UNIVERSALITY OF THE RIEMANN ZETA FUNCTION: TWO REMARKS

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Dedicated to Professor Karl-Heinz Indlekofer on his seventieth anniversary

Communicated by Imre Kátai
(Received December 26, 2012; accepted January 14, 2013)

Abstract. In this article, we precise two results of Bagchi, the first one on the universality of the Riemann zeta function the second on its relation with the Riemann hypothesis.

1. Introduction

Originating with the discovery by Voronin of a property of universality for the Riemann Zeta function [8], the field of research on the universality of Dirichlet series and their distribution in various function spaces has been considerably developed, with the production of a lot of articles and at least two books [4] , [7]. Since most of the authors seem to consider that the ultimate formulation, the "standard version", of this property of universality is provided by the contribution of Bagchi, we shall refer essentially to his work [1].

The purpose of this paper will be to reconsider two of the results obtained in the case of the Riemann Zeta function by Bagchi [1], taking account of the fact that as proved in [5] and [6], some functions appearing naturally in the context of the universality of Dirichlet series are almost-periodical, a property which has interesting consequences, and so, the present contribution can be situated in the continuity of the approach initiated by Bohr to study some statistical properties of the Riemann Zeta function and other Dirichlet series.

Key words and phrases: Riemann Zeta function, Riemann hypothesis, universality.
2010 Mathematics Subject Classification: 11 M 06.
https://doi.org/10.71352/ac.39.311

2. Notations and definitions

2.1. The space $\mathcal{V}$

The context is as follows: we have some series $\Sigma_{n\geq 0} A_n e^{-\lambda_n s}$, $\lambda_{n+1} > \lambda_n \geq 0$, which is absolutely convergent for $\operatorname{Re} s = \sigma > \alpha$, defines a function $f(s)$ meromorphic in a wider half-plane $\operatorname{Re} s = \sigma > \alpha_1$, $\alpha_1 \leq \alpha$, with eventually a finite set Z of poles and it satisfies the following conditions:

(i) $\sup(1/T) \int_{-T}^{+T} |f(\sigma+it)|^2 \, dt = K_\sigma$ is finite for all $\sigma > \alpha_1$, $\sigma \notin \operatorname{Re} Z$,

(ii) there exist a positive integer m and positive constants D and U_0 such that

$$|f(\sigma+it)| < D \left| \frac{t}{2} \right|^{m-\frac{1}{2}}$$

for

$$\alpha_1 < \sigma, \quad |t| > U_0.$$

The set of the functions satisfying such requirements will be denoted by $\mathcal{V}$. We remark that Z can be empty.

2.2. Universality in $\mathcal{V}$

Given $f(s)$ in $\mathcal{V}$, we shall say that $f(s)$ has the STRONG universality property (in the Bagchi sense [1]) in the strip

$$S = S_{(\beta_1,\beta_2)} = \{ s \in \mathbb{C} \mid \beta_1 < \operatorname{Re} s < \beta_2, \alpha_1 \leq \beta_1 < \beta_2 \ \}$$

if it is holomorphic in this open strip $S_{(\beta_1,\beta_2)}$ and for any $\varepsilon > 0$, any compact set K contained in $S_{(\beta_1,\beta_2)}$ with connected complement, and any given complex-valued continuous function $h(s)$ supported on K, analytic in the interior of K, the relation

$$\liminf_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T \ ; \ \sup_{s\in K} |f(s+it) - h(s)| \leq \varepsilon \right\} > 0$$

holds, $meas$ denoting the Lebesgue measure on $\mathbb{R}$.

If the function $h(s)$ is assumed to be non-zero in the strip $S_{(\beta_1,\beta_2)}$, we shall say that $f(s)$ has the WEAK universality property in this strip.

3. A first remark

In the case of the Riemann Zeta function $\zeta(s)$, we have the following extension of the original Voronin Theorem by Bagchi [1], which is some kind of "standard formulation":

Theorem 1. *Let K be a compact subset of the strip* $\mathrm{Re}\, s \in \left]\frac{1}{2}, 1\right[$ *with connected complement, and $f(s)$ be any non-vanishing function analytic in $\overset{o}{K}$, the interior of K, and continuous on K. Then for every $\varepsilon > 0$,*

$$\liminf_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \le t \le T \; ; \; \sup_{s\in K} |\zeta(s+it) - f(s)| \le \varepsilon \right\} > 0.$$

This result clearly implies that the Riemann Zeta function is weakly universal and the remark is that such a result provides rather few, if no, information on the set

$$\left\{ t \in \mathbb{R}, 0 \le t \; ; \; \sup_{s\in K} |\zeta(s+it) - f(s)| \le \varepsilon \right\}$$

and it can be clarified and ameliorated quite a lot using the simple fact that the function

$$t \mapsto \sup_{s\in K} |\zeta(s+it) - f(s)|$$

is almost periodic (in the Besicovitch sense) for the exponent 2 ($B^2.A.P.$).

