Annales Univ. Sci. Budapest., Sect. Comp. 39 (2013) 355–364

ON ADDITIVE FUNCTIONS WITH VALUES IN ABELIAN GROUPS

Bui Minh Phong (Budapest, Hungary)

Dedicated to Professor K.-H. Indlekofer on his 70th anniversary

Communicated by Imre Kátai

(Received December 30, 2012; accepted January 21, 2013)

Abstract. Let $\mathbb{G}_0 \subseteq \mathbb{G}$ be Abelian groups. We prove that if $\Gamma \in \mathbb{G}$, $f_0, f_1, f_2, f_3, f_4, f_5$ are $\mathbb{G}$-valued completely additive functions and

$$\sum_{j=0}^{5} f_j(n+j) + \Gamma \in \mathbb{G}_0 \quad \text{for all} \quad n \in \mathbb{Z},$$

then $\Gamma \in \mathbb{G}_0$ and $f_j(n) \in \mathbb{G}_0$ for all $n \in \mathbb{Z}$, $j \in \{0, 1, \cdots, 5\}$.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ be the set of primes, positive integers, integers, rational and real numbers, respectively. For each real number z we define $\| z \|$ as follows:

$$\| z \| = \min_{k \in \mathbb{Z}} | z - k | .$$

Key words and phrases: Additive function, Abelian group, $\mathbb{G}$-valued completely additive functions.
2010 Mathematics Subject Classification: 11A05, 11A25.
This work is supported by the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645)
https://doi.org/10.71352/ac.39.355

An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if $(n, m) = 1$ implies
$$f(nm) = f(n) + f(m),$$
and completely additive if this relation holds for all positive integers n and m. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the class of all real-valued additive and completely additive functions, respectively.

First we list the following conjectures due to I. Kátai.

Conjecture 1. *If* $f_0, f_1, \dots, f_k \in \mathcal{A}^*$ *and*
$$\| f_0(n) + f_1(n+1) + \dots + f_k(n+k) \| \to 0, \quad as \quad n \to \infty,$$
then there are $\tau_0, \dots, \tau_k \in \mathbb{R}$ *such that*
$$\tau_0 + \dots + \tau_k = 0$$
and
$$\| f_0(n) - \tau_0 \log n \| = \dots = \| f_k(n) - \tau_k \log n \| = 0$$
for all $n \in \mathbb{N}$.

Conjecture 2. *If* $f_0, f_1, \dots, f_k \in \mathcal{A}^*$ *and*
$$f_0(n) + f_1(n+1) + \dots + f_k(n+k) \in \mathbb{Z} \quad for\ all \quad n \in \mathbb{N},$$
then
$$f_j(n) \in \mathbb{Z} \quad for\ all \quad n \in \mathbb{N} \quad and \quad j = 0, 1, \cdots, k.$$

Conjecture 2 is known for $k = 2,\ 3$ (see [5] and [6]). R. Styer [12] determined all those $f_0, f_1, f_2 \in \mathcal{A}$ for which
$$f_0(n) + f_1(n+1) + f_2(n+2) \in \mathbb{Z} \quad (n \in \mathbb{N}).$$
In [7] it was proved that for arbitrary $a, b \in \mathbb{N}$, all solutions $f_1,\ f_2,\ f_3 \in \mathcal{A}^*$ of
$$f_1(n-a) + f_2(n) + f_3(n+b) \in \mathbb{Z} \quad (n \in \mathbb{N},\ n \geq a+1)$$
form a finite dimensional space. If $f_j(q) \equiv 0 \pmod 1$ $(i = 1, 2, 3)$ holds for all primes $q \leq \max(3,\ a+b)$, then $f_j(n) \equiv 0 \pmod 1$ $(j = 1,\ 2,\ 3,\ n \in \mathbb{N})$.

I. Kátai stated a weaker conjecture in [4]:

Conjecture 3. *If* $P(x) = 1 + A_1x + A_2x^2 + \dots + A_kx^k \in \mathbb{R}[x] \setminus \mathbb{Q}[x]$ *and* $f \in \mathcal{A}^*$ *satisfy*
$$f(n) + A_1f(n+1) + A_2f(n+2) + \dots + A_kf(n+k) \in \mathbb{Z},$$
then $f(n) = 0$ *for all* $n \in \mathbb{N}$.

This is true for $k = 2$ and for $k = 3$ (see [4, 5]). It is clear that Conjecture 2 implies Conjecture 3. In [9] A. Kovács and B. M. Phong proved Conjecture 3 for $k = 4$.

