SOME REMARKS ON A RESULT OF TIMOFEEV AND KHRIPUNOVA

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Dedicated to Professor Karl-Heinz Indlekofer on his seventieth anniversary

Communicated by Bui Minh Phong

(Received September 24, 2012; accepted November 10, 2012)

Abstract. The sum $\sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} \tau(n-1)\omega(n+1)$ is investigated where $\tau(n) =$ $=$ number of divisors of n, $\omega(n) =$ number of prime divisors of n, $\Omega(n) =$ $=$ number of prime power divisors of n, $\mathcal{N}_k = \{n|\Omega(n) = k\}$.

1. Introduction

1.1. Notation

$\mathcal{P}$= set of primes, p and q with and without indices always denote prime numbers. $\omega(n)$= number of distinct prime factors of n; $\Omega(n)$= number of prime power divisors of n; $\tau(n)$= number of divisors of n; $\tau_k(n)$= number of positive integers $x_1, x_2, \ldots, x_k$ satisfying $n = x_1 \cdots x_k$. Let $p(n)$ be the smallest and $P(n)$ be the largest prime factor of n. For some integer $k \ge 1$ let $\mathcal{P}_k := \{n \mid \omega(n) = k\}$; $\mathcal{N}_k := \{n \mid \Omega(n) = k\}$, $\pi_k(x) := \#\{n \le x \mid n \in \mathcal{P}_k\}$, $N_k(x) := \#\{n \le x \mid \Omega(n) = k\}$.

Key words and phrases: Divisor function, prime divisor function, square-full integers.
2010 Mathematics Subject Classification: 11N37.
Research supported by the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645).
https://doi.org/10.71352/ac.39.239

Furthermore we shall write x_k instead of the k-fold iterate of $\log x$, i.e. $x_1 = \log x,\ \ x_2 = \log x_1,\ x_3 = \log x_2, \dots$. (We shall use this abbreviation only for the variable x.)

1.2. Preliminaries

Sathe [1] and A. Selberg [2] showed that for $x \geq 3, \quad 1 \leq k \leq (2-\xi)\, x_2$, where $0 < \xi < 1$, we have

$$N_k(x) = \frac{x}{x_1} F\left(\frac{k}{x_2}\right) \frac{x_2}{(k-1)!} \left(1 + \mathcal{O}_\xi\left(\frac{1}{x_2}\right)\right).$$

Here

$$F(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{1}{p}\right)^z \left(1 - \frac{z}{p}\right)^{-1},$$

$\Gamma(x)$ is the Euler gamma function.

For $(2+\varepsilon)\, x_2 \leq k \leq x_1/\log 2$ the behaviour of $N_k(x)$ was studied by J.-L. Nicolas [3]. He proved, that in this range of k,

$$N_k(x) = \frac{Cx}{2^k} \log \frac{x}{2^k} + \mathcal{O}\left(\frac{x}{2^k}\left(\log \frac{3x}{2^k}\right)^\beta\right)$$

where $0 < \beta < 1$, and

$$C = \frac{1}{4} \prod_{p>2} \left(1 + \frac{1}{p(p-2)}\right).$$

Similar theorem is valid for $\pi_k(x)$. Let

$$\lambda(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z.$$

Let $A > 0$ be an arbitrary constant. Then, uniformly as $x \geq 3, \quad 1 \leq k \leq Ax_2$ we have

$$\pi_k(x) = \frac{x}{x_1} \cdot \frac{x_2^{k-1}}{(k-1)!} \left\{\lambda\left(\frac{k-1}{x_2}\right) + \mathcal{O}\left(\frac{k}{x_2^2}\right)\right\},$$

where the constant, implied by the error term may depend on A.

N.M. Timofeev and M.B. Khripunova in their paper [4] proved a theorem of Titchmarsh type, and of a Vinogradov–Bombieri type for the integers in $\mathcal{N}_k$ which is quoted in this paper as Lemma 1 and Lemma 2.

Let $t \geq 2, \quad Q(t) = \prod_{p<t} p, \quad p \in \mathcal{P}$. Let $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ be arbitrary small positive numbers. Denote

$$\mu(x,k,t,a,d) := \#\{n \leq x | n \in \mathcal{N}_k, \quad (n, Q(t)) = 1, \quad n \equiv a \pmod{d}\}.$$

Lemma 1. *Let* $2 \leq t \leq \sqrt{x}, \quad k \leq x_2^2$, *and let*

$$\Delta_k(t) =$$

$$\sum_{d \leq Q} \max_{y \leq x} \max_{(a,d)=1} \left| \mu(y,k,t,a,d) - \frac{1}{\varphi(d)} \#\{n \leq y | n \in \mathcal{N}_k, (n, dQ(t)) = 1\} \right|.$$

Then

$$\Delta_k(t) \ll Q\sqrt{x} \exp\left(x_2^{2+\varepsilon}\right) + \frac{x}{x_1^B},$$

where $\varepsilon > 0$ *and* B *are arbitrary positive constants.*

Lemma 2. *Suppose* $k \leq (2-\varepsilon)x_2, \ 0 < \varepsilon < 1, d \leq x^{\frac{1}{2}+\alpha(k)}, \ 2 \leq t \leq x^{\beta(k)}$, $\alpha(k) = \frac{1}{3k}$ *and* $\beta(k) = \frac{1}{10}\exp\left(-\frac{k}{2}\right)$. *Let* $0 < \varepsilon_1 < 1$. *Then there exists a constant* $c(\varepsilon, \varepsilon_1)$ *such that*

$$\mu(x,k,t,a,d) \leq c(\varepsilon, \varepsilon_1) \frac{x}{\varphi(d) x_1} (1+\varepsilon_1)^k \frac{\left(\log \frac{x_1}{\log t}\right)^{k-1}}{(k-1)!}.$$

They used their results to prove the asymptotic of the sums

$$\sum_{\substack{n \in \mathcal{N}_k \\ n \leq x}} \tau(n-1) \quad \text{in [4], and} \quad \sum_{\substack{n < N \\ n \in \mathcal{N}_k}} \tau(N-n) \quad \text{in [5].}$$

Solving a weakened conjecture of Ivič [6] I proved in [7] that

$$\sum_{n \leq x} \tau(n + \tau(n)) = \mathcal{D} x x_1 + \mathcal{O}\left(\frac{x x_1}{x_2}\right). \tag{1.1}$$

(1.1) is an easy consequence of Lemma 1. The proof is going on the usual way with the help of Lemma 1. Finally we proved that the contribution of the integers $n \in \bigcup_{k \geq (2-\varepsilon)x_2} \mathcal{N}_k$ to (1.1) is less than $\mathcal{O}\left(\frac{x x_1}{x_2}\right)$.

