POWER SERIES – THE STRUCTURE OF H.-O.-GAPS

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Dedicated to an excellent mathematician
and my real good friend Karl-Heinz Indlekofer

Communicated by Imre Kátai
(Received December 13, 2012; accepted January 24, 2013)

Abstract. Suppose that a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu$ with positive radius of convergence has a sequence of H.-O.-gaps. Then there exists a neighborhood $U(z_0)$ of z_0, such that the rearrangement $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-\zeta)^\nu$ also has H.-O.-gaps for all $\zeta \in U(z_0)$.

1. Introduction

Let be given a power series $\sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu$ with radius of convergence R, where $0 < R < \infty$. We say that this series has a sequence $\{p_k, q_k\}$ of Hadamard–Ostrowski-gaps (H.-O.-gaps for short) if the following conditions hold:

- p_k, q_k are natural numbers with $p_1 < q_1 < p_2 < q_2 < \dots$,
- there exists $\lambda > 1$ such that $\frac{q_k}{p_k} > \lambda$ for all $k \in \mathbb{N}$,
- for $I := \bigcup_{k=1}^{\infty} [p_k, q_k]$ we have $\overline{\lim\limits_{\substack{\nu\to\infty\\ \nu\in I}}} |a_\nu|^{1/\nu} < \frac{1}{R}$.

Key words and phrases: H.-O.-gaps, overconvergence, rearrangements.
2010 Mathematics Subject Classification: 30B30, 30B40.
https://doi.org/10.71352/ac.39.303

In the case that

$$\frac{q_k}{p_k} \to \infty \text{ and } \lim_{\substack{\nu\to\infty \\ \nu\in I}} |a_\nu|^{1/\nu} = 0$$

we say that this series has Ostrowski-gaps (O.-gaps for short).

Approximately 90 years ago it was shown by Ostrowski ([3], [4], [5], [6]) that there exists a deep interdependence with the occurence of those gaps and the phenomenon of overconvergence i. e. the convergence of a subsequence of the partial sums $s_n(z) = \sum_{\nu=0}^{n} a_\nu(z-z_0)^\nu$ in a bigger domain than the circle of convergence.

Ostrowski's main results on overconvergence are the following.

Theorem $\mathbf{O_1}$ *Suppose that the power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu(z-z_0)^\nu$ with radius of convergence $R, 0 < R < \infty$ possesses H.-O.-gaps $\{p_k, q_k\}$ and let the function f be holomorphic at $z_1, |z_1 - z_0| = R$.*

Then there exists a neighborhood $U(z_1)$ of z_1 such that the sequence $\{s_{p_k}(z)\}$ of partial sums converges compactly on $U(z_1)$.

If the power series has O.-gaps $\{p_k, q_k\}$, then the function f is holomorphic in a simply connected domain $G(f) \supset \{z : |z-z_0| < R\}$ and $\{s_{p_k}(z)\}$ converges compactly on $G(f)$.

Theorem $\mathbf{O_2}$ *Suppose that the power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu(z-z_0)^\nu$ has radius of convergence $R, 0 < R < \infty$ and that $\{s_{p_k}(z)\}$ is an overconvergent subsequence of its partials sums.*

Then the series has a sequence of H.-O.-gaps of the type $\{p_k, q_k\}$.

For a good treatise on the theory of overconvergence we refer to Hille's book [2], section 16.7.

In recent years the investigation of overconvergence has got a revival since its connection with universal properties of power series has been detected. For details we refer to the excellent survey of Große-Erdmann [1], where also a synopsis of the relevant literatur is given.

In this paper we are dealing with a long outstanding problem: If $f(z) = \sum_{\nu=0}^{\infty} a_\nu(z-z_0)^\nu$ has a sequence of H.-O.-gaps, does there exist a neighborhood $U(z_0)$ of z_0, such that the series $f(z) = \sum_{\nu=0}^{\infty} b_\nu(z-\zeta)^\nu$ has also H.-O.-gaps for all $\zeta \in U(z_0)$?