In [10] we stated the following

Conjecture 4. *If the functions $f_j \in \mathcal{A}^*$ $(j = 0, 1, \ldots, k)$ and the real number Γ satisfy the condition*

$$f_0(n) + f_1(n+1) + \ldots + f_k(n+k) + \Gamma \in \mathbb{Z} \quad \text{for all} \quad n \in \mathbb{N},$$

then $\Gamma \in \mathbb{Z}$ and

$$f_j(n) \in \mathbb{Z} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad j = 0, 1, \cdots, k.$$

It is obvious that Conjecture 2 follows from Conjecture 4.

Next, let $\mathbb{G}$ be an Abelian group with identity element 0 and let $\mathcal{A}^*_{\mathbb{G}}$ denote the set of those functions $f : \mathbb{N} \to \mathbb{G}$, for which $f(nm) = f(n) + f(m)$ holds for all $n,\ m \in \mathbb{N}$. The domain of $f \in \mathcal{A}^*_{\mathbb{G}}$ can be extended to $\mathbb{Q}_+$ (the multiplicative group of positive rational numbers) by $f(\frac{n}{m}) = f(n) - f(m)$. If we define $f(0) := 0$ and $f(-\alpha) := f(\alpha)$ for $\alpha \in \mathbb{Q}_+$, then the domain of $f \in \mathcal{A}^*_{\mathbb{G}}$ can be extended to $\mathbb{Q}$ and the equation $f(\alpha\beta) = f(\alpha) + f(\beta)$ remains valid for arbitrary nonzero rational numbers α, β.

It is obvious that if $\mathbb{G} = \mathbb{R}$, then $\mathcal{A}^*_{\mathbb{R}} = \mathcal{A}^*$.

Recently, we proved in [2] that Conjecture 2 is true for the case $k = 4$ by assuming that the relation

$$f_0(n) + f_1(n+1) + f_2(n+1) + f_3(n+1) + f_4(n+1) \in \mathbb{Z}$$

holds for all $n \in \mathbb{Z}$.

We shall prove Conjecture 2 and Conjecture 4 for $k = 5$.

Theorem. *If $\mathbb{G}_0 \subseteq \mathbb{G}$ are Abelian groups, $\Gamma \in \mathbb{G}$, $\{f_0, f_1, f_2, f_3, f_4, f_5\} \subseteq$ $\subseteq \mathcal{A}^*_{\mathbb{G}}$ and*

$$f_0(n) + f_1(n+1) + f_2(n+2) + f_3(n+3) + f_4(n+4) + f_5(n+5) + \Gamma \in \mathbb{G}_0$$

is true for all $n \in \mathbb{Z}$, then

$$\Gamma \in \mathbb{G}_0 \quad \text{and} \quad f_j(n) \in \mathbb{G}_0 \quad \text{for all} \quad n \in \mathbb{Z},\ j \in \{0, 1, \cdots, 5\}.$$

Corollary 1. *If* $\Gamma \in \mathbb{R}$, $\{f_0, f_1, f_2, f_3, f_4, f_5\} \subseteq \mathcal{A}^*$ *and*

$$f_0(n)+f_1(n+1)+f_2(n+2)+f_3(n+3)+f_4(n+4)+f_5(n+5)+\Gamma \equiv 0 \pmod 1$$

is true for all $n \in \mathbb{Z}$, *then* $\Gamma \in \mathbb{Z}$ *and*

$$f_j(n) \in \mathbb{Z} \quad \text{for all} \quad n \in \mathbb{Z},\ j \in \{0, 1, \cdots, 5\}.$$

2. Lemmata

We shall prove our theorem by using the following lemmas:

Lemma 1. *Let* $\mathbb{G}$ *be an Abelian group,* $\mathbb{G}_0$ *be an arbitrary subgroup of* $\mathbb{G}$ *and let* $\varphi_0,\ \varphi_1,\ \varphi_2 \in \mathcal{A}^*_{\mathbb{G}}$. *Assume that*

$$\text{(2.1)} \quad \varphi_0(n)+\varphi_1(n+1)+\varphi_2(n+2)-\varphi_2(n+4)-\varphi_1(n+5)-\varphi_0(n+6) \in \mathbb{G}_0$$

holds for all $n \in \mathbb{N}$. *If*

$$\text{(2.2)} \qquad \varphi_0(n) \in \mathbb{G}_0, \quad \varphi_1(n) \in \mathbb{G}_0, \quad \varphi_2(n) \in \mathbb{G}_0 \quad \text{for} \quad n \le 12,$$

then

$$\text{(2.3)} \qquad \varphi_0(n) \in \mathbb{G}_0, \quad \varphi_1(n) \in \mathbb{G}_0, \quad \varphi_2(n) \in \mathbb{G}_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Remark. We note that this result was proved by I. Kátai and M. van Rossum-Wijsmuller [8] for the case $\mathbb{G}_0 = \mathbb{Z}$. Now we prove this lemma for any Abelian group $\mathbb{G}_0$.