As we mentioned in [7] one can prove similarly

$$\sum_{n \leq x} \tau(n + f(n)) = \mathcal{D}_f x x_1 + \mathcal{O}\left(\frac{x x_1}{x_2}\right) \tag{1.2}$$

with some constants $\mathcal{D}_f > 0$, if $f(n) = \omega(n), \Omega(n), \tau(\tau(n)), 2^{\omega(n)}, \tau_k(n)$, where $\tau_k(n)$ is the number of solutions of $n = u_1 \dots u_k$ in positive integers $u_1, \dots, u_k$. Similar theorems can be proved if we substitute τ on the left hand side of (1.2) by $2^{\omega(m)}$. Even one can prove the asymptotic of (1.2) if we sum only on the set $n \in \mathcal{N}_k$ for a given k uniformly as $1 \le k \le (2-\varepsilon)x_2$. Changing τ into τ_3 in (1.2), we stock. We are able to prove only the exact order of $\sum\limits_{n \le x} \tau_3(n + f(n))$.

1.3. On sums of form $\sum \tau(f(n)n)$

Assume that f is a multiplicative function taking on positive integer values, $1 \le f(p^a) < ca^{c_1}$ with suitable constants c, c_1, and $f(p) = A \in \mathcal{P}$ for every prime p.

Let

$$A_y(n) := \prod_{\substack{p^\alpha || n \\ p < y}} p^\alpha, \quad B_y(n) := \prod_{\substack{p^\alpha || n \\ p \ge y}} p^\alpha.$$

Then $n = A_y(n) \cdot B_y(n)$.

Let $y = x_2$, and A_x be a monotonically increasing sequence tending to infinity as $x \to \infty$.

One can observe that the contribution of $\sum_{n \le x} \tau(f(n)n)$ for those n for which $B_{x_2}(n)$ is not square-free, or $A_{x_2} > x_2^{A_x}$ is $o(xx_1x_2)$. For the other integers n we can write $f(n) = f(A_{x_2}(n)) \cdot A^{\omega\left(B_{x_2}(n)\right)}$, $nf(n) = A_{x_2}(n) f(A_{x_2}(n)) \cdot A^{\omega\left(B_{x_2}(n)\right)} \cdot B_{x_2}(n)$.

Let K run over the integers up to $x_2^{A_x}$ satisfying $P(K) \le x_2$, and m run over the square free integers m satisfying $p(m) > x_2$. Let $Kf(K) = A^{\alpha(K)} R_K$, where $(R_K, A) = 1$. Thus we have

$$\sum_{n \le x} \tau(f(n)n) = \sum_K \tau(R_K) \sum_{m \le x/K} (\omega(m) + \alpha(K) + 1)\tau(m) + o(xx_1x_2). \tag{1.3}$$

It remained to estimate the sums

$$\sum_{\substack{m \le y \\ p(m) \ge x_2}} \tau(m)|\mu(m)|, \quad \sum_{m \le y} \tau(m)|\mu(m)|\omega(m)$$

for $x/x_2^{A_x} \le y \le x$, which can be done on routine way.

Thus we have

$$\sum_{n\le x}\tau\left(f\left(n\right)n\right)=\left(1+o_x\left(1\right)\right)cxx_1x_2.$$

we do not want to give a complete proof of this relation.

1.4. Theorems

We shall prove

Theorem 1. *Let $r\ge 2$ be an integer. Then*

$$S\left(x\right):=\sum_{n\le x}\tau_r\left(\tau\left(n\right)n\right)=\left(1+o\left(1\right)\right)cxx_1x_2^{r-1} \tag{1.4}$$

holds, where c is a suitable positive constant.

Theorem 2. *We have*

$$T\left(x\right):=\sum_{n\le x}\tau\left(n\tau\left(n-1\right)\right)=C\left(1+o_x\left(1\right)\right)xx_1x_2, \tag{1.5}$$

where C is a positive constant.

1.5.

In the proof of Theorem 2 we shall use Lemma 3. For some integer $\mathcal{D}>0$ let $\mathcal{B}_{\mathcal{D}}$ be the semigroup generated by $\{1,p_1,\dots,p_r\}$ where $p_1,\dots,p_r$ are the prime factors of $\mathcal{D}$, i.e. $\mathcal{B}_{\mathcal{D}}=\{1,p_1^{\alpha_1},\dots,p_r^{\alpha_r}|\alpha_j=0,1,2,\dots;\quad j=1,\dots,r\}$. Let $\alpha_p\left(n\right)$ be that exponent k for which $p^k|n$ and $p^{k+1}\nmid n$.

Lemma 3. *Let $A,B,C\le x_1$ be positive integers, $(A,B)=1$, q run over the primes in $\mathcal{I}=[x_1^2,x^\eta]$, where $0<\eta<1/10$. Then*

$$\sum_{q\in\mathcal{I}}\sum_{\substack{A\nu\equiv 1 \pmod{Bq}\\ p\nu\le x}}\tau\left(CA\nu\right)=E\left(A,B,C\right)xx_1x_2+\mathcal{O}\left(xx_2\right), \tag{1.6}$$

where

$$E\left(A,B,C\right)=\frac{2}{A\varphi\left(B\right)}\prod_{p|(B,C)}\tau\left(p^{\alpha_p(C)}\right)\prod_{\substack{p|AC\\ p\nmid B}}\left(1-\frac{1}{p}\right)^2\sum_{\mu=0}^{\infty}\frac{p^{\alpha_p(C)+\alpha_p(A)}}{p^\mu},$$

and the constant implied by the $\mathcal{O}$ term may depend on A,B,C.

