

EXISTENCE OF MOMENTS IN THE HSU–ROBBINS–ERDŐS THEOREM

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Abstract. We consider the so-called empirical version of the Hsu–Robbins series and find conditions for the existence of its moments.

1. Introduction

Let X_k , $k \geq 1$, be a sequence of independent identically distributed random variables and let S_n , $n \geq 1$, be the sequence of their partial sums. According to Hsu and Robbins [5], the sequence $\{S_n/n\}$ is said to converge completely to a constant μ if

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

In a more convenient form, the latter condition is written as

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n - n\mu| \geq n\varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

Hsu and Robbins [5] found the sufficient condition for the complete convergence of $\{S_n/n\}$ to μ , namely

$$\mathbb{E}[X_1] = \mu, \quad \mathbb{E}[X_1^2] < \infty.$$

Later, Erdős [2] proved the converse.

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Motivated by their results we assume throughout that the first moment exists and is zero and we introduce the random variable

$$\xi \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mathbb{I}_{\{|S_n| \geq \varepsilon n\}},$$

where \mathbb{I}_A is the indicator of a random event A . Note that the right hand side depends on ε but since our results do not depend on this quantity we suppress this variable in the variable ξ . Then the Hsu–Robbins–Erdős theorem is stated as follows

$$\mathbb{E}[\xi] < \infty \quad \text{for all } \varepsilon > 0 \quad \Longleftrightarrow \quad \mathbb{E}[X_1] = 0, \quad \mathbb{E}[X_1^2] < \infty.$$

The aim of this note is to find necessary and sufficient conditions for

$$(1) \quad \mathbb{E}[\xi^r] < \infty \quad \text{for all } \varepsilon > 0$$

if $r > 0$. It turns out that this question is related to a Baum–Katz [1] result extending the Hsu–Robbins–Erdős theorem. Below is a particular case of the Baum–Katz result.

Theorem (Baum and Katz [1]). *If $r > 0$, then*

$$\sum_{n=1}^{\infty} n^{r-1} P(|S_n| \geq \varepsilon n) < \infty \quad \text{for all } \varepsilon > 0$$

if and only if

$$(2) \quad \mathbb{E}[X_1] = 0 \quad \text{and} \quad \mathbb{E}[|X_1|^{r+1}] < \infty.$$

2. Main result

Theorem 1. *The following implications hold:*

- (a) *if $r \geq 1$, then (2) implies (1);*
- (b) *if $0 < r \leq 1$, then (1) implies (2).*

Remark 1. In the case $r = 1$, we obtain that (1) is equivalent to (2). This result is, in fact, the Hsu–Robbins–Erdős theorem.

Remark 2. The case (a) is easy to treat for integer r . We show this for a particular case of $r = 2$. Then

$$\begin{aligned}\xi^2 &= \sum_{i,j \geq 1} \mathbb{I}_{\{|S_i| \geq \varepsilon i\}} \mathbb{I}_{\{|S_j| \geq \varepsilon j\}} = \\ &= \sum_{i=1}^{\infty} \sum_{j \leq i} \mathbb{I}_{\{|S_i| \geq \varepsilon i\}} \mathbb{I}_{\{|S_j| \geq \varepsilon j\}} + \sum_{i=1}^{\infty} \sum_{j > i} \mathbb{I}_{\{|S_i| \geq \varepsilon i\}} \mathbb{I}_{\{|S_j| \geq \varepsilon j\}}.\end{aligned}$$

Denoting the terms on the right hand side by ξ_1^2 and ξ_2^2 , respectively, we have

$$\xi_1^2 = \sum_{i=1}^{\infty} \mathbb{I}_{\{|S_i| \geq \varepsilon i\}} \left[\sum_{j \leq i} \mathbb{I}_{\{|S_j| \geq \varepsilon j\}} \right] \leq \sum_{i=1}^{\infty} i \mathbb{I}_{\{|S_i| \geq \varepsilon i\}}.$$

Passing to the expectations

$$\mathbb{E}[\xi_1^2] \leq \sum_{i=1}^{\infty} i \mathbb{P}(|S_i| \geq \varepsilon i).$$

By the Baum–Katz theorem with $r = 2$, the latter series is finite if and only if $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[|X_1|^3] < \infty$. The same holds for ξ_2^2 and thus case (a) follows.

3. Proof of the main result

We start with the following elementary lemma.