Proof. Assume that $\varphi_0,\ \varphi_1,\ \varphi_2 \in \mathcal{A}^*_{\mathbb{G}}$ satisfy the conditions (2.1) and (2.2), and that (2.3) is not true. Set

$$\mathcal{H}(n) := \varphi_0(n)+\varphi_1(n+1)+\varphi_2(n+2)-\varphi_2(n+4)-\varphi_1(n+5)-\varphi_0(n+6).$$

Then there is a minimal positive integer n_0, $n_0 > 12$ for which $\varphi_i(n_0) \notin \mathbb{G}_0$. Then n_0 should be a prime $P \ge 13$ and

$$\text{(2.4)} \qquad \varphi_2(P) \notin \mathbb{G}_0,\ \varphi_0(P) \in \mathbb{G}_0,\ \varphi_1(P) \in \mathbb{G}_0.$$

Let $\xi := \varphi_2(P) \in \mathbb{G}$ and $\xi \notin \mathbb{G}_0$. From $\mathcal{H}(P-4) \in \mathbb{G}_0$, we have that $\varphi_0(P+2) + \xi \in \mathbb{G}_0$, $P+2 \in \mathcal{P}$. Thus

$$\text{(2.5)} \qquad P \equiv 2 \pmod 3.$$

By using (2.4), (2.5) and $P \in \mathcal{P},\ P \geq 13$, we obtain

$$2|P+\ell,\ \frac{P+\ell}{2} < P\ (\ell = 1,\ 3,\ 5) \quad \text{and} \quad 3|P+k,\ \frac{P+k}{3} < P \ \ (k=-2,\ 4).$$

Consequently it follows from $\mathcal{H}(P-2) \in \mathbb{G}_0$ and $\mathcal{H}(P) \in \mathbb{G}_0$ that

$$\varphi_2(P+2) - \xi \in \mathbb{G}_0,\ \ \varphi_0(P+6) - \xi \in \mathbb{G}_0,\ P+6 \in \mathcal{P}. \tag{2.6}$$

Since

$$\mathcal{H}(P+2) = \varphi_0(P+2) + \varphi_1(P+3) + \varphi_2(P+4) - \\ - \varphi_2(P+6) - \varphi_1(P+7) - \varphi_0(P+8) \in \mathbb{G}_0$$

and $2|P+3,\ 3|P+4,\ 2|P+7$, we have $\varphi_2(P+6) + \varphi_0(P+8) + \xi \in \mathbb{G}_0$. If $\varphi_0(P+8) \in \mathbb{G}_0$, then $\varphi_2(P+6) + \xi \in \mathbb{G}_0$. Consequently from $\mathcal{H}(P+4) \in \mathbb{G}_0$ it follows that $\varphi_2(P+8) + \xi \in \mathbb{G}_0,\ P+8 \in \mathcal{P}$. If $\varphi_0(P+8) \notin \mathbb{G}_0$, then we also have $P+8 \in \mathcal{P}$. Thus we have proved that $P, P+2, P+6, P+8 \in \mathcal{P}$, which implies

$$P \equiv 1 \pmod 5. \tag{2.7}$$

Next, we prove the following assertion:

$$\varphi_1(4P+7) \in \mathbb{G}_0. \tag{2.8}$$

From (2.5) we have $4P+7 \equiv 0 \pmod 3$. Therefore, let $Q := \frac{4P+7}{3}$. If $Q \notin \mathcal{P}$, then $Q = Q_1Q_2,\ 2 \leq Q_1, Q_2 \leq \frac{4P+7}{6} < P$. Consequently $\varphi_1(Q) = \varphi_1(Q_1Q_2) = = \varphi_1(Q_1) + \varphi_1(Q_2) \in \mathbb{G}_0$. Assume now that $Q \in \mathcal{P}$. Since