Remark. One can prove better assertions, by using known results of D.I. Tolev [9] or Heath–Brown [10], but this lemma is sufficient for our purposes.

Theorem 3. *Let*

$$S_k(x) := \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} \tau(n-1)\,\omega(n+1).$$

Let $0<\xi<1$. *Then, for* $1\le k\le(2-\xi)\,x_2$ *we have*

$$\begin{aligned} S_k(x) &= (1+o_x(1))\,x x_2 \prod_p\left(1+\frac{1}{p(p-1)}\right)\times \\ &\times\prod_p\left(1-\frac{k-1}{(p^2-p+1)\,x_2}\right)F\left(\frac{k}{x_2}\right)\frac{x_2^{k-1}}{(k-1)!}. \end{aligned} \tag{1.7}$$

Here F *is defined in Section* 1.1.

Especially, for $k=1$:

$$S_1(x) := \sum_{p\le x}\tau(p-1)\,\omega(p+1) = (1+o_x(1))\,Cxx_2,$$

where

$$C=\prod\left(1+\frac{1}{p(p-1)}\right).$$

Remark. Timofeev and Khripunova proved in [4] that

$$\begin{aligned} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}}\tau(n-1) &= x\prod_p\left(1+\frac{1}{p(p-1)}\right)\prod_p\left(1-\frac{k-1}{(p^2-p+1)\,x_2}\right)\times \\ &\times F\left(\frac{k}{x_2}\right)\frac{x_2^{k-1}}{(k-1)!}\left(1+\mathcal{O}_\xi\left(\frac{1}{\sqrt{x_2}}\right)\right). \end{aligned} \tag{1.8}$$

2. Proof of Theorem 1

It is known that $\sum_{m\le y}\tau_r(m)\le C_1 y(\log y)^{r-1}$, if $y\ge 2$, furthermore that $\tau_r(ab)\le\tau_r(a)\cdot\tau_r(b)$.

Let us write every n in the form $n = Km$, where K is the square-full part and m is the square-free part of n.

Since $n\tau(n) = K \cdot \tau(K) \cdot 2^{\omega(n)} \cdot m$, we obtain that

$$\tau_r(n\tau(n)) \leq \tau_r(K) \cdot \tau_r(\tau(K))\, \tau_r(m) \binom{\omega(m) + r - 1}{r - 1} \leq$$
$$\leq C_2 \tau_r(K)\, \tau_r(K)\, \omega(m)^{r-1}.$$

It is clear, furthermore, that

$$\begin{aligned}\sum_{m \leq y} \tau_r(m)\, \omega(m)^{r-1} \leq & C_3 \sum_{j=1}^{r} \sum_{p_1 < \ldots < p_j} \tau_r(p_1 \ldots p_j \nu) \leq \\ \leq & C_4(r)\, y (\log y)^{r-1} \sum_{j=1}^{r-1} \sum_{p_1 < \ldots p_j < y} \frac{\tau_r(p_1) \ldots \tau_r(p_j)}{p_1 \ldots p_j} \leq \\ \leq & C_5(r)\, y (\log y)^{r-1} \left(1 + \sum_{p < y} \frac{\tau_r(p)}{p}\right)^{r-1} \leq \\ \leq & C_6(r)\, y (\log y)^{r-1} (\log\log 10y)^{r-1}.\end{aligned}$$

It is obvious that

$$\sum \frac{\tau_r(K)\, \tau_r(\tau(K))}{K}$$

is convergent, where K runs over the square-full integers.

Let

$$T_K(x) = \sum_{n \leq x}{}^{*} \tau_r(n\tau(n)), \tag{2.1}$$

where $*$ indicates that we sum over those n the square-full part of which is K.

Let $Y_x \to \infty$ arbitrarily slowly. Then

$$S(x) = \sum_{K \leq Y_x} T_K(x) + o_{Y_x}(1)\, x \cdot x_1^{r-1} x_2^{r-1}. \tag{2.2}$$

Let us fix some $K\,(\leq Y_x)$. Write $K\tau(K) = 2^{\alpha_K} \cdot R$, R odd, $m = 2^{\delta_0} m_1 m_2$, where $\delta_0 \in \{0, 1\}$; m_1, m_2 are coprime odd integers, m_2 is the largest odd divisor of m coprime to R (consequently $m_1 | R$).

We have

$$T_K(x) = \sum_{\delta_o=0}^{1} \sum_{m_1|R} \sideset{}{^{**}}\sum_{m_2 \le \frac{x}{Km_1 \cdot 2^{\delta_0}}} \tau_r\left(2^{\alpha_k+\delta_0+\omega(m)}\right) \tau_r(Rm_1)\, \tau_r(m_2) =$$

$$= \sum_{\delta_o=0}^{1} \sideset{}{^{+}}\sum_{m_1|R} \tau_r(Rm_1) \times$$

$$\times \sideset{}{^{**}}\sum_{m_2 \le \frac{x}{Km_1 \cdot 2^{\delta_0}}} \binom{\alpha_k+\delta_0+\omega(m_1)+\omega(m_2)+r-1}{r-1} \tau_r(m_2),$$

where $**$ indicates that we sum over those square-free integers which are coprime to $2R$, $+$ indicates that m_1 runs over the square-free divisors of R.

Since the contribution of those m_2 for which $\omega(m_2) < \frac{1}{2}x_2$ is very small, and $\alpha_K + \delta_0 + \omega(m_1)$ is less than $\mathcal{O}(Y_x)$, say, therefore the binomial coefficient on the right hand side of (2.3) can be substituted by $\frac{\omega(m_2)^{r-1}}{(r-1)!}$.

