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THE LAW OF LARGE NUMBERS WITH RESPECT TO EWENS PROBABILITY

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Dedicated to Professor Karl-Heinz Indlekofer on his 70th birthday

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Abstract. We explore the value distribution problem of sequences of additive functions defined on the symmetric group endowed with the Ewens probability measure. Necessary and sufficient conditions for the weak law of large numbers are obtained if the parameter is not less than one. The result can be applied to linear statistics of a random permutation matrix.

1. Introduction and the result

Throughout the paper, $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ will denote the sets of natural, integer, real, complex numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $n, i, j, \in \mathbb{N}$ while $k, k_i, s, s_i \in \mathbb{Z}_+$. For a vector $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n$, we set $\ell(\bar{s}) = 1s_1 + \cdots + ns_n$.

We deal with asymptotic value distribution problems of mappings defined on the symmetric group $\mathbb{S}_n$. Let $\sigma \in \mathbb{S}_n$ be an arbitrary permutation and $\sigma = \varkappa_1 \cdots \varkappa_w$ be its representation as the product of independent cycles $\varkappa_i$ and $w := w(\sigma)$ be their number. If $k_j(\sigma)$, $1 \le j \le n$, denotes the number of cycles of length j in this decomposition, then $\bar{k}(\sigma) := \big(k_1(\sigma), \ldots, k_n(\sigma)\big)$ is called the cycle structure vector. The Ewens probability measure on the subsets $A \subset \mathbb{S}_n$ is defined by

$$\nu_n(A) := \nu_{n,\theta}(A) = \frac{1}{\theta^{(n)}} \sum_{\sigma \in A} \theta^{w(\sigma)},$$

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where $\theta > 0$ is a fixed parameter and $\theta^{(n)} := \theta(\theta+1)\cdots(\theta+n-1)$. One easily finds the distribution of the cycle structure vector, namely,

$$\nu_n\big(\bar{k}(\sigma)=\bar{s}\big) = \frac{n!}{\theta^{(n)}}\prod_{j=1}^{n}\Big(\frac{\theta}{j}\Big)^{s_j}\frac{1}{s_j!}, \tag{1.1}$$

where $\bar{s}=(s_1,\dots,s_n)\in\mathbb{Z}_+^n$ and $\ell(\bar{s})=n$. The probabilities on the right-hand side of (1.1) define the so-called Ewens Sampling Formula on $\ell^{-1}(n)$ which is very important in the statistical problems (see, for instance, [1]). If ξ_j, $j\ge 1$, denote independent Poisson r.vs given on some probability space $\{\Omega,\mathcal{F},P\}$ with $\mathbf{E}\xi_j=\theta/j$ and $\bar{\xi}:=(\xi_1,\dots,\xi_n)$, then

$$\nu_n\big(\bar{k}(\sigma)=\bar{s}\big) = P\big(\bar{\xi}=\bar{s}\big|\,\ell(\bar{\xi})=n\big), \quad \bar{s}\in\ell^{-1}(n). \tag{1.2}$$

It is known [1] that

$$\big(k_1(\sigma),\dots k_n(\sigma),0,\dots\big)\Rightarrow\big(\xi_1,\dots,\xi_n,\xi_{n+1},\dots\big)$$

in the sense of convergence of finite dimensional distributions. Here and in what follows we assume that $n\to\infty$. We have even more. The total variation distance

$$\frac{1}{2}\sum_{s_1,\dots,s_r\ge 0}\Big|\nu_n\big(k_1(\sigma)=s_1,\dots,k_r(\sigma)=s_r\big)-P(\xi_1=s_1,\dots,\xi_r=s_r)\Big| = o(1) \tag{1.3}$$

if and only if $r=o(n)$. This and more precise results of the remainder in terms of r/n can be found in [1].

By definition, an additive (completely additive) function $h:\mathbb{S}_n\to\mathbb{R}$ is defined via a real array $\{a_j, j\ge 1\}$, by setting

$$h(\sigma) := \sum_{j=1}^{n} a_j k_j(\sigma). \tag{1.4}$$

If $a_j=a_{nj}$, $1\le j\le n$, depends on n, we obtain a sequence of functions but we will not add the index n, where no misunderstanding arises. Dealing with the distribution $\nu_n\circ h^{-1}$ we may also assume that $a_j=0$ if $j>n$. The main problem in the field is the following question:

Under what conditions posed on the array $\{a_j:=a_{nj}\}$, $1\le j\le n$, *there exist a centralizing sequence* $\alpha(n)\in\mathbb{R}$ *and a distribution function* $F(x)$ *such that* $\nu_n\big(h(\sigma)-\alpha(n)<x\big)$ *converges to* $F(x)$ *at the points of continuity of the latter* (*weakly converges*)?

Unfortunately, so far we could not afford to establish necessary and sufficient conditions even for $\theta=1$. The most general results have been obtained by the second author (see, e.g., [12], [13], and the references therein). In [11], he has been more successful for $\theta = 1$ in the case of the weak law of large numbers (then $F(x) = F_0(x)$ is the degenerate at the zero point distribution function). We now extend this result for $\theta \geq 1$.