In fact, the following statement holds, which implies a more precise formulation of the weak universality of the Riemann zeta function :

Theorem 2. *Let K be a compact subset of the strip* $\mathrm{Re}\, s \in \left]\frac{1}{2}, 1\right[$ *with connected complement, and $f(s)$ be any non-vanishing function analytic in $\overset{o}{K}$, the interior of K, and continuous on K. Then, apart for an atmost countable set S of ε,*

$$\lim_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \le t \le T \; ; \; \sup_{s\in K} |\zeta(s+it) - f(s)| \le \varepsilon \right\}$$

exists and is positive.

Proof. We have the following proposition:

Proposition 3. *Let K be a compact subset of the strip* $\operatorname{Re} s \in]1/2, 1[$, *and $h(s)$ be a function continuous on K. Then, apart for an almost countable set S of ε, the function*

$$I_{\zeta,h,K,\varepsilon,}(t) = 1 \quad \textit{if} \quad \sup_{s\in K} |h(s) - \zeta(s+it)| \ \le \varepsilon, \qquad = 0 \ \textit{if not,}$$

is B-almost-periodic.

This is a consequence of Corollary 5.2 p.185–186 of [6] which gives immediately that

$$t \mapsto \sup_{s\in K} |\zeta(s+it) - h(s)| \, .$$

is $B^2 - almost - periodic$ $(B^2.A.P)$, only the continuity of $h(s)$ on K being required, and so, it has an asymptotic distribution. A consequence is that, apart for an atmost countable set S of ε, the function $I_{\zeta,h,K,\varepsilon,}(t)$ is $B.A.P.$ (see [3], p.78).

Now, for ε not in S, since

$$\int_0^T I_{\zeta,h,K,\varepsilon,}(t)dt = meas \, \{t \in \mathbb{R}, 0 \le t \le T \ ; \ I_{\zeta,h,K,\varepsilon,}(t) = 1\} =$$

$$= meas \left\{ t \in \mathbb{R}, 0 \le t \le T \ ; \ \sup_{s\in K} |\zeta(s+it) - h(s)| \le \varepsilon \right\}$$

we get that

$$\lim_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \le t \le T \ ; \ \sup_{s\in K} |\zeta(s+it) - h(s)| \le \varepsilon \right\} =$$

$$= \quad \lim_{T\to+\infty} \frac{1}{T} \int_0^T I_{\zeta,h,K,\varepsilon,}(t)dt,$$

which exists since $I_{\zeta,h,K,\varepsilon,}(t)$ is $B.A.P.$

So, in the formula appearing in Theorem 1, we have replaced the lim inf by a lim. The positivity of the limit is provided by the "standard version" due to Bagchi of the Voronin Theorem quoted above.

The proof is now complete. ■

As a corollary, we can formulate the application of this result to the Bagchi theorem in the following way:

Corollary 4. *Given any compact subset K of the strip* $\operatorname{Re} s \in \left]\frac{1}{2}, 1\right[$ *with connected complement, any non-vanishing function $f(s)$ analytic in $\overset{o}{K}$, the interior of K, and continuous on K, apart for an atmost countable S of $\varepsilon > 0$,*

$$\lim_{T \to +\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \leq t \leq T \ ; \ \sup_{s \in K} |\zeta(s+it) - f(s)| \leq \varepsilon \right\}$$

exists and is positive.

Remark 1. This result is related to a general property of functions belonging to $\mathcal{V}$ since with our notations, for any f belonging to $\mathcal{V}$, any complex-valued function $h(s)$ continuous on K, the function $t \mapsto f_{K,h,k}(t)$ defined by

$$\begin{aligned} f_{K,h,k}(t) &= \sup_{s \in K} |h(s) - f(s+it)| \ \text{ if } \ |t| \geq 2(k+1), \\ &= \rho, \ \text{ where } \rho \text{ is any complex number elsewhere,} \end{aligned}$$

(where k is defined by $k = \max\left(U_0, (|\operatorname{Im} z|, z \in K \cup Z)\right)$), is $B^2.A.P.$ (see [5], Theorem 6, p. 112).