Thus we have

$$T_K(x) = \sum_{\delta_o=0}^{1} \sideset{}{^{+}}\sum_{m_1|R} \tau_r(Rm_1) \sideset{}{^{**}}\sum_{m_2 \le \frac{x}{Km_1 \cdot 2^{\delta_0}}} \frac{\omega(m_2)^{r-1}}{(r-1)!} \tau_r(m_2) +$$

$$+\mathcal{O}\left(\sum_{m_1|R} \tau_r(Rm_1) \sideset{}{^{**}}\sum_{m_2 \le \frac{x}{Km_1}} \omega(m_2)^{r-2} \cdot \tau_r(m_2)\right).$$

Since

$$\sum_{(\nu,\mathcal{D})=1} \frac{\tau_r(\nu)\,|\mu(\nu)|}{\nu^s} = \prod_{p\nmid\mathcal{D}} \left(1+\frac{\tau_r(p)}{p^s}\right) = \prod_{p|\mathcal{D}} \frac{1}{1+\frac{r}{p^s}} \zeta^r(s)\, A_r(s),$$

where $A_r(s) = \prod_p \left(1+\frac{r}{p^s}\right)\left(1-\frac{1}{p^s}\right)^r$, $A_r(s)$ is bounded in the halfplane $\operatorname{Re} s > \frac{1}{2}+\varepsilon$, ($\varepsilon > 0$ constant), we can deduce that

$$\sum_{\substack{\nu \le x \\ (\nu,\mathcal{D})=1}} \tau_r(\nu)\,|\mu(\nu)| = (1+o_x(1)) \prod_{p|\mathcal{D}} \frac{1}{1+\frac{r}{p}} A_r(1)\, x \cdot x_1^{r-1}, \tag{2.4}$$

which is valid for $\mathcal{D} \le x^{1/3}$, say.

Let $\tilde\omega(n) := \sum_{\substack{p|n \\ p < x^{\frac{1}{10r}}}} 1$. Then $0 \le \omega(n) - \tilde\omega(n) \le 10r$, if $n \le x$, i.e.

$\omega(n)^{r-1} = \tilde{\omega}(n)^{r-1} + \mathcal{O}\left(\omega(n)^{r-2}\right)$. Hence

$$\frac{\omega(m_2)^{r-1}}{(r-1)!} = \sum_{\substack{p_1 \cdots p_{r-1} | m_2 \\ p_1 < \ldots < p_{r-1} < x^{\frac{1}{10r}}}} 1 + \mathcal{O}\left(\omega(m_2)^{r-2}\right). \tag{2.5}$$

Thus

$$\sum_{m_2 \le \frac{x}{Km_1 \cdot 2^{\delta_0}}}^{**} \frac{\omega(m_2)^{r-1}}{(r-1)!}\tau_r(m_2) = \sum_{m_2 \le \frac{x}{Km_1 \cdot 2^{\delta_0}}}^{**} \frac{\tilde{\omega}(m_2)^{r-1}}{(r-1)!}\tau_r(m_2) + \\ + \mathcal{O}\left(\frac{x}{Km_1} x_1^{r-1} \cdot x_2^{r-2}\right). \tag{2.6}$$

From (2.5) we obtain that the sum on the right hand side of (2.6) is

$$\sum_{\substack{p_1 < \ldots < p_{r-1} < x^{\frac{1}{10r}} \\ (p_1 \cdots p_{r-1}\nu, 2Km_1) = 1 \\ \nu \le \frac{x}{K \cdot m_1 \cdot 2^{\delta_0} p_1 \cdots p_{r-1}}}} \tau_r(p_1 \ldots p_{r-1}) \left|\mu(p_1 \ldots p_{r-1}\nu)\right| \tag{2.7}$$

which can be estimated by using (2.4). Thus (2.7) equals to

$$\sum_{\substack{p_1 < \ldots < p_{r-1} < x^{\frac{1}{10r}} \\ (p_1 \cdots p_{r-1}\nu, 2Km_1) = 1}} (1 + o_x(1))\, x x_1^{r-1} A_r(1) \times \\ \times \frac{\tau_r(p_1 \ldots p_{r-1})}{Km_1 \cdot 2^{\delta_o} p_1 \ldots p_{r-1}} \prod_{p | 2Kp_1 \ldots p_{r-1}} \frac{1}{1 + \frac{r}{p}}.$$

We can observe that

$$\sum_{\substack{p_1 < \ldots < p_{r-1} < x^{\frac{1}{10r}} \\ (p_1 \cdots p_{r-1}\nu, 2Km_1) = 1}} \prod_{j=1}^{r-1} \frac{\tau_r(p_j)}{p_j + (r+1)} = \frac{1 + o_x(1)}{(r-1)!} \left\{ \sum_{p < x^{\frac{1}{10r}}} \frac{\tau_r(p)}{p} \right\}^{r-1} = \\ = \frac{1 + o_x(1)}{(r-1)!} r^{r-1} \cdot x_2^{r-1}. \tag{2.9}$$

Collecting our estimates we obtain our theorem. ■

3. Proof of Lemma 3

The left hand side of (1.6) can be written as

$$\sum_{\sigma\in\mathcal{B}_{AC}}\tau\left(CA\sigma\right)\sum_{q\in\mathcal{I}}\sum_{\substack{A\sigma\mu\equiv 1 \pmod{Bq}\\ \mu\le x/A\sigma}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)$$

where

$$\chi_{AC}^{(0)}\left(n\right)=\begin{cases}1, & \text{if}\quad (n,AC)=1,\\ 0, & \text{if}\quad (n,AC)>1.\end{cases}$$

