n A(n,k) = 0$, for any $k \geq 1$. Thus, assumption 1) of Theorem 2.1 holds.

To prove that assumption 2) of Theorem 2.1 holds, we shall follow *the contraction principle* in the space of all convergent sequences (for more details see [6, 7]).

Consider the set of random variables $(w_{n,k};\ \ n,k \geq 1)$, where

$$w_{n,k} = c_n A(n,k)\theta_k, \quad k,n \geq 1.$$

This set possesses the following properties:

a) for any $n \geq 1$, the series $\sum_{k=1}^{\infty} w_{n,k}$ almost surely converges in the space $\mathbb{R}$;

b) the sequences $W_k = (w_{n,k}, \quad n \geq 1), \quad k \geq 1$, are independent and symmetric as elements of the space of sequences $\mathbb{R}^{\infty}$.

Moreover,

$$c_n S_n = \sum_{k=1}^{n} c_n A(n,k)\theta_k = \sum_{k=1}^{\infty} c_n A(n,k)\theta_k = \sum_{k=1}^{\infty} w_{n,k}, \tag{4}$$

for any $n \geq 1$.

Fix an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^{\infty}$. Along with the set $(w_{n,k}; n,k \geq\ 1)$ consider the nonrandom set $(\lambda_{n,k};\ \ n,k \geq 1)$, where

$$\lambda_{n,k} = \begin{cases} 1, & n = m_{j+1}, \quad m_j < k \leq m_{j+1}, \quad j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, for the set $(\lambda_{n,k};\ \ n,k \geq 1)$ the following condition holds

$$Var(\lambda_{n,k}; n,k \geq 1) = \sup_{n\geq 1} \sup_{m\geq 2} \left[\left(\sum_{k=1}^{m-1} |\lambda_{n,k} - \lambda_{n,k+1}| \right) + |\lambda_{n,m}| \right] = 2 < \infty.$$

Hence $(\lambda_{n,k};\ \ n,k \geq 1)$ is a contraction set of bounded variation (see [6, 7]). Besides, according to the definition of sets $(w_{n,k};\ \ n,k \geq 1)$ and $(\lambda_{n,k};\ \ n,k \geq 1)$ one has that for any $k \geq 1$,

$$|\lambda_{n,k} w_{n,k}| \underset{n\to\infty}{\to} 0, \quad \text{almost surely.}$$

Further, since $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, then by virtue of (4) the sequence of random variables $(\sum_{k=1}^{\infty} w_{n,k},\ \ n \geq 1)$ converges almost surely in the space $\mathbb{R}$.

Thus, all the assumptions of *the contraction principle* in the space of convergent sequences (see [6, 7]) hold. The latter implies that

$$\Big|\sum_{k=1}^{\infty} \lambda_{n,k} w_{n,k}\Big| \underset{n\to\infty}{\to} 0, \quad \text{almost surely.}$$

This relation in our terms may be rewritten as follows:

$$\Big|\sum_{k=m_j+1}^{m_{j+1}} c_{m_{j+1}} A(m_{j+1}, k)\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \text{almost surely.}$$

Thus, condition 2) of Theorem 2.1 holds. This completes the proof of necessary part of Theorem 2.1.

Now let us focus on the sufficiency part of Theorem 2.1. First note that the sequence of partial sums (S_n) obeys the system of following second-order recurrence equations:

$$S_{-1} = S_0 = 0, \quad S_n = (1 + \alpha_n)S_{n-1} - \alpha_n S_{n-2} + \beta_n \theta_n, \ \ n \geq 1. \tag{5}$$