$$\mathcal{H}(Q-5) = \varphi_0(Q-5) + \varphi_1(Q-4) + \varphi_2(Q-3) - \varphi_2(Q-1) - \varphi_1(Q) - \varphi_0(Q+1).$$

and

$$\mathcal{H}(Q-1) = \varphi_0(Q-1) + \varphi_1(Q) + \varphi_2(Q+1) - \varphi_2(Q+3) - \varphi_1(Q+4) - \varphi_0(Q+5)$$

and $2|Q+\ell,\ \frac{Q+\ell}{2} < P$ if $\ell = -5, -3, 1, 3, 5$, therefore

$$\varphi_1(Q-4) - \varphi_1(Q) \in \mathbb{G}_0 \quad \text{and} \quad \varphi_1(Q) - \varphi_1(Q+4) \in \mathbb{G}_0. \tag{2.9}$$

It is clear that $(Q-4)Q(Q+4) \equiv 0 \pmod 3$. This together with (2.9) show that $\varphi_1(Q) \in \mathbb{G}_0$. Thus (2.8) is proved.

From $\mathcal{H}(4P+6) \in \mathbb{G}_0$ and (2.8), we have

$$\varphi_0(4P+6) + \varphi_2(4P+8) - \varphi_2(4P+10) - \varphi_1(4P+11) - \varphi_0(4P+12) \in \mathbb{G}_0. \tag{2.10}$$

It is obvious from (2.5) and (2.7) that

$$10|4P+6, \quad 6|4P+10, \quad 5|4P+11, \quad 8|4P+12.$$

Therefore (2.10) shows that $\varphi_2(4P+8) \in \mathbb{G}_0$, and so $\varphi_2(P+2) \in \mathbb{G}_0$. This contradicts (2.6). The proof of Lemma 1 is complete. ∎

In the following let

$$L_n := \left(\frac{n}{n+6}, \frac{n+1}{n+5}, \frac{n+2}{n+4}\right) \quad (n \in \mathbb{N}).$$

The elements L_n belong to the multiplicative group $\mathbb{Q}_+^3$. Let $\mathcal{L}$ be the group generated by the elements L_n $(n \in \mathbb{N})$. It is obvious that

$$\varphi_0(\alpha) + \varphi_1(\beta) + \varphi_2(\gamma) \in \mathbb{G}_0 \quad \text{holds for all} \quad (\alpha, \beta, \gamma) \in \mathcal{L}.$$

We prove the following

Lemma 2. *If $\mathcal{L}$ is the subgroup of $\mathbb{Q}_+^3$ generated by the sequence L_n, then $\mathcal{L} = \mathbb{Q}_+^3$. Therefore* (2.3) *holds, i.e.*

$$\varphi_0(n) \in \mathbb{G}_0, \quad \varphi_1(n) \in \mathbb{G}_0, \quad \varphi_2(n) \in \mathbb{G}_0 \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. For each prime p we shall use the following notations:

$$a_p := (p, 1, 1), \; b_p := (1, p, 1) \quad \text{and} \quad c_p := (1, 1, p).$$

We shall prove that

$$\text{(2.11)} \qquad a_p \in \mathcal{L}, \; b_p \in \mathcal{L} \; \text{ and } \; c_p \in \mathcal{L} \; \text{ for all } \; p \in \{2, 3, 5, 7, 11\}.$$

Lemma 2 is a direct consequence of (2.11) and Lemma 1. ∎

Using a simple Maple program for

$$n \in \{1, 2, 3, 4, 5, 6, 8, 12, 50, 9, 7, 11, 14, 18, 21, 19, 13, 15, 17, 24, 26\},$$

we can give $a_p, \; b_q, \; c_r$ for primes $p \le 23, q \le 31$ and $r \le 23$ in terms of L_n and $a_2, \; a_3, \; a_5, \; b_2, \; b_3, \; c_2, \; c_3, \; c_5$.