The contribution of $\sigma > x_1$ can be ignored. For fixed σ, $(\sigma, B)=1$, we can use the theorem of D. Wolke [8], according to

$$\sum_{q\in\mathcal{I}}\left|\sum_{\substack{\mu\le x/A\sigma\\ A\sigma\mu\equiv 1 \pmod{Bq}}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)-\frac{1}{\varphi\left(Bq\right)}\sum_{\substack{\mu\le x/A\sigma\\ (\mu,Bq)=1}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)\right|\ll\frac{x}{AC}x_1^{-20}.$$

Thus

$$\sum_{q\in\mathcal{I}}\sum_{\substack{\mu\le x/A\sigma\\ A\sigma\mu\equiv 1 \pmod{Bq}}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)=\frac{1}{\varphi\left(B\right)}\left(\sum_{q\in\mathcal{I}}\frac{1}{q-1}\right)\sum_{\substack{\mu\le x/A\sigma\\ (\mu,Bq)=1}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)+$$

$$+\mathcal{O}\left(\frac{x}{AC}x_1^{-20}\right)+\text{Error}.$$

$$\text{Error}\ll\sum_{q\in\mathcal{I}}\frac{1}{\varphi\left(B\right)q}\sum_{\substack{q^l m\le x/A\sigma\\ l\ge 1\\ (m,q)=1}}\tau\left(q^l m\right)\ll$$

$$\ll\sum_{q\in\mathcal{I}}\frac{1}{\varphi\left(B\right)}\left\{\sum_{l\ge 1}\frac{\tau\left(q^l\right)}{q^{l+1}}\frac{x}{A\sigma}x_1+\mathcal{O}\left(\sqrt{x}\right)\right\}\ll\frac{xx_1}{A\sigma\varphi\left(B\right)}1/x_1.$$

We can write

$$\sum_{\substack{\mu\le x/A\sigma\\ (\mu,Bq)=1}}\tau\left(\mu\right)\chi_{AC}^{(0)}\left(\mu\right)=\sum_{\mu\le x/A\sigma}\tau\left(\mu\right)\chi_{ABC}^{(0)}\left(\mu\right)$$

and the right hand side

$$=\frac{\varphi^2\left(ABC\right)}{\left(ABC\right)^2}\frac{\chi}{A\sigma}\{x_1+\mathcal{O}\left(1\right)\}.$$

Collecting our estimates, Lemma 3 easily follows. ∎

4. Proof of Theorem 2

Let t be a multiplicative function defined on prime powers p^{α}, $\alpha \geq 2$ to be $t(p^{\alpha}) = p^{\alpha}$. Furthermore let $t(2) = 2$, and $t(p) = 1$ if p is odd prime. It is clear that the set $\{t(n) \,|n \in \mathbb{N}\} = \mathcal{E}$ is the union of the set of square-full numbers and the twice of the square-full numbers. Let $e(n)$ be defined from the equation $= t(n)\, e(n)$. We say that $e(n)$ is the odd square-free part, and $t(n)$ be the quasi square-free part of n.

Let $K, L \in \mathcal{E}$,

$$\Sigma_{K,L} := \sum_{\substack{n \leq x \\ t(n-1)=K \\ t(n)=L \\ n \leq x}} \tau(n\tau(n-1)). \tag{4.1}$$

If the sum (4.1) is nonempty, then $(K, L) = 1$, $2|KL$. Let us write $n - 1 = = Km$, $n = L\nu$, where m is the odd square-free part of $n-1$, and ν is the odd square-free part of n. We have $(K, m) = 1$, $(L, \nu) = 1$.

Let $\tau^{(2)}(n) := \tau(\tau(n))$. Since $\tau(ab) \leq \tau(a) \cdot \tau(b)\,, \tau(a) < a$ holds for every $a, b \in \mathbb{N}$, therefore

$$\tau(n\tau(n-1)) \leq \tau(n)\, \tau^{(2)}(n-1) \leq \tau^2(n) + \tau^2(n-1)\,.$$

We shall prove that

$$\sum_{\max(K,L) > x_1^5} \Sigma_{K,L} = o_x(1)\, x x_1 x_2. \tag{4.2}$$

Indeed,

$$\sum_{K > x_1^5} \sum_{L} \sum_{m \leq x/K} \{\tau^2(m)\, \tau^2(K) + \tau^2(Km+1)\} = \Sigma_1 + \Sigma_2, \tag{4.3}$$

where in Σ_1 we sum over $K \in [x_1^5, x^{1/4}]$, and in Σ_2 over $K > x^{1/4}$. Σ_2 is small, since $\tau(m)\,, \tau(K)\,, \tau(Km+1)$ are less then $c_\varepsilon x^{\varepsilon/2}$, therefore $\Sigma_2 \ll \ll x^{\varepsilon} \sum_{K > x^{1/4}} 1/K \ll x^{0,9}$, say.

Since $\sum_{m \leq x/k} \tau^2(m) \ll \frac{x}{K} x_1^3$, and $\sum_{Km+1 \leq x} \tau^2(Km+1) \ll \frac{x}{K} x_1^3$ for $K \leq x^{1/4}$ (say), therefore

$$\Sigma_1 \ll x \cdot x_1^3 \sum_{x_1^5 \leq K \leq x^{1/4}} \frac{\tau^2(K)}{K}. \tag{4.4}$$

Since $\tau^2(K) \ll K^{\varepsilon}$, for an arbitrary small $\varepsilon > 0$, therefore the sum on the right hand side of (4.3) is less than $x_1^{-5/2+\varepsilon}$. Thus $\Sigma_1 \ll x \cdot x_1^{0,9}$ say, consequently (4.4) is less than $o_x(1)\, x \cdot x_1 x_2$.

We can overestimate the contribution of those n for which $L > x_1^5$, similarly. We omit the details.