Now we pass from the relations (5) to the first-order recurrence relations in the space $\mathbb{R}^2$, namely

$$\widetilde{S}_1 = \Theta_1, \quad \widetilde{S}_n = B_n \widetilde{S}_{n-1} + \Theta_n, \ \ n \geq 2, \tag{6}$$

where

$$\widetilde{S}_n = \begin{pmatrix} S_n \\ S_{n-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} 1+\alpha_n & -\alpha_n \\ 1 & 0 \end{pmatrix}, \quad \Theta_n = \begin{pmatrix} \beta_n\theta_n \\ 0 \end{pmatrix}, \quad n \geq 1.$$

Now we transform recurrence relation (6) to the following form

$$\widetilde{S}_n = \Big(\prod_{j=n}^{2} B_j\Big)\Theta_1 + \Big(\prod_{j=n}^{3} B_j\Big)\Theta_2 + \cdots + B_n\Theta_{n-1} + \Theta_n, \quad n \geq 1, \tag{7}$$

where $\prod_{j=n}^{k} B_j = B_n B_{n-1} \cdots B_k, \ \ k \leq n$. Using the mathematical induction principle and (3) one may obtain that

$$\prod_{j=n}^{k} B_j = \begin{pmatrix} a(n, k-1) & 1 - a(n, k-1) \\ a(n-1, k-1) & 1 - a(n-1, k-1) \end{pmatrix}, \quad 2 \leq k \leq n.$$

Now we prove the sufficiency part of the theorem. We show that assumptions 1)—2) provide $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely. We distinguish between the following two cases:

I) all coefficients (α_k) are nonzero, that is $\alpha_k \neq 0, \quad k \geq 2$;

II) there are some zero coefficients among (α_k).

Case I). Let all coefficients (α_k) be nonzero, that is $\alpha_k \neq 0, \quad k \geq 2$. Then all matrices $B_n, n \geq 1$, involved in representation (7) are non-singular, that is $\det B_n \neq 0, \quad n \geq 1$. This means that one can pass from relation (7) to the following one

$$\widetilde{S}_n = \Big(\prod_{j=n}^{2} B_j\Big)\Big(\Theta_1 + B_2^{-1}\Theta_2 + (B_2^{-1}B_3^{-1})\Theta_3 + \cdots + (B_2^{-1}B_3^{-1}\cdots B_n^{-1})\Theta_n\Big),$$
$$n \geq 1,$$

where B_k^{-1} is the inverse matrix for $B_k, \quad k \geq 1$. Now we represent the sequence $(\widetilde{S}_n)$ as a sequence of sums of independent random vectors with an operator normalization (see [6, 7]), namely

$$\widetilde{S}_n = \mathcal{A}_n \sum_{k=1}^{n} X_k = \mathcal{A}_n \Xi_n, \quad n \geq 1,$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_n = \prod_{j=n}^{2} B_j = \begin{pmatrix} a(n,1) & 1-a(n,1) \\ a(n-1,1) & 1-a(n-1,1) \end{pmatrix}, \quad n \geq 2,$$

$$\Xi_n = \sum_{k=1}^{n} X_k, \quad n \geq 1,$$

$$X_1 = \Theta_1, \qquad X_k = \Big(\prod_{j=2}^{k} B_j^{-1}\Big)\Theta_k, \quad k \geq 2.$$

Note that (X_n) is a sequence of independent symmetric random vectors in the space $\mathbb{R}^2$. Thus, sequence $(\widetilde{S}_n)$ is represented in the form of the sequence $(\mathcal{A}_n \Xi_n)$ of sums of independent random vectors with an operator normalization in $\mathbb{R}^2$.

For $n \geq 1$ set

$$C_n = \begin{pmatrix} c_n & 0 \\ 0 & c_{n-1} \end{pmatrix},$$

and consider the sequence $(C_n\mathcal{A}_n\Xi_n)$ which is also the sequence of sums of independent symmetric random vectors in the space $\mathbb{R}^2$.

Therefore, $c_nS_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if $(C_n\mathcal{A}_n\Xi_n) \in c_0(\mathbb{R}^2)$, almost surely. Now to complete the proof of sufficiency part of the theorem in case I) one should check assumptions A) and B) of Proposition 2.1 for the sequence $(C_n\mathcal{A}_n\Xi_n)$.