Lemma. *Let $a_n \in \{0, 1\}$ for each n .*

(i) *Let $r \geq 1$. Then, for all $n \geq 1$,*

$$\left(\sum_{k=1}^n a_k \right)^r \leq r \sum_{k=1}^n k^{r-1} a_k.$$

(ii) *Let $0 < r \leq 1$. Then, for all $n \geq 1$,*

$$\left(\sum_{k=1}^n a_k \right)^r \geq \frac{r}{r+1} \sum_{k=1}^n k^{r-1} a_k.$$

Proof of Lemma. (i) It is clear that for $r \geq 1$

$$(k-1)^{r-1} \leq \int_{k-1}^k x^{r-1} dx \leq k^{r-1}, \quad k \geq 1,$$

whence

$$r \int_0^n x^{r-1} dx = n^r \leq r \sum_{k=1}^n k^{r-1} \quad \text{for all } n \geq 1.$$

Next, for $0 < r \leq 1$

$$r \int_0^n x^{r-1} dx = n^r \geq r \left(\sum_{k=1}^n (k-1)^{r-1} + n^{r-1} - n^{r-1} \right) \geq r \left(\sum_{k=1}^n k^{r-1} \right) - rn^r$$

which in turn implies

$$n^r \geq \frac{r}{r+1} \sum_{k=1}^n k^{r-1}.$$

Now fix n and let

$$I_n = \{k \leq n : a_k = 1\}, \quad m = m_n = \text{card}(I_n).$$

Then

$$I_n = \{i_1, \dots, i_m\}, \quad 1 \leq i_1 < \dots < i_m \leq n.$$

It is clear that

$$i_1 \geq 1, \quad \dots, \quad i_m \geq m.$$

Therefore

$$\begin{aligned} (a_1 + \dots + a_n)^r &= m^r \leq r(1^{r-1} + \dots + m^{r-1}) \leq \\ &\leq r(i_1^{r-1} + \dots + i_m^{r-1}) = \\ &= r(i_1^{r-1}a_{i_1} + \dots + i_m^{r-1}a_{i_m}) = \\ &= r \sum_{k=1}^n k^{r-1}a_k \end{aligned}$$

which proves the case (i).

(ii) We use the same notation I_n , m , and i_1, \dots, i_m as in the proof of case (i). Since $0 < r \leq 1$ we find

$$\begin{aligned} (a_1 + \dots + a_n)^r &= m^r \geq \\ &\geq \frac{r}{r+1} \sum_{k=1}^m k^{r-1} \geq \\ &\geq \frac{r}{r+1} (i_1^{r-1}a_{i_1} + \dots + i_m^{r-1}a_{i_m}) \geq \\ &\geq \frac{r}{r+1} \sum_{k=1}^n k^{r-1}a_k \end{aligned}$$

which proves the case (ii). ■

Proof of Theorem 1. Let $a_n = \mathbb{I}_{\{|S_n| \geq \varepsilon n\}}$. For (a), we apply case (i) of the above Lemma:

$$\xi^r \leq r \sum_{k=1}^{\infty} k^{r-1} \mathbb{I}_{\{|S_k| \geq \varepsilon k\}}.$$

Passing to the expectations

$$\mathbb{E}[\xi^r] \leq r \sum_{k=1}^{\infty} k^{r-1} \mathbb{P}(|S_k| \geq \varepsilon k).$$

By the Baum–Katz theorem, the right hand side is finite if $\mathbb{E}[|X|^{r+1}] < \infty$.

For (b), we apply case (ii) of the above Lemma:

$$\xi^r \geq \frac{r}{r+1} \sum_{k=1}^{\infty} k^{r-1} \mathbb{I}_{\{|S_k| \geq \varepsilon k\}}$$

hence, (1) implies that the expectation of the right hand side is finite which implies $\mathbb{E}[|X|^{r+1}] < \infty$ by the Baum–Katz theorem. Now standard arguments imply that random variable X has expectation zero. \blacksquare

Remark 3. We do not know whether the implications (1) \implies (2) for $0 < r < 1$ and (2) \implies (1) for $r \geq 1$ hold but we conjecture that the two statements are equivalent for all $r > 0$.

4. Extensions

Here we will consider the multiindex case. Therefore, let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$, X be i.i.d. random variables, that is, we discuss a random field with index set \mathbb{Z}_+^d , $d \geq 2$, denoting the positive integer d -dimensional lattice with coordinate-wise partial ordering \leq . As before we discuss partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{Z}_+^d$. Finally, let $|\mathbf{n}| = n_1 \cdots n_d$.

In the following we assume again that the expectation of X exists and is zero. Now we investigate the random variable

$$\xi_d \stackrel{\text{def}}{=} \sum_{\mathbf{k} \in \mathbb{Z}_+^d} \mathbb{I}_{\{|S_{\mathbf{k}}| \geq \varepsilon |\mathbf{k}|\}}.$$

Similarly to the case of $d = 1$, we are interested in moment conditions implying the existence of the r -th moment of ξ_d , i.e.,

$$(3) \quad \mathbb{E}[\xi_d^r] < \infty \quad \text{for all } \varepsilon > 0.$$

Typically in these kind of results one needs a somewhat stronger moment condition on X in case $d > 1$ as compared to $d = 1$, namely

$$(4) \quad \mathbb{E}[|X|^{r+1}(\log_+ |X|)^{(d-1)r}] < \infty,$$

here $\log_+ x = \log(1+x)$ for $x \geq 0$. Now, we are ready to formulate our second result.