Let $\gamma > 1/\log 2$ be a constant. Let $\mathcal{B}$ be the set of those $n \le x$ for which $\omega(m) > \gamma x_2$. We shall observe that the contribution of those integers to $T(x)$ for which $n \in \mathcal{B}$ is $o_x(1)\, x x_1 x_2$. Observe that $u/2^u$ is monotonically decreasing, therefore $\frac{\omega(m)}{2^{\omega(m)}} \le \frac{\gamma x_2}{2^{\gamma x_2}}$, if $\omega(m) \ge \gamma x_2$. Furthermore

$$\begin{aligned} \tau(n\tau(n-1)) &\le \tau(n)\,\tau^{(2)}(K)\,(\omega(m)+1) \le \\ &\le 2\tau(Km+1)\,\tau^{(2)}(K)\,\tau(m)\,\frac{\gamma x_2}{2^{\gamma x_2}} \le \\ &\le 2\tau(Km+1)\,\tau(Km)\,\frac{\gamma x_2}{2^{\gamma x_2}}. \end{aligned}$$

Thus

$$\sum_{n\in\mathcal{B}} \tau(n\tau(n-1)) \ll \frac{x_2}{2^{\gamma x_2}} \sum_{n\le x} \tau(n)\,\tau(n-1) \ll \frac{x \cdot x_1^2 x_2}{2^{\gamma x_2}}. \tag{4.5}$$

Since $x_1/2^{\gamma x_2} \ll x_1^{-\varepsilon}$, therefore we can drop the integers $n \in \mathcal{B}$.

Let $\Sigma_{K,L}^{(1)}$ be the sum of $\tau(n\tau(n-1))$ appearing in $\Sigma_{K,L}$, and additionally satisfying $\omega(m) \le \gamma x_2$. Then, we have $\tau(n\tau(n-1)) \le 2\gamma x_2 \tau(n)\,\tau^{(2)}(K)$, and so

$$\Sigma_{K,L}^{(1)} \le 2\gamma x_2 \tau^{(2)}(K) \sum_{\substack{\nu \le x/L \\ L\nu \equiv 1 \pmod{K}}} \tau(L\nu) \ll x_2 \tau^{(2)}(K)\,\tau(L)\,\frac{x}{KL} x_1.$$

Since $\sum_{K\in\mathcal{E}} \frac{\tau^{(2)}(K)}{K} < \infty$, $\sum_{L\in\mathcal{E}} \frac{\tau(L)}{L} < \infty$, therefore, if Y_x is tending to infinity, then

$$\sum_{\max(K,L)\ge Y_x} \Sigma_{K,L}^{(1)} = o_x(x x_1 x_2). \tag{4.6}$$

Now we assume that $K, L \le Y_x$, $(K,L) = 1$. Let $L\tau(K) = 2^{\beta} R$, R is odd. We have $2^{\beta} \le Y_x^2$. Then $n\tau(n-1) = 2^{\beta} R \cdot 2^{\omega(m)} \nu$, consequently

$$\tau(n\tau(n-1)) = (\omega(m)+\beta+1)\,\tau(R\nu). \tag{4.7}$$

Here we used that ν is odd. Thus

$$\Sigma_{K,L}^{(1)} = \sum \omega(m)\,\tau(R\nu) + \sum (\beta+1)\,\tau(R\nu) = \Sigma_{K,L}^{(2)} + \Sigma_{K,L}^{(3)}. \tag{4.8}$$

The second sum is less than

$$\ll (\beta+1)\,\tau(R) \sum_{\substack{\nu \le x/L \\ L\nu \equiv 1 (K)}} \tau(\nu) \ll (\beta+1)\,\frac{\tau(R)}{KL} x x_1.$$

Let us choose $Y_x \to \infty$ so that $Y_x = \mathcal{O}(x_3)$. We obtain that

$$\sum_{K,L < x_1^{5/2}} \Sigma_{K,L}^{(3)} = o_x(1)\, x x_1 x_2. \tag{4.9}$$

Let η be a small positive number, $\omega_1(n) = \sum_{\substack{p \mid n \\ p \in [x_1^2, x^\eta]}} 1$. Let

$$\Sigma_{K,L}^{(2,1)} = \sum \omega_1(m)\,\tau(R\nu). \tag{4.10}$$

Since $0 \le \omega(m) - \omega_1(m) \le 1/\eta$, therefore

$$\sum_{K,L} \left(\Sigma_{K,L}^{(2)} - \Sigma_{K,L}^{(2,1)}\right) \ll \left(\frac{1}{\eta} + x_3\right) x x_1.$$

It remains to estimate $\Sigma_{K,L}^{(2,1)}$.

Since $(\nu, 2) = 1$, therefore $\tau(R\nu) = \frac{1}{(\beta+1)} \tau(\tau(K)\, L\nu)$. Taking into account that $(\nu, \mu) = 1, \quad (K, m) = 1, \quad (L, \nu) = 1$, therefore

$$\Sigma_{K,L}^{(2,1)} = \sum_{q \in [x_1^2, x^\eta]} \sum_{\delta_1 \mid K} \sum_{\kappa_1 \mid L} \sum_{(\delta_2, K) = 1} \sum_{(\kappa_2, L) = 1} \mu(\delta_1)\,\mu(\delta_2)\,\mu(\kappa_1)\,\mu(\kappa_2) \times$$
$$\times U_q(\delta_1, \delta_2, \kappa_1, \kappa_2),$$

where

$$U_q(\delta_1, \delta_2, \kappa_1, \kappa_2) := \sum_{\substack{L\kappa_1\kappa_2^2 \equiv 1 \pmod{\delta_1 \delta_2^2 K q} \\ \nu \le \frac{x}{L\kappa_1\kappa_2^2}}} \tau\left(\tau(K)\,\kappa_1 L \kappa_2^2 \nu\right). \tag{4.11}$$

We have

$$U_q(\delta_1, \delta_2, \kappa_1, \kappa_2) \le \tau(\tau(K))\,\tau(\kappa_1)\,\tau(L)\,\tau\left(\kappa_2^2\right) \sum_{\substack{L\kappa_1\kappa_2^2 \equiv 1 \pmod{\delta_1 \delta_2^2 K q} \\ L\kappa_1\kappa_2^2 \le x}} \tau(\nu).$$

The sum on the right hand side is

$$\ll \frac{x x_1}{L\kappa_1\kappa_2^2\delta_1\delta_2^2 K q} \quad \text{if} \quad \max\left(L\kappa_1\kappa_2^2, \delta_1\delta_2^2 K q\right) \le x^{3/4},$$

and $\ll \frac{x^\varepsilon}{L\kappa_1\kappa_2^2 K q \delta_1 \delta_2^2}$ in general.