Case II). Without loss of generality we will assume that there is an infinite number of zeros among α_k's, $k \geq 2$. This immediately implies that the limit $\lim\limits_{n\to\infty} A(n,k)$ exists for any $k \geq 1$. Therefore, assumption 1) of Theorem 2.1 holds if and only if $\lim\limits_{n\to\infty} c_n = 0$, whence $\sup_{n\geq 1}|c_n| < \infty$. Note also that if in assumption 2) of Theorem 2.1 we set $m_j = j$, $j \geq 1$, then we obtain that $\lim\limits_{j\to\infty} |c_j\beta_j\theta_j| = 0$. This implies that $\sup\limits_{n\geq 1}|c_n\beta_n\theta_n| < \infty$, almost surely.

To prove the theorem in case II) we apply *the disturbed coefficients method* similar to that one introduced in [6, 7, 18]. Namely, we will construct an auxiliary regressive sequence $(\widehat{\eta}_k)$ which would obey representation (1) involving nonzero coefficients $\widehat{\alpha}_n$, $n \geq 1$, such that all the assumptions of Theorem 2.1 hold for it, and

$$\left|c_n(\widehat{S}_n - S_n)\right| \underset{n\to\infty}{\longrightarrow} 0, \quad \text{almost surely},$$

where $\widehat{S}_n = \sum_{k=1}^n \widehat{\eta}_k$, $n \geq 1$.

Thus, along with sequence (1) consider a regressive sequence $(\widehat{\eta}_k)$ which obeys the system of recurrence equations

$$\widehat{\eta}_1 = \beta_1\theta_1, \quad \widehat{\eta}_k = \widehat{\alpha}_k\widehat{\eta}_{k-1} + \beta_k\theta_k, \quad k \geq 2,$$

where (β_k) and (θ_k) are the same as in (1), while

$$\widehat{\alpha}_k = \begin{cases} \alpha_k, & \text{if } \alpha_k \neq 0, \\ \varepsilon_k, & \text{if } \alpha_k = 0. \end{cases}$$

The sequence (ε_k) is constructed such that $\varepsilon_k > 0$, $k \geq 1$, and

$$\left|\prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j\right| \leq 2^{-k} \cdot 2^{-l} \cdot |c_k|, \quad k, l \geq 1. \tag{8}$$

First we show that such a sequence (ε_k) exists. We enumerate the indices $k \geq 1$ such that $\alpha_k = 0$. The resulting sequence is denoted by (n_i), $i \geq 1$, that is $\alpha_{n_i} = 0$, $i \geq 1$. Now we choose the sequence (ε_{n_i}) such that

$$\varepsilon_{n_i} \leq \frac{2^{-n_{(i+1)}} \prod_{j=1}^{n_i} |\widehat{c}_j|}{\delta_{n_i}}, \quad i \geq 1,$$

where

$$\delta_{n_i} = \begin{cases} 1, & \text{if } \Delta_{n_i} \in (0,1), \\ \Delta_{n_i}, & \text{if } \Delta_{n_i} \in [1,\infty), \end{cases}$$

$$\Delta_{n_i} = \sup_{s \geq n_{(i-1)}+1, t \leq n_{(i+1)}-1} \Big| \prod_{m=s}^{n_i-1} \alpha_m \cdot \prod_{m=n_i+1}^{t} \alpha_m \Big|, \quad i \geq 1,$$

and

$$\widehat{c}_k = \begin{cases} 1, & \text{if } |c_k| > 1, \\ c_k, & \text{if } |c_k| \leq 1. \end{cases}$$