n	L_n	$a_p,\ b_q,\ c_r$
1	$\left(\frac{1}{7}, \frac{1}{3}, \frac{3}{5}\right)$	$a_7 = \frac{c_3}{L_1 b_3 c_5}$
2	$\left(\frac{1}{2^2}, \frac{3}{7}, \frac{2}{3}\right)$	$b_7 = \frac{b_3 c_2}{L_2 a_2^2 c_3},$
3	$\left(\frac{1}{3}, \frac{1}{2}, \frac{5}{7}\right)$	$c_7 = \frac{c_5}{L_3 a_3 b_2}$
4	$\left(\frac{2}{5}, \frac{5}{3^2}, \frac{3}{2^2}\right)$	$b_5 = \frac{L_4 a_5 b_3^2 c_2^2}{a_2 c_3}$
5	$\left(\frac{5}{11}, \frac{3}{5}, \frac{7}{3^2}\right)$	$a_{11} = \frac{a_5 b_3 c_7}{L_5 b_5 c_3^2} = \frac{c_5 a_2}{L_3 L_4 L_5 a_3 b_2 b_3 c_2^2 c_3}$
6	$\left(\frac{1}{2}, \frac{7}{11}, \frac{2^2}{5}\right)$	$b_{11} = \frac{b_7 c_2^2}{L_6 a_2 c_5} = \frac{b_3 c_2^3}{L_2 L_6 a_2^3 c_3 c_5}$
8	$\left(\frac{2^2}{7}, \frac{3^2}{13}, \frac{5}{2.3}\right)$	$b_{13} = \frac{a_2^2 b_3^2 c_5}{L_8 a_7 c_2 c_3} = \frac{L_1 a_2^2 b_3^3 c_5^2}{L_8 c_2 c_3^2}$
12	$\left(\frac{2}{3}, \frac{13}{17}, \frac{7}{2^3}\right)$	$b_{17} = \frac{a_2 b_{13} c_7}{L_{12} a_3 c_2^3} = \frac{L_1 a_2^3 b_3^3 c_5^3}{L_3 L_8 L_{12} c_3^2 c_2^4 a_3^2 b_2}$
50	$\left(\frac{5^2}{2^2.7}, \frac{3.17}{5.11}, \frac{2.13}{3^3}\right)$	$c_{13} = \frac{L_{50} a_2^2 a_7 b_5 b_{11} c_3^3}{a_5^2 b_3 b_{17} c_2} = \frac{L_3 L_4 L_8 L_{12} L_{50} a_3^2 b_2 c_2^8 c_3^4}{L_1^2 L_2 L_6 a_2^5 a_5 b3^2 c_5^5}$
9	$\left(\frac{3}{5}, \frac{5}{7}, \frac{11}{13}\right)$	$c_{11} = \frac{L_9 a_5 b_7 c_{13}}{a_3 b_5} = \frac{L_3 L_8 L_9 L_{12} L_{50} a_3 b_2 c_2^7 c_3^4}{L_1^2 L_2^2 L_6 a_5 a_2^6 b_3^3 c_5^5}$
7	$\left(\frac{7}{13}, \frac{2}{3}, \frac{3^2}{11}\right)$	$a_{13} = \frac{a_7 b_2 c_3^2}{L_7 b_3 c_{11}} = \frac{L_1 L_2^2 L_6 a_2^6 a_5 b_3 c_5^4}{L_3 L_7 L_8 L_9 L_{12} L_{50} a_3 c_2^7 c_3}$
11	$\left(\frac{11}{17}, \frac{3}{2^2}, \frac{13}{3.5}\right)$	$a_{17} = \frac{a_{11} b_3 c_{13}}{L_{11} b_2^2 c_3 c_5} = \frac{L_8 L_{12} L_{50} a_3 c_2^6 c_3^2}{L_1^2 L_2 L_5 L_6 L_{11} a_2^4 a_5 b_2^2 b_3^2 c_5^5}$
14	$\left(\frac{7}{2.5}, \frac{3.5}{19}, \frac{2^3}{3^2}\right)$	$b_{19} = \frac{a_7 b_3 b_5 c_2^3}{L_{14} a_2 a_5 c_3^2} = \frac{L_4 b_3^2 c_2^5}{L_1 L_{14} a_2^2 c_3^2 c_5}$
18	$\left(\frac{3}{2^2}, \frac{19}{23}, \frac{2.5}{11}\right)$	$b_{23} = \frac{a_3 b_{19} c_2 c_5}{L_{18} a_2^2 c_{11}} = \frac{L_1 L_4 L_2^2 L_6 a_2^2 a_5 b_3^5 c_5^5}{L_3 L_8 L_9 L_{12} L_{14} L_{18} L_{50} b_2 c_2 c_3^6}$
21	$\left(\frac{7}{3^2}, \frac{11}{13}, \frac{23}{5^2}\right)$	$c_{23} = \frac{L_{21} a_3^2 b_{13} c_5^2}{a_7 b_{11}} = \frac{L_1^2 L_2 L_6 L_{21} a_2^5 a_3^2 b_3^3 c_5^6}{L_8 c_3^2 c_2^4}$
19	$\left(\frac{19}{5^2}, \frac{5}{2.3}, \frac{3.7}{23}\right)$	$a_{19} = \frac{L_{19} a_5^2 b_2 b_3 c_{23}}{b_5 c_3 c_7} = \frac{L_1^2 L_2 L_3 L_6 L_{19} L_{21} a_2^6 a_3^3 a_5 b_2^2 b_3^2 c_5^5}{L_4 L_8 c_3^2 c_2^6}$
13	$\left(\frac{13}{19}, \frac{7}{3^2}, \frac{3.5}{17}\right)$	$c_{17} = \frac{a_{13} b_7 c_3 c_5}{L_{13} a_{19} b_3^2} = \frac{L_4 c_3}{L_1 L_3^2 L_7 L_9 L_{12} L_{13} L_{19} L_{21} L_{50} a_2^2 a_3^4 b_2^2 b_3^2}$
15	$\left(\frac{5}{7}, \frac{2^2}{5}, \frac{17}{19}\right)$	$c_{19} = \frac{a_5 b_2^2 c_{17}}{L_{15} a_7 b_5} = \frac{c_3 c_5}{L_3^2 L_7 L_9 L_{12} L_{13} L_{15} L_{19} L_{21} L_{50} a_2 a_3^4 b_3^3 c_2^2}$
17	$\left(\frac{17}{23}, \frac{3^2}{11}, \frac{19}{3.7}\right)$	$a_{23} = \frac{a_{17} b_3^2 c_{19}}{L_{17} b_{11} c_3 c_7} =$ $= \frac{L_8 c_2 c_3^3}{L_1^2 L_3 L_5 L_7 L_9 L_{11} L_{13} L_{15} L_{17} L_{19} L_{21} a_2^2 a_3^2 a_5 b_2 b_3^4 c_5^4}$