Theorem 2. *The following implications hold:*

- (a) *if $r \geq 1$, then (4) implies (3);*
- (b) *if $0 < r \leq 1$, then (3) implies (4).*

Proof. The proof follows similar arguments as the one for Theorem 1. Let $a_{\mathbf{k}} = \mathbb{1}_{\{|S_{\mathbf{k}}| \geq \varepsilon |\mathbf{k}|\}}$, $\mathbf{k} \in \mathbb{Z}_+^d$, and for any positive integer n let

$$I_n = \{\mathbf{k} : |\mathbf{k}| \leq n \text{ and } a_{\mathbf{k}} = 1\}, \quad m = m_n = \text{card}(I_n).$$

Then

$$I_n = \{\mathbf{k}_1, \dots, \mathbf{k}_m\}$$

where the indices are ordered along the hyperbolas $|\mathbf{k}| = \ell$, $\ell = 1, 2, \dots$, and therein in lexicographic order. In general we do not have any more that $|\mathbf{k}_\nu| \geq \nu$ since several indices may be incident to the same “hyperbola” $|\mathbf{k}| = \nu$. We write $d(\ell) = \#\{\mathbf{k} \in \mathbb{Z}_+^d \text{ with } |\mathbf{k}| = \ell\}$. The terms $d(\ell)$ themselves do not have a nice asymptotic behavior, but their partial sums $M(n) = \sum_{\ell=1}^n d(\ell)$ have the asymptotic

$$M(n) \sim \frac{1}{(d-1)!} n (\log n)^{d-1}, \quad n \rightarrow \infty,$$

see [7]. This asymptotic is well known in the generalized Dirichlet divisor problem. In fact, we do not need the precise asymptotic for the proof below, while the asymptotic relation

$$M(n) \asymp n(\log n)^{d-1}$$

is sufficient for our purposes. The latter relation can be proved by comparing $M(n)$ and the volume of the domain

$$x_1 \geq 1, \dots, x_d \geq 1, \quad x_1 \dots x_d \leq n$$

in the space \mathbb{R}^d .

Now we conclude that if $|\mathbf{k}_\nu| = \ell$, then

$$(5) \quad \nu \leq M(\ell) = M(|\mathbf{k}_\nu|) \leq c |\mathbf{k}_\nu| (\log |\mathbf{k}_\nu|)^{d-1}$$

with some positive $c > 0$. As above, we conclude that for $r \geq 1$ and any positive integer n

$$\begin{aligned}
 \left(\sum_{|\mathbf{k}| \leq n} a_{\mathbf{k}} \right)^r &= m^r \leq \\
 &\leq r \left(1^{r-1} + \cdots + m^{r-1} \right) \leq \\
 &\leq c r \left(|\mathbf{k}_1|^{r-1} (\log |\mathbf{k}_1|)^{(d-1)(r-1)} + \cdots + \right. \\
 &\quad \left. + |\mathbf{k}_m|^{r-1} (\log |\mathbf{k}_m|)^{(d-1)(r-1)} \right) = \\
 &= c r \left(|\mathbf{k}_1|^{r-1} (\log |\mathbf{k}_1|)^{(d-1)(r-1)} a_{\mathbf{k}_1} + \cdots + \right. \\
 &\quad \left. + |\mathbf{k}_m|^{r-1} (\log |\mathbf{k}_m|)^{(d-1)(r-1)} a_{\mathbf{k}_m} \right) = \\
 &= c r \sum_{|\mathbf{k}| \leq n} |\mathbf{k}|^{r-1} (\log |\mathbf{k}|)^{(d-1)(r-1)} a_{\mathbf{k}}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mathbb{E}[\xi_d^r] &\leq c r \sum_{|\mathbf{k}| \geq 1} |\mathbf{k}|^{r-1} (\log |\mathbf{k}|)^{(d-1)(r-1)} \mathbb{P}(|S_{\mathbf{k}}| > |\mathbf{k}| \varepsilon) = \\
 &= c r \sum_{\ell \geq 1} d(\ell) \ell^{r-1} (\log \ell)^{(d-1)(r-1)} \mathbb{P}(|S_{\ell}| > \ell \varepsilon).
 \end{aligned}$$

In sums with otherwise smooth summands we may replace $d(\ell)$ by the asymptotic derivative of $M(\ell)$, i.e., $(\log \ell)^{d-1}$. Hence, the last sum above is finite if and only if (4) holds, as it can formally be seen using arguments similar to those in the proof Lemma 3.1 in [3] or on page 2448 in [6]. The converse implication follows as in the proof of Theorem 1 (b) and the arguments in the proof of Lemma 3.1 in [3] or [6] again (the case of an arbitrary slowly varying function is treated in [4] for $d = 1$). In doing so we use the inequality

$$\nu^{r-1} \geq M(\ell)^{r-1} \geq c(|\mathbf{k}_{\nu}|(\log |\mathbf{k}_{\nu}| + 1)^{d-1})^{r-1}, \quad 0 < r \leq 1,$$

that follows from (5). ■

Remark 4. Again we conjecture that the moment condition (4) is equivalent to the moment condition (3).

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