It is obvious that the corresponding problem cannot hold for power series with Hadamard-gaps or Fabry-gaps.

2. Power series with H.-O.-gaps

We first deal with power series having H.-O.-gaps and prove the following result.

Theorem 1. *Suppose that the power series* $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ *with radius of convergence* R, $0 < R < \infty$ *has a sequence of H.-O.-gaps.*

Then there is a neighborhood $U(z_0)$ *of* z_0 *such that* $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu$ *has also H.-O.-gaps for all* $\zeta \in U(z_0)$.

Proof. Without loss of generality we may assume that $z_0 = 0$ and $R = 1$ and that $z = 1$ is a singularity of the function f. We denote by $\{p_k, q_k\}$ the sequence of H.-O.-gaps and in addition we can suppose that $a_\nu = 0$ for $p_k \leq \nu \leq q_k$.

We choose another sequence $\{p_k^*, q_k^*\}$ of H.-O.-gaps and constants $\gamma > 1$ and $\lambda^* > 1$ in the following way

$$p_k < p_k^* < q_k^* < q_k \, ; \quad \frac{p_k^*}{p_k} \geq \gamma \, , \quad \frac{q_k}{q_k^*} \geq \gamma \, ; \quad \frac{q_k^*}{p_k^*} \geq \lambda^* \, .$$

The partial sums of the considered power series are denoted by

$$s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \, .$$

Let be given a point $\zeta \in \mathbb{D}$, then the expansion of f around ζ:

$$f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu$$

has radius of convergence at most $|1 - \zeta|$.

1. For $p_k \leq n \leq q_k$ we have

$$s_n(z) = \sum_{\nu=0}^{n} b_\nu^{(\zeta)} (z - \zeta)^\nu$$

and we obtain for $0 < r < 1 - |\zeta|$ and $p_k \leq \nu \leq q_k$

$$b_\nu - b_\nu^{(\zeta)} = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=r} \frac{f(z) - s_\nu(z)}{(z - \zeta)^{\nu+1}} \, dz = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=r} \frac{f(z) - s_{q_k}(z)}{(z - \zeta)^{\nu+1}} .$$

We choose $0 < \varepsilon < \min\{1 - |\zeta| - r, r^{1/\gamma} - r\}$ and for all sufficiently great k it follows

$$\big|b_\nu - b_\nu^{(\zeta)}\big| \le \frac{1}{r^\nu} \cdot \max_{|z-\zeta|\le r} \big|f(z) - s_{q_k}(z)\big| \le \frac{1}{r^\nu} \cdot \max_{|z|\le|\zeta|+r} \big|f(z) - s_{q_k}(z)\big| \le$$

$$\le \frac{1}{r^\nu} \cdot \big(|\zeta| + r + \varepsilon\big)^{q_k}.$$

Therefore, we have for $p_k^* \le \nu \le q_k^*$

$$\big|b_\nu - b_\nu^{(\zeta)}\big|^{1/\nu} \le \frac{1}{r} \cdot \big(|\zeta| + r + \varepsilon\big)^{\frac{q_k}{\nu}} \le \frac{1}{r} \cdot \big(|\zeta| + r + \varepsilon\big)^{\frac{q_k}{q_k^*}} \le$$

$$\le \frac{1}{r}\big(|\zeta| + r + \varepsilon\big)^{\gamma}.$$

For $\zeta = 0$ we have $(r + \varepsilon)^\gamma < r$ and by continuity there exists $\zeta_1 \ne 0$ such that

$$\big(1 + |\zeta|\big)^2 \cdot \big(|\zeta| + r + \varepsilon\big)^\gamma < r$$

for all ζ with $|\zeta| < |\zeta_1|$. We therefore get for $0 < |\zeta| < |\zeta_1|$ and sufficiently great k

$$\big|b_\nu - b_\nu^{(\zeta)}\big|^{1/\nu} < \frac{1}{(1 + |\zeta|)^2} < \frac{1}{1 + |\zeta|} \le \frac{1}{|1 - \zeta|}.$$