Thus

$$\sum_{x_1^2<q<x^\eta}\ \sum_{\max(\kappa_2,\delta_2)>Y_x} U_q(\delta_1,\delta_2,\kappa_1,\kappa_2) \ll$$
$$\ll \frac{\tau^2(K)\,\tau(L)\,xx_1x_2}{LK}\sum_{\kappa_1|L}\frac{\tau(\kappa_1)\,|\mu(\kappa_1)|}{\kappa_1}\sum_{\delta_1|K}\frac{|\mu(\delta_1)|}{\delta_1}\times$$
$$\times \sum_{\max(\kappa_2,\delta_2)>Y_x}\frac{1}{\kappa_2^2\delta_2^2}+\mathcal{O}\left(x^{0,9}\right)\ll$$
$$\ll \frac{1}{Y_x}\frac{\tau^2(K)\,\tau(L)\,xx_1x_2}{LK}\prod_{p|L}\left(1+\frac{2}{p}\right)\cdot\frac{K}{\varphi(K)}+\mathcal{O}\left(x^{0,9}\right).$$

Summing over all possible K, L the contribution of these sums is $o_x(1)\,xx_1x_2$. It remains to estimate the sums (4.11) under the condition $\max(\kappa_2,\delta_2)\le Y_x$. To estimate (4.10) let us write $A = L\kappa_1\kappa_2^2,\ B_q=\delta_1\delta_2^2Kq,\ C=\tau(K)$.

Let

$$H_q := \sum_{\substack{\nu\le Y\\ A\nu\equiv 1 \pmod{Bq}}}\tau(CA\nu),\quad \text{where } Y=\frac{x}{A}. \tag{4.12}$$

Then the right hand side of (4.11) equals to H_q. Let us write $\nu=\sigma\mu$, where $(\mu, CA)=1$, and all the prime factors of σ divide CA. Then

$$H_q=\sum_\sigma \tau(CA\sigma)\sum_{\substack{\mu\le Y/\sigma\\ (A\sigma)\mu\equiv 1 \pmod{Bq}\\ (\mu,CA)=1}} = \sum_\sigma \tau(CA\sigma)\,T_\sigma.$$

T_σ can be estimated by Lemma 3. Lemma 3 is valid if $K, L, \delta_1, \delta_2, \kappa_1, \kappa_2$ are fixed. Then there exists a suitable sequence $Y_x\to\infty$, such that it remains valid uniformly as $\max(K,L,\delta_1,\delta_2,\kappa_1,\kappa_2)\le Y_x$. Arguing as earlier, we can get that

$$\sum_{\substack{K,L\in\mathcal{E}\\ \max(K,L)>Y_x}}|\Sigma_{K,L}^{(2)}| + \sum_{\substack{K,L\in\mathcal{E}\\ \max(K,L)\le Y_x}}\ \sum_{q\in\left(x_1^2,x_2^\eta\right)}\ \sum_{\delta_1|K}\sum_{\kappa_1|L}\ \sum_{\substack{(\delta_2,K)=1\\ \max(\delta_1,\delta_2,\kappa_1,\kappa_2)\ge Y_x}}\times$$
$$\times\sum_{(\kappa_2,L)=1}U_q(\delta_1,\delta_2,\kappa_1,\kappa_2)=o(xx_1x_2).$$

Hence our theorem follows. ■

5. Proof of Theorem 3

The assertion is based on Lemma 1 and 2. Let $\varepsilon_x \to 0$ (slowly). We distinguish two cases:

(A) $3k < \varepsilon_x \cdot x_2$,

(B) $3k \geq \varepsilon_x \cdot x_2$.

In the case (B) let $\mathcal{I} = \left[x^{\frac{1}{3k}}, x^{\frac{1}{\varepsilon_x x_2}} \right]$. It is clear that for $n \leq x$,

$$\omega_o(n) := \sum_{\substack{p \mid n \\ p > x^{\frac{1}{\varepsilon_x \cdot x_2}}}} 1 \leq \varepsilon_x \cdot x_2. \tag{5.1}$$

Let $\omega_1(n) = \sum_{\substack{p \mid n \\ p < x^{\frac{1}{3k}}}} 1$, and in the case (A) let $\omega_2(n) = \sum_{\substack{p \mid n \\ p \in \mathcal{I}}} 1$.

Let $S_k^{(j)}(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} \tau(n-1)\, \omega_j(n+1) \quad (j = 0, 1, 2)$, where $S_k^{(2)}(x) = 0$ in the case (B).

From (5.1), by (1.7) we obtain that

$$S_k^{(0)}(x) \ll \varepsilon_x x_1 N_k(x). \tag{5.2}$$

Assume that we are in the case (A). We shall estimate $S_k^{(2)}(x)$.