24	$(\frac{2^2}{5}, \frac{5^2}{29}, \frac{13}{2.7})$	$b_{29} = \frac{a_2^2 b_5^2 c_{13}}{L_{24} a_5 c_2 c_7} = \frac{L_3^2 L_4^3 L_8 L_{12} L_{50} a_3^3 b_2^2 b_3^2 c_2^{11} c_3^2}{L_1^2 L_2 L_6 L_{24} a_2^5 c_5^6}$
26	$(\frac{13}{2^4}, \frac{3^3}{31}, \frac{2.7}{3.5})$	$b_{31} = \frac{a_{13} b_3^3 c_2 c_7}{L_{26} a_2^4 c_3 c_5} = \frac{L_1 L_2^2 L_6 a_2^2 a_5 b_3^4 c_5^4}{L_3^2 L_7 L_8 L_9 L_{12} L_{26} L_{50} a_3^2 b_2 c_2^6 c_3^2}$

Table 1

Now, by using the above relations for $n = 10,\ 16,\ 20,\ 22,\ 28\ 30,\ 34,\ 44$ and $n = 64$, we get that the following 9 expressions are elements of $\mathcal{L}$ and they are of the form

$$a_2^{\alpha_2} a_3^{\alpha_3} a_5^{\alpha_5} b_2^{\beta_2} b_3^{\beta_3} c_2^{\gamma_2} c_3^{\gamma_3} c_5^{\gamma_5},$$

where $\alpha_2, \alpha_3, \alpha_5, \beta_2, \beta_3, \gamma_2, \gamma_3, \gamma_5$ are suitable integers. We have

$$F_1 := \frac{L_2 L_4 L_6 L_{10}}{L_3} = \frac{a_3 b_2 c_2^2 c_3}{a_2^5 b_3^2 c_5^2} \in \mathcal{L},$$

$$F_2 := \frac{L_8 L_{12} L_{16}}{L_1 L_2 L_4 L_5} = \frac{a_2^7 b_3^2 c_3^2 c_5}{a_3 c_2^4} \in \mathcal{L},$$

$$F_3 := \frac{L_1^3 L_2^5 L_4^2 L_6^2 L_{20}}{L_3^2 L_7 L_8^2 L_9^2 L_{12}^2 L_{50}^2} = \frac{a_3^2 b_2 c_2^9 c_3^5}{a_2^{11} a_5^3 b_3^6 c_5^9} \in \mathcal{L},$$