Fix $k \geq 1$, and note that

$$\Big| \prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j \Big| = 0, \quad l \geq 1,$$

if $\alpha_j \neq 0$ for all $k+1 \leq j \leq k+l$. Thus we may restrict the consideration to the case where there exists at least one number $n_i \in \{k+1, k+2, ..., k+l\}$ such that $\alpha_{n_i} = 0$. Then

$$\Big| \prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j \Big| = \Big| \prod_{j=k+1}^{k+l} \widehat{\alpha}_j \Big| = \Big| \prod_{k+1 \leq j \leq k+l : \alpha_j \neq 0} \alpha_j \Big| \cdot \prod_{i: k+1 \leq n_i \leq k+l} \varepsilon_{n_i} \leq$$

$$\leq \Big| \prod_{k+1 \leq j \leq k+l : \alpha_j \neq 0} \alpha_j \Big| \cdot \prod_{i: k+1 \leq n_i \leq k+l} \frac{2^{-n_{(i+1)}} \prod_{j=1}^{n_i} |\widehat{c}_j|}{\delta_{n_i}} \leq$$

$$\leq |c_k| \cdot \prod_{i: k+1 \leq n_i \leq k+l} 2^{-n_{(i+1)}} \leq 2^{-(k+l)} |c_k|,$$

for all $l \geq 1$. This means that inequalities (8) hold if the sequence (ε_{n_i}) is chosen as indicated above.

We continue the proof of Theorem 2.1 for the case II) and show that assumptions 1)-2) of this theorem hold for the sequence $(\widehat{\eta}_k)$. Let $\widehat{A}(n,k)$, $n,k \geq 1$, be defined as $A(n,k)$, $n,k \geq 1$, with coefficients $\widehat{\alpha}_k$, $k \geq 2$, instead of α_k, $k \geq 2$, involved. Since

$$\Big| c_n \Big(\widehat{A}(n,k) - A(n,k) \Big) \Big| = \Big| c_n \Big(\sum_{l=1}^{n-k} \Big(\prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j \Big) \beta_k \Big) \Big| \leq$$

$$\leq |c_n| \cdot \sum_{l=1}^{n-k} \Big| \prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j \Big| \beta_k \leq |c_n| \cdot \sum_{l=1}^{n-k} 2^{-l} \cdot 2^{-k} |c_k| \beta_k =$$

$$= |c_n| (2^{-k} - 2^{-n}) \cdot |c_k| \beta_k \underset{n \to \infty}{\longrightarrow} 0,$$

for any $k \geq 1$, then by virtue of assumption 1) of Theorem 2.1, $\lim\limits_{n\to\infty} c_n \widehat{A}(n,k) = 0$, for any $k \geq 1$. Thus, assumption 1) of this Theorem holds for the sequence $(\widehat{\eta}_k)$.

Fix an arbitrary sequence (m_j) belonging to $\mathfrak{R}^\infty$. Then

$$\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} \Big(\widehat{A}(m_{j+1},k) - A(m_{j+1},k)\Big)\theta_k\Big| =$$

$$= \Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} \sum_{l=1}^{m_{j+1}-k} \Big(\prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j\Big)\beta_k\theta_k\Big| \leq$$

$$\leq |c_{m_{j+1}}| \sum_{k=m_j+1}^{m_{j+1}} \sum_{l=1}^{m_{j+1}-k} \Big|\prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j\Big| \cdot \beta_k|\theta_k| \leq$$

$$\leq |c_{m_{j+1}}| \sum_{k=m_j+1}^{m_{j+1}} \sum_{l=1}^{m_{j+1}-k} 2^{-k-l}|c_k| \cdot \beta_k|\theta_k| =$$

$$= |c_{m_{j+1}}| \sum_{k=m_j+1}^{m_{j+1}} 2^{-k}|c_k\beta_k\theta_k| \sum_{l=1}^{m_{j+1}-k} 2^{-l} \leq$$

$$\leq |c_{m_{j+1}}| \sum_{k=m_j+1}^{m_{j+1}} 2^{-k}|c_k\beta_k\theta_k| \leq |c_{m_{j+1}}| \cdot \sup_{n\geq 1}|c_n\beta_n\theta_n| \sum_{k=m_j+1}^{m_{j+1}} 2^{-k} \underset{j\to\infty}{\longrightarrow} 0.$$

Hence according to assumption 2) of Theorem 2.1, one has that assumption 2) of this theorem also holds for the sequence $(\widehat{\eta}_k)$.