2. For $p_k \le \nu \le q_k$ and $R > 1$ we have

$$b_\nu^{(\zeta)} = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=R} \frac{s_\nu(z)}{(z - \zeta)^{\nu+1}}\, dz = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=R} \frac{s_{p_k}(z)}{(z - \zeta)^{\nu+1}}\, dz.$$

If we choose $\zeta \ne 0$ and $\varepsilon > 0$ so small that $R + |\zeta| + \varepsilon < R^\gamma$, we get for all sufficiently large k

$$|b_\nu^{(\zeta)}| \le \frac{1}{R^\nu} \cdot \max_{|z-\zeta|\le R} \big|s_{p_k}(z)\big| \le \frac{1}{R^\nu} \cdot \max_{|z|\le|\zeta|+R} \big|s_{p_k}(z)\big| \le$$

$$\le \frac{1}{R^\nu} \cdot (R + |\zeta| + \varepsilon)^{p_k}.$$

Therefore, we have for $p_k^* \le \nu \le q_k^*$

$$\big|b_\nu^{(\zeta)}\big|^{1/\nu} \le \frac{1}{R} \cdot \big(R + |\zeta| + \varepsilon\big)^{\frac{p_k}{\nu}} \le \frac{1}{R} \cdot \big(R + |\zeta| + \varepsilon\big)^{\frac{p_k}{p_k^*}} \le$$

$$\le \frac{1}{R} \cdot \big(R + |\zeta| + \varepsilon\big)^{1/\gamma}$$

and again as above we find a $\zeta_2 \neq 0$ such that for $0 < |\zeta| < |\zeta_2|$ and all sufficiently great k

$$\left|b_\nu^{(\zeta)}\right|^{1/\nu} < \frac{1}{(1+|\zeta|)^2} < \frac{1}{1+|\zeta|} \leq \frac{1}{|1-\zeta|}.$$

3. It follows that for $I^* = \bigcup_{k=1}^{\infty}[p_k^*, q_k^*]$ and all ζ with

$$0 < |\zeta| < \min\left\{|\zeta_1|, |\zeta_2|\right\} =: |\zeta_0|$$

$$\overline{\lim_{\substack{\nu\to\infty \\ \nu\in I^*}}} |b_\nu|^{1/\nu} \leq \frac{1}{(1+|\zeta|)^2} < \frac{1}{|1-\zeta|},$$

so, that the series $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-\zeta)^\nu$ has H.-O.-gaps. ■

Remark 2.1. By slightly changing the proof one can choose $\zeta_0 \neq 0$ so small, that for $m \geq 2$

$$\overline{\lim_{\substack{\nu\to\infty \\ \nu\in I^*}}} |b_\nu|^{1/\nu} \leq \frac{1}{(1+|\zeta|)^m}$$

for all ζ with $0 < |\zeta| < |\zeta_0|$. Therefore, the coefficients in the intervals $[p_k^*, q_k^*]$ become arbitrary small, when ζ is near to zero.

3. Power series with O.-gaps

We now consider power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\infty$ with radius of convergence $R, 0 < R < \infty$ and O.-gaps. It is well known [4] that such a series defines a function f, which is holomorphic in a simply connected domain $G(f)$ and every boundary point of $G(f)$ is a singularity for f.

We prove the following result.

Theorem 2. *Suppose that the power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu$ with radius of convergence R, $0 < R < \infty$, has a sequence of O.-gaps.*

Then the power series $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-\zeta)^\nu$ has also O.-gaps for all $\zeta \in G(f)$.