$$F_4 := \frac{L_3^3 L_4 L_5 L_8^2 L_9 L_{12}^2 L_{14} L_{18} L_{22} L_{50}^2}{L_1^4 L_2^3 L_6^2} = \frac{a_2^7 a_5^2 b_3^4 c_5^{12}}{a_3^3 b_2^3 c_2^9 c_3^{11}} \in \mathcal{L},$$

$$F_5 := \frac{L_1 L_{24} L_{28}}{L_2 L_3^2 L_4^3 L_5 L_6 L_{11}} = \frac{a_2^3 a_3^2 a_5 b_2^4 b_3 c_3^3}{c_2^2} \in \mathcal{L},$$

$$F_6 := \frac{L_4^2 L_8 L_{26} L_{30}}{L_1^2 L_2^3 L_6 L_{13} L_{19} L_{21}} = \frac{a_2^6 a_3 a_5 b_2 b_3^3 c_5^4}{c_2^5 c_3} \in \mathcal{L},$$

$$F_7 := \frac{L_1^3 L_2^2 L_5 L_6 L_{11} L_{34}}{L_3^2 L_4 L_7 L_8^2 L_9 L_{12}^2 L_{13} L_{15} L_{19} L_{21} L_{50}^2} = \frac{a_3^5 c_2^{13} c_3^3}{a_2^{10} a_5 b_2^2 c_5^8} \in \mathcal{L},$$

$$F_8 := \frac{L_3 L_5 L_8 L_{44}}{L_1^2 L_2^3 L_6 L_{21}} = \frac{a_2^{10} a_3 b_3^4 c_5^7}{a_5 b_2 c_2^9 c_3^3} \in \mathcal{L},$$

$$F_9 := \frac{L_2^4 L_4 L_6^2 L_{64}}{L_3^4 L_7 L_8 L_9^3 L_{12}^3 L_{13} L_{14} L_{18} L_{19} L_{21} L_{50}^3} = \frac{a_3^5 b_2^4 c_2^8 c_3^6}{a_5^2 b_3 c_5^7} \in \mathcal{L}.$$

This system has solutions in $a_2,\ a_3,\ a_5,\ b_2,\ b_3,\ c_2,\ c_3,\ c_5$, which are given in terms of $F_1, \cdots, F_9$. Thus, $a_2,\ a_3,\ a_5,\ b_2,\ b_3,\ c_2,\ c_3,\ c_5$ are elements of $\mathcal{L}$.

By means of Maple program, we have

$$a_2 = \frac{F_6^{15} F_8^{10} F_9^3}{F_1^{18} F_2^{60} F_3^{35} F_4^{44} F_5^{32} F_7^{16}}, \qquad a_3 = \frac{F_6^{23} F_8^{17} F_9^5}{F_1^{28} F_2^{98} F_3^{58} F_4^{72} F_5^{52} F_7^{26}},$$

$$a_5 = \frac{F_6^{42}F_8^{19}F_9^5}{F_1^{45}F_2^{132}F_3^{72}F_4^{96}F_5^{71}F_7^{35}}, \qquad b_2 = \frac{F_1^{108}F_2^{353}F_3^{205}F_4^{258}F_5^{184}F_7^{93}}{F_6^{84}F_8^{60}F_9^{16}},$$
$$b_3 = \frac{F_1^{170}F_2^{561}F_3^{327}F_4^{410}F_5^{292}F_7^{148}}{F_6^{131}F_8^{96}F_9^{26}}, \qquad c_2 = \frac{F_6^{121}F_8^{91}F_9^{24}}{F_1^{162}F_2^{529}F_3^{308}F_4^{386}F_5^{273}F_7^{139}},$$
$$c_3 = \frac{F_6^{192}F_8^{145}F_9^{38}}{F_1^{257}F_2^{841}F_3^{490}F_4^{614}F_5^{434}F_7^{221}}, \qquad c_5 = \frac{F_6^{280}F_8^{213}F_9^{56}}{F_1^{376}F_2^{1233}F_3^{719}F_4^{900}F_5^{636}F_7^{324}}.$$

Finally, we infer from Table 1 that $a_7,\ a_{11},\ b_5,\ b_7,\ b_{11},\ c_7,\ c_{11}$ are elements of $\mathcal{L}$. This shows that (2.11) is true, consequently Lemma 2 is proved. ■