Consequently, defining

$$I_{f,h,K,\varepsilon,}(t) = 1 \quad \text{if} \quad \sup_{s \in K} |h(s) - f(s+it)| \ \ \leq \varepsilon, \quad = 0 \text{ if not,}$$

we get as above that apart for an almost countable set S, $I_{f,h,K,\varepsilon,}$ has a mean-value.

So, the universality theorem as usually formulated and its various avatars allow essentially to assert the positivity of all these mean-values in the special cases where it applies, but not at all their existences which are consequences of the almost periodicity. This is a general fact and it applies in the case of discrete, joint, etc. cases of universality.

4. Second remark

It can be proved that the Riemann Zeta function is not strongly universal (see [7], chap.8, p.156 et seq.), as for many other similar series, and this is presented as *a consequence of the distribution* of their zeroes in the critical strip. As written in [7], p.156, ”the location of the complex zeroes of Riemann's Zeta function is closely connected with the universality property”. Moreover, a feature which seems to have attracted the attention of some authors is that if in the formulation of the ”standard version” of the universality theorem one can approximate ζ by itself, then, the Riemann hypothesis is true. More precisely, as stated by Bagchi,

if for any given compact subset K of the strip $\operatorname{Re} s \in \left]\frac{1}{2}, 1\right[$ with connected complement and for any $\varepsilon > 0$, the relation

$$\liminf_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \le t \le T \; ; \; \sup_{s\in K} |\zeta(s+it) - \zeta(s)| \le \varepsilon \right\} > 0$$

holds, then the Riemann hypothesis is true.

Now, using the approach developed in [6] , this statement turns to be a simple consequence of a well-known theorem of Hurwitz. In fact, we have the following general result:

Theorem 5. *Let $Z(s) = \Sigma_{n\in\mathbb{N}}\alpha(n)n^{-s}$ defined for $\operatorname{Re} s = \sigma > 1$ by the absolutely convergent product*

$$Z(s) = \prod_{p \ prime} \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{rs}},$$

$Z(s)$ not identical to zero and belonging to the class $\mathcal{V}$ as defined above, with $\alpha = 1$, $\alpha_1 = 1/2$.

Assume that for some complex-valued continuous function $h(s)$ supported on K, a compact set contained in $S_{(1/2,1)}$, $h(s)$ is analytic in $\overset{o}{K}$, the interior of K, and for any $\varepsilon > 0$, the inequality

$$\limsup_{T\to+\infty} \frac{1}{T} meas \left\{ t \in \mathbb{R}, 0 \le t \le T \; ; \; \sup_{s\in K} |Z(s+it) - h(s)| \le \varepsilon \right\} > 0$$

holds, and moreover, that for all p prime,

$$\min_{s\in S_{(1/2,1)}} \left| \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{rs}} \right| > 0.$$

Then, h is identically 0 in $\overset{o}{K}$ or h has no zero in $\overset{o}{K}$.

Remark 2. If we assume that we can have $Z(s) = h(s)$, this will imply immediately that $Z(s)$ has no zero in $\overset{o}{K}$. And if $Z(s)$ has a zero in $\overset{o}{K}$, we cannot have such an hypothesis satisfied, and this result holds *independently of the distribution of the zeroes of $Z(s)$ in the strip $S_{(1/2,1)}$.*

Remark 3. A consequence of this result is that no element of the class $\mathcal{V}$ satisfying the hypothesis of the above theorem can be strongly universal, i.e. the Linnik–Ibragimov conjecture is not exact for this set of Dirichlet series. The only possibility for Dirichlet series of this class is the weak universality.

Proof. Due to Theorem 8.1 of [6], we have the following:

Proposition 6. *Assume that K is a compact set contained in the open strip*

$$S = \{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s < 1\},$$

and $h(s)$ is a complex-valued continuous function defined on K.