From (1.7) we obtain that

$$S_k^{(2)}(x) \ll \delta_x x_2 x_1 N_k(x) + \Sigma_1, \tag{5.3}$$

where

$$\Sigma_1 = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k \\ \omega_2(n+1) \geq \delta_x \cdot x_2}} \tau(n-1)\, \omega_2(n+1). \tag{5.4}$$

Here we assume that $\delta_x \to 0$, slowly. Let n be counted in (5.4). Assume that $p_1 < \ldots < p_T$ are all the distinct prime divisors of $n+1$ located in $\mathcal{I}$. It is clear that $T \geq [\delta_x x_2]$. Let $Q = p_1 \ldots p_T = Q_1 Q_2$, where $Q_1 = p_1 \ldots p_{[T/2]}$. Since $Q_1 < Q_2$, $Q \leq x$, therefore $Q_1 \leq \sqrt{x}$. Furthermore $\omega(Q) \leq 3\omega(Q_1)$. Consequently

$$\Sigma_1 \leq \sum_{\substack{\omega(Q_1) \geq \frac{\delta_x x_2}{3} \\ Q_1 \leq \sqrt{x}}} \omega(Q_1) \sum_{\substack{n \leq x \\ n+1 \equiv 0 \pmod{Q_1}}} \tau(n-1). \tag{5.5}$$

It is known (see [11]) that

$$\sum_{En+R\le x}\tau\left(En+R\right)\ll\frac{xx_1}{E},\tag{5.6}$$

uniformly as $(1\le)\,E\ll x^{1-\delta}$, $0<R<E$, $(E,R)=1$, the constant implied by $\ll$ may depend on δ. From (5.5), (5.6) we deduce that

$$\Sigma_1\ll xx_1\sum_{\omega(Q_1)\ge j_0}\frac{\omega\left(Q_1\right)}{Q_1},\tag{5.7}$$

$j_0=\frac{\delta_x\cdot x_2}{3}$. Let

$$U:=\sum_{p\in\mathcal{I}}1/p\le\log\frac{1/\varepsilon_x x_2}{1/3k}+1\le\log\frac{6e}{\varepsilon_x}=\tau_x.\tag{5.8}$$

We may assume that $\tau_x\to\infty$ arbitrarily slowly, if ε_x has been chosen appropriately to tend to 0.

It is clear that

$$\frac{\omega\left(Q_1\right)}{Q_1}=\sum_{p|Q_1}\frac{1}{p}\cdot\frac{1}{\left(Q_1/p\right)},$$

consequently

$$\sum_{\omega(Q_1)\ge j_0}\frac{\omega\left(Q_1\right)}{Q_1}=\sum_{p\in\mathcal{I}}\frac{1}{p}\sum_{\omega(Q_3)\ge j_0-1}\frac{1}{Q_3},\tag{5.9}$$

where Q_3 run over the square free integers all prime factors of which belong to $\mathcal{I}$. Then the right hand side of (5.9) is less than

$$U\cdot\sum_{l=j_0-1}\frac{1}{l!}U^l\le\frac{cU^{j_0}}{(j_0-1)!}\quad(=:M)\,.$$

Observe that

$$\log M\le\log c+j_0\log U-j_0/2\log j_0\le-\frac{\varepsilon_x x_2x_3}{3},$$

if x is large enough, i.e.

$$M\ll\exp\left(-\frac{\varepsilon_x x_2x_3}{3}\right).$$

Hence, and from (5.7) we obtain that

$$\Sigma_1\ll o_x\left(1\right)x_1N_k\left(x\right)$$

uniformly as $k\le\left(2-\xi\right)x_2$.

To complete the proof of Theorem 3 it remains to show that $S_k^{(1)}(x)$ asymptotically equals to the right hand side of (1.6). This can be done by applying the method of Timofeev and Khripunova.

We have

$$S_k^{(1)}(x) = \sum_{p<x^{1/3k}} A_p(x), \tag{5.10}$$

where

$$A_p(x) = \sum_{\substack{n\le x\\ n\in\mathcal{N}_k\\ n+1\equiv 0 \pmod p}} \tau(n-1). \tag{5.11}$$

We have

$$\begin{aligned} A_p(x) = 2\sum_{u\le\sqrt{x}} \#\{n\equiv 1 \pmod u, u^2<n<x, n\equiv -1 \pmod p\}+\\ +O(\#\{n\le x | n-1=\text{square}\}). \end{aligned} \tag{5.12}$$

As in [4] we can drop the contribution of the error term, and even those integers which are counted for $u \ge \sqrt{x}\exp\left(-x_2^4\right)$. For the summands for $u<\sqrt{x}\exp\left(-x_2^4\right)$ we can apply Lemma 1:

$$\begin{aligned} \#\{n\equiv 1 \pmod{u^2<n<x}, n\equiv -1 \pmod p\} =\\ = \#\{n\equiv l_{u,p} \pmod{pu}, n\le x\} - \#\{n\equiv l_{u,p} \pmod{pu}, n<u^2\} \end{aligned}$$

if $(u,p)=1$, where $l_{u,p}$ is determined from $n\equiv 1 \pmod u$, $n\equiv -1 \pmod p$.

$$A_p(x) = 2\sum_{u\le\sqrt{x}} B_u(x) + \mathcal{O}(\#\{n\le x | n-1=\text{square}\}), \tag{5.13}$$

$$B_u(x) = \#\{n\in\mathcal{N}_k, n\equiv -1 \pmod p, n\equiv 1 \pmod u, u^2<n<x\}. \tag{5.14}$$

As in [4] we can drop $\sum\limits_{u>\sqrt{x}\exp\left(-x_2^4\right)} B_u(x)$. The contribution of the error term is small, $\ll\sqrt{x}$. The contribution of $A_2(x)$ is not larger than (1.7). Let $p>2$. If $B_u(x)\ne 0$, then $(u,p)=1$. For such a pair let $l_{u,p}$ be the residue $\pmod{pu}$ such that $n\equiv 1 \pmod u$, $n\equiv -1 \pmod p$. We have

$$\begin{aligned} B_u(x) = \#\{n\in\mathcal{N}_k, n\equiv l_{u,p} \pmod{pu}, n\le x\}-\\ - \#\{n\in\mathcal{N}_k, n\equiv l_{u,p} \pmod{pu}, n<u^2\} =\\ = \mu(x,k,2,l_{u,p},pu) - \mu\left(u^2,k,2,l_{u,p},pu\right). \end{aligned}$$

Applying Lemma 1 and Lemma 2, as in [4], we obtain the theorem. We do not give the details. ■

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