Therefore the sequence $(\widehat{\eta}_k)$ satisfies all the assumptions of Theorem 2.1, whence we conclude that $c_n\widehat{S}_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, in view of the case I) proved above.

Finally, let us show that $\Big|c_n(\widehat{S}_n - S_n)\Big| \underset{n\to\infty}{\longrightarrow} 0$, almost surely. Indeed,

$$\Big|c_n(\widehat{S}_n - S_n)\Big| = \Big|c_n\Big(\sum_{k=1}^{n}(\widehat{A}(n,k) - A(n,k))\theta_k\Big)\Big| \leq$$

$$\leq |c_n| \sum_{k=1}^{n}\sum_{l=1}^{n-k} \Big|\prod_{j=k+1}^{k+l} \widehat{\alpha}_j - \prod_{j=k+1}^{k+l} \alpha_j\Big| \cdot \beta_k|\theta_k| \leq |c_n| \sum_{k=1}^{n}\sum_{l=1}^{n-k} 2^{-k-l}|c_k\beta_k\theta_k| =$$

$$= |c_n| \sum_{k=1}^{n}(2^{-k} - 2^{-n})|c_k\beta_k\theta_k| \leq |c_n| \sum_{k=1}^{n} 2^{-k}|c_k\beta_k\theta_k| \leq \sup_{k\geq 1}|c_k\beta_k\theta_k| \cdot |c_n| \underset{n\to\infty}{\longrightarrow} 0.$$

Since $c_n\widehat{S}_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, we conclude that $c_nS_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, too. Thus, Theorem 2.1 is valid for the case II), as well, and the proof is complete. ■

Remark 2.1. Except extremely degenerate cases of the sequence (η_k), assumption 1) of Theorem 2.1 may be replaced by $\lim_{n\to\infty} c_n a(n,k) = 0$, for any $k \geq 1$.

Remark 2.2. Assumption 2) of Theorem 2.1 holds if and only if for all the sequences (m_j) of the class $\mathfrak{R}^\infty$ and any $\varepsilon > 0$

$$\sum_{j=1}^{\infty} \mathbb{P}\Big\{\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} A(m_{j+1},k)\theta_k\Big| > \varepsilon\Big\} < \infty.$$

Remark 2.3. In assumption 2) of Theorem 2.1 we do not have to demand that the appropriate condition holds for all the sequences (m_j) belonging to class $\mathfrak{R}^\infty$. Primarily the class $\mathfrak{R}^\infty$ appeared in works [16, 1, 2]. In particular, in these works assumptions similar to assumption 2) of Theorem 2.1 may be found. In [1, 2] was shown that the corresponding assumptions may be restricted to involve only some specially constructed so-called "testing sequences" instead of all sequences (m_j) of the class $\mathfrak{R}^\infty$. The construction algorithm of the explicit form of the "testing sequences" may be found in [2].

The following Corollary shows that the result of Theorem 2.1 corresponds to the well-known Prokhorov-Loeve type strong law of large numbers for scalar-normed sums of independent random variables (see [16, 2]).

Corollary 2.1. *Let* $\alpha_k = 0$, $k \geq 2$, *i.e.* $\eta_k = \beta_k\theta_k$, $k \geq 1$. *Then* $c_nS_n \underset{n\to\infty}{\longrightarrow}$ $\underset{n\to\infty}{\longrightarrow} 0$, *almost surely, if and only if* $\lim_{n\to\infty} c_n = 0$, *and for all the sequences* (m_j) *of the class* $\mathfrak{R}^\infty$

$$\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} \beta_k\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \textit{almost surely.}$$

Let us consider some examples.