If for any positive ε, we have

$$\limsup_{T\to+\infty} \frac{1}{2T} meas \left\{ t \in \mathbb{R}, |t| \leq T \; ; \; \sup_{s\in K} |Z(s+it) - h(s)| \leq \varepsilon \right\} > 0,$$

then, for any *sequence $\{E_l\}_{l\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that*

$$\{n \in \mathbb{N}; 1 \leq n \leq l\} \subseteq E_l,$$

and

$$\sum_{n\in E_l} \frac{|\alpha(n)|}{n^{\sigma_K}} < +\infty$$

where $\frac{1}{2} < \sigma_K = \min\{\operatorname{Re} s, s \in K\}$, the condition

$$\liminf_{l\to+\infty} \inf_{t\in\mathbb{R}} \sup_{s\in K} \left| \sum_{n\in E_l} \frac{\alpha(n)}{n^{(s+it)}} - h(s) \right| = 0$$

holds.

Now, due to Theorem 9.1 of [6], we have the following:

Proposition 7. *If the condition*

$$\liminf_{y\to+\infty} \inf_{t\in\mathbb{R}} \sup_{s\in K} \left| Z_{y_-}(s+it) - h(s) \right| = 0,$$

is fulfilled, where K is a compact subset contained in the open strip

$$S = \{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s < 1\},$$

$h(s)$ is a function continuous on K and $Z_{y_-}(s)$ is defined on S by

$$Z_{y_-}(s) = \prod_{\substack{p\leq y \\ p \text{ prime}}} \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{rs}},$$

then, given any $\varepsilon > 0$, the set

$$\left\{ t \in \mathbb{R} \ ; \ \sup_{s \in K} |Z(s+it) - h(s)| \leq \varepsilon \right\}$$

has a positive lower density.

As a consequence of these two statements, we have

Lemma 8.

$$\liminf_{y \to +\infty} \inf_{t \in \mathbb{R}} \sup_{s \in K} \left| Z_{y_-}(s+it) - h(s) \right| = 0$$

and

$$\limsup_{T \to +\infty} \frac{1}{2T} meas \left\{ t \in \mathbb{R}, |t| \leq T \ ; \ \sup_{s \in K} |Z(s+it) - h(s)| \leq \varepsilon \right\} > 0$$

are equivalent.

Now, to use this lemma, we remark that

$$\liminf_{y \to +\infty} \inf_{t \in \mathbb{R}} \sup_{s \in K} \left| Z_{y_-}(s+it) - h(s) \right| = 0$$

means that there exists a sequence t_y of real numbers such that for some subsequence $t_{\varphi(y)}$

$$\lim_{y \to +\infty} \sup_{s \in K} \left| Z_{\varphi(y)_-}(s+it_{\varphi(y)}) - h(s) \right| = 0.$$

So, the sequence $s \mapsto Z_{\varphi(y)_-}(s+it_{\varphi(y)})$ of analytic functions on K converges uniformly to $h(s)$.

Now, remark that $Z_{\varphi(y)_-}(s+it_{\varphi(y)})$ is never equal to zero for some s in K since for any given y, we have

$$\left| Z_{\varphi(y)_-}(s+it_{\varphi(y)}) \right| = \left| \prod_{\substack{p \leq \varphi(y) \\ p \text{ prime}}} \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{r(s+it_{\varphi(y)})}} \right| \geq \min_{s \in K} \left| \prod_{\substack{p \leq \varphi(y) \\ p \text{ prime}}} \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{r(s+it_{\varphi(y)})}} \right| \geq$$

$$\geq \prod_{\substack{p \leq \varphi(y) \\ p \text{ prime}}} \min_{s \in K} \left| \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{r(s+it_{\varphi(y)})}} \right| \geq$$

$$\geq \prod_{\substack{p \leq \varphi(y) \\ p \text{ prime}}} \min_{s \in S_{(1/2,1)}} \left| \sum_{r=0}^{+\infty} \frac{\alpha(p^r)}{p^{rs}} \right| > 0.$$

So, as a consequence of the Hurwitz theorem, either $h(s) = 0$ for all s in $\overset{o}{K}$, or $h(s) \neq 0$ for all s in $\overset{o}{K}$.

This ends the proof of the theorem. ■

If the hypothesis $h(s) = Z(s)$ can be fulfilled, this applies independently of the distribution of the zeroes of $Z(s)$ in the critical strip.

Consequence: if we can have $h(s) = Z(s)$,we get that $Z(s)$ has no zero in $\overset{o}{K}$. And applying this result in the cas $Z(s) = \zeta(s)$, we get immediately the result of Bagchi quoted above.

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