3. Proof of the Theorem

Assume that the conditions of the theorem are satisfied, i.e. $\mathbb{G}_0 \subseteq \mathbb{G}$ are Abelian groups, $\Gamma \in \mathbb{G}$,

$$f_0(n) + f_1(n+1) + f_2(n+2) + f_3(n+3) + f_4(n+4) + f_5(n+5) + \Gamma \in \mathbb{G}_0$$

is true for all $n \in \mathbb{Z}$. Since $f_j(-m) = f_j(m)$ $(m \in \mathbb{N})$, we infer from the last relation that

$$f_5(n+1) + f_4(n+2) + f_3(n+3) + f_2(n+4) + f_1(n+5) + f_0(n+6) + \Gamma \in \mathbb{G}_0$$

and

$$\varphi_0(n) + \varphi_1(n+1) + \varphi_2(n+2) - \varphi_2(n+4) - \varphi_1(n+5) - \varphi_0(n+6) \in \mathbb{G}_0,$$

where

$$\varphi_0(n) = f_0(n), \quad \varphi_1(n) = f_1(n) - f_5(n), \quad \varphi_2(n) = f_2(n) - f_4(n).$$

Hence, Lemma 2 implies that

$$\varphi_0(n) = f_0(n) \in \mathbb{G}_0, \ \ \varphi_1(n) = f_1(n) - f_5(n) \in \mathbb{G}_0, \ \ \varphi_2(n) = f_2(n) - f_4(n) \in \mathbb{G}_0.$$

Therefore

$$f_1(n+1) + f_2(n+2) + f_3(n+3) + f_2(n+4) + f_1(n+5) + \Gamma \in \mathbb{G}_0 \tag{3.1}$$

is satisfied for all $n \in \mathbb{Z}$. It is easy to deduce from (3.1) that

$$f_1(n) + f_2(n+1) + [f_3(n+2) - f_1(n+2)] - \\ -[f_3(n+4) - f_1(n+4)] - f_2(n+5) - f_1(n+6) \in \mathbb{G}_0.$$

This together with Lemma 2 also imply

$$f_1(n) \in \mathbb{G}_0, \ \ f_2(n) \in \mathbb{G}_0 \ \text{ and } \ f_3(n) - f_1(n) \in \mathbb{G}_0.$$

Thus $f_j(n) \in \mathbb{G}_0$ for all $n \in \mathbb{N}$, $j \in \{0, 1, \cdots, 5\}$, consequently $\Gamma \in \mathbb{G}_0$.

Our Theorem is proved. ■

References

[1] **Elliott, P.D.T.A.,** *Arithmetic Functions and Integer Products*, Springer-Verlag, New York, (1985).

[2] **Chakraborty, K., I. Kátai and B.M. Phong,** On additive functions (mod 1), *Annales Univ. Sci. Budapest., Sect. Comp.*, **36** (2012), 355–373.

[3] **Erdős, P.,** On the distribution function of additive functions, *Ann. Math.*, **47** (1946), 1–20.

[4] **Kátai, I.,** On arithmetical functions with regularity properties, *Acta Sci. Math.*, **45** (1983), 253–260.

[5] **Kátai, I.,** On additive functions satisfying a congruence, *Acta Sci. Math.*, **47** (1984), 85–92.

[6] **Kátai, I.,** A remark on additive functions satisfying a relation, *Annales Univ. Sci. Budapest., Sect. Math.*, **27** (1985), 229–233.

[7] **Kátai, I.,** Arithmetical functions satisfying some relations, *Acta Sci. Math.*, **55** (1991), 249–268.

[8] **Kátai, I. and M. van Rossum-Wijsmuller,** Additive functions satisfying congruences, *Acta Sci. Math.*, **56** (1992), 63–72.

[9] **Kovács, A. and B.M. Phong,** On completely additive functions satisfying a congruence, *Annales Univ. Sci. Budapest., Sect. Comp.*, **19** (2000), 237–249.

[10] **Phong, B.M.,** On additive functions at consecutive integers,(manuscript).

[11] **van Rossum-Wijsmuller, M.,** Additive functions on the Gaussian integers, *Publ. Math. Debrecen*, **38** (1991), 255–262.

[12] **Styer, R.,** A problem of Kátai on sums of additive functions, *Acta Sci. Math.*, **55** (1991), 269–286.

[13] **Wirsing, E., Tang Yuansheng and Shao Pintsung,** On a conjecture of Kátai for additive functions, *J. Number Theory*, **56** (1996), 391–395.

Bui Minh Phong
Eötvös Loránd University
Department of Computer Algebra
H-1117 Budapest
Pázmány Péter sétány 1/C
Hungary
bui@compalg.inf.elte.hu