Proof. Similar to the situation in the proof of Theorem 1 we again may assume that $z_0 = 0$ and $R = 1$, so that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$. The sequence of O.-gaps is denoted by $\{p_k, q_k\}$ and we may suppose that $a_\nu = 0$ for $p_k \leq \nu \leq q_k$, where $\frac{q_k}{p_k} \to \infty$ for $k \to \infty$.

We choose p_k^*, q_k^* with $p_k < p_k^* < q_k^* < q_k$ in such way that

$$\frac{p_k^*}{p_k} \to \infty, \qquad \frac{q_k^*}{p_k^*} \to \infty, \qquad \frac{q_k}{q_k^*} \to \infty \qquad \text{for } k \to \infty.$$

Let be given a point $\zeta \in G(f)$, consider the power series

$$f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-\zeta)^\nu$$

and suppose that its radius of convergence is $2r$.

1. We consider

$$s_{q_k^*}(z) = \sum_{\nu=0}^{q_k^*} a_\nu z^\nu = \sum_{\nu=0}^{q_k^*} b_\nu^{(\zeta)} (z-\zeta)^\nu.$$

For all sufficiently great k we get with a constant $\vartheta, 0 < \vartheta < 1$

$$\max_{|z-\zeta| \le r} \left| f(z) - s_{q_k^*}(z) \right| = \max_{|z-\zeta| \le r} \left| f(z) - s_{q_k}(z) \right| \le \vartheta^{q_k}.$$

For $p_k \le \nu \le q_k$ we have

$$b_\nu - b_\nu^{(\zeta)} = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=r} \frac{f(z) - s_{q_k^*}(z)}{(z-\zeta)^{\nu+1}} \, dz$$

and it follows for $p_k^* \le \nu \le q_k^*$

$$\left| b_\nu - b_\nu^{(\zeta)} \right|^{1/\nu} \le \frac{1}{r} \cdot \vartheta^{\frac{q_k}{\nu}}$$

which implies that for $I^* = \bigcup_{k=1}^{\infty} [p_k^*, q_k^*]$

$$\lim_{\substack{\nu \to \infty \\ \nu \in I^*}} \left| b_\nu - b_\nu^{(\zeta)} \right|^{1/\nu} = 0.$$

2. For $R > 1$ and $p_k^* \le \nu \le q_k^*$ we have for sufficiently great k

$$b_\nu^{(\zeta)} = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=R} \frac{s_{q_k^*}(z)}{(z-\zeta)^{\nu+1}} \, dz = \frac{1}{2\pi i} \int\limits_{|z-\zeta|=R} \frac{s_{p_k}(z)}{(z-\zeta)^{\nu+1}} \, dz$$

and therefore we get for sufficiently great k

$$\left| b_\nu^{(\zeta)} \right|^{1/\nu} \le \frac{1}{R} \cdot \max_{|z| \le |\zeta| + R} \left| s_{p_k}(z) \right|^{1/\nu} \le \frac{1}{R} \cdot \big(|\zeta| + R + 1 \big)^{\frac{p_k}{\nu}} \le$$

$$\le \frac{1}{R} \cdot \big(|\zeta| + R + 1 \big)^{\frac{p_k}{p_k^*}}$$

which implies

$$\varlimsup_{\substack{\nu\to\infty\\ \nu\in I^*}} \left|b_\nu^{(\zeta)}\right|^{1/\nu} \le \frac{1}{R}$$

and, since $R > 1$ was arbitrary, it follows

$$\lim_{\substack{\nu\to\infty\\ \nu\in I^*}} \left|b_\nu^{(\zeta)}\right|^{1/\nu} = 0.$$

3. It follows that for all $\zeta \in G(f)$ the power series $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z-\zeta)^\nu$ has a sequence of O.-gaps $\{p_k^*, q_k^*\}$ (which is independent of ζ). In addition the sequence $s_{p_k^*}(z) = \sum_{\nu=0}^{p_k^*} b_\nu^{(\zeta)} (z-\zeta)^\nu$ converges to $f(z)$ compactly on $G(f)$. ■

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