Example 2.1. Let $c_n = \frac{1}{n}$, $n \geq 1$. Then according to Theorem 2.1

$$\frac{1}{n}(\eta_1 + \eta_2 + \ldots + \eta_n) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{almost surely,}$$

if and only if the following two conditions hold:

1) $\lim\limits_{n\to\infty} \frac{1}{n} A(n,k) = 0$, for any $k \geq 1$;

2) for all the sequences (m_j) of the class $\mathfrak{R}^\infty$

$$\frac{1}{m_{j+1}}\Big|\sum_{k=m_j+1}^{m_{j+1}} A(m_{j+1},k)\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \textit{almost surely.}$$

Example 2.2. Consider an autoregressive sequence (η_k):

$$\eta_1 = \beta_1\theta_1, \quad \eta_k = q\eta_{k-1} + \beta_k\theta_k, \quad k \geq 2,$$

where q is some real constant. Let us distinguish between 4 cases:

a) $|\mathbf{q}| < \mathbf{1}$. In this case $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if and only if $\lim\limits_{n\to\infty} c_n = 0$, and for all the sequences (m_j) of the class $\mathfrak{R}^\infty$

$$\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} \beta_k\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \text{almost surely.}$$

Thus, in the case of "weak dependence" the strong law of large numbers is the same as in independent case.

b) $\mathbf{q} = \mathbf{1}$. In this case $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if and only if $\lim\limits_{n\to\infty} nc_n = 0$, and for all the sequences (m_j) of the class $\mathfrak{R}^\infty$

$$\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} (m_{j+1} - k + 1)\beta_k\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \text{almost surely.}$$

c) $\mathbf{q} = -\mathbf{1}$. In this case $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if and only if $\lim\limits_{n\to\infty} c_n = 0$, and for all the sequences (m_j) of the class $\mathfrak{R}^\infty$

$$\Big|c_{m_{j+1}} \sum_{\substack{m_j+1\leq k\leq m_{j+1}: \\ (m_{j+1}-k)-odd}} \beta_k\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \text{almost surely,}$$

where the sum is taken over all $m_j + 1 \leq k \leq m_{j+1}$, such that $(m_{j+1} - k)$ is an odd integer.

d) $|\mathbf{q}| > \mathbf{1}$. In this case $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if and only if $\lim\limits_{n\to\infty} c_n q^n = 0$, and for all the sequences (m_j) of the class $\mathfrak{R}^\infty$

$$\Big|c_{m_{j+1}} \sum_{k=m_j+1}^{m_{j+1}} (1 - q^{m_{j+1}-k+1})\beta_k\theta_k\Big| \underset{j\to\infty}{\to} 0, \quad \text{almost surely.}$$

Thus, these four cases illustrate how the rate of the dependence between the elements of autoregressive sequence (η_k) influences on the form of the strong law of large numbers and the normalizing sequence (c_k).

For the Gaussian Markov sequence (η_k) the result of Theorem 2.1 specializes as follows.

Corollary 2.2. *Let (θ_k) be a standard Gaussian sequence, i.e. (η_k) be a Gaussian Markov sequence. Then $c_n S_n \underset{n\to\infty}{\longrightarrow} 0$, almost surely, if and only if the following two conditions hold:*

1) $\lim\limits_{n\to\infty} c_n A(n,k) = 0$, *for any* $k \geq 1$;

2) *for all the sequences (m_j) of the class $\mathfrak{R}^\infty$ and any $\varepsilon > 0$*

$$\sum_{j=1}^{\infty} \exp\left\{-\frac{\varepsilon}{c_{m_{j+1}}^2 \sum_{k=m_j+1}^{m_{j+1}} (A(m_{j+1},k))^2}\right\} < \infty.$$

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