In virtue of (1.3), one could expect that the conditions are close to that for the sums of independent r.vs $X_j := a_j\xi_j$, $j \leq n$. The instance of $\lambda\ell(\bar{k}(\sigma)) \equiv \lambda n$, with an arbitrary sequence $\lambda := \lambda_n \in \mathbb{R}$ shows that this is not the case, however. This sequence of functions obeys the degenerated limit law at the zero point if centralized by λn, while the corresponding sum of X_j does not in general. This shows that an additive function can have a deterministic summand to be extracted in the first step of the problem solving. If we are successful in doing this, the difference

$$h(\sigma;\lambda) := h(\sigma) - \lambda\ell(\sigma) = \sum_{j=1}^{n} a_j(\lambda)k_j(\sigma),$$

where $a_j(\lambda) := a_j - \lambda j$, demonstrates closer behavior in some stochastic sense to the sums of independent r.vs $a_j(\lambda)\xi_j$, $j \leq n$. That have been established to be true if permutations are taken with equal probabilities or even according to a generalized Ewens measure, provided that the influence of long cycles is negligible (see [13], [5]). If the latter does not hold and $\theta \neq 1$, more bias appears. As it is seen in the below formulated result, this gives an extra factor $(1 - j/n)^{\theta-1}$ in the conditions. Firstly, we present a quantitative result.

Define the Lévy distance of the random variable $h(\cdot)$ from the set of constants

$$L(h;\nu_n) := \inf\big\{\varepsilon + \nu_n(|h(\sigma) - a| \geq \varepsilon)\colon\ a \in \mathbb{R}, \varepsilon > 0\big\}.$$

Denote

$$\psi_n(m) = \frac{\theta^{(m)}}{m!}\frac{n!}{\theta^{(n)}}$$

if $m \in \mathbb{Z}_+$. Let $u \vee v := \max\{u, v\}$, $u \wedge v := \min\{u, v\}$, $u^* := (1 \wedge |u|)\operatorname{sgn} u$ if $u, v \in \mathbb{R}$,

$$U_n(h,\lambda) := \sum_{j\leq n} \frac{\theta}{j}\big(a_j(\lambda)\big)^{*2}\psi_n(n-j)$$

and $U_n(h) = \min\{U_n(h,\lambda)\colon\ \lambda \in \mathbb{R}\}$. In the sequel, $\ll$ is used as an analog of $\mathrm{O}(\cdot)$, moreover, dependence on θ in the involved constants is allowed.

Theorem 1.1. *If $\theta \geq 1$, then*

$$L(h;\nu_n) \leq 2\big(1 \wedge (2U_n(h))^{1/3}\big)$$

and

$$U_n(h) \ll (1/n) \vee L(h;\nu_n)$$

for all $n \geq 1$.

We now give the answer to the above question in the case of the degenerate limit distribution modifying a bit the conditions of the theorem.

Corollary 1.1. *Let $\theta \ge 1$ and $h_n(\sigma)$ be a sequence of additive functions on $\mathbb{S}_n$ defined via $\{a_j = a_{nj}\}$, $j \le n$, in (1.4). The distributions $\nu_n\big(h_n(\sigma) - \alpha(n) < < x\big)$ converge to $F_0(x)$ if and only if*

$$\sum_{j<n} \frac{a_{nj}(\lambda)^{*2}}{j}\Big(1-\frac{j}{n}\Big)^{\theta-1} = \mathrm{o}(1)$$

for some $\lambda = \lambda_n \in \mathbb{R}$ and

$$\alpha(n) = n\lambda + \sum_{\substack{j<n\\ |a_{nj}|<1}} \frac{\theta a_{nj}(\lambda)}{j}\Big(1-\frac{j}{n}\Big)^{\theta-1} + \mathrm{o}(1).$$

We now present some motivation. In the last decade, much attention was paid to the random permutation matrix ensemble with the Ewens probability measure endowed in it. Let $M := M(\sigma) := \big(\mathbf{1}\{i = \sigma(j)\}\big)$, $1 \le i, j \le n$ and $\sigma \in \mathbb{S}_n$, be such a matrix taken with the weighted frequency $\nu_n(\{M\}) = = \nu_n(\{\sigma\}) = \theta^{w(\sigma)}/\theta^{(n)}$,

$$Z_n(x;\sigma) := \det\big(I - xM(\sigma)\big) = \prod_{j\le n}(1-x^j)^{k_j(\sigma)}, \quad x \in \mathbb{C}, \tag{1.5}$$

be its characteristic polynomial, and let $\mathrm{e}^{2\pi i \varphi_j(\sigma)}$, where $\varphi_j(\sigma) \in [0,1)$ and $j \le n$, be its eigenvalues. The papers [6], [16], and [17] or many preprints put in the AMS arXiv (see, for instance, [2] and [7] and the references therein) concern $\log|Z_n(x;\sigma)|$, $\Im \log Z_n(x;\sigma)$ or the linear statistics

$$\mathrm{Trf}(\sigma) := \sum_{j\le n} f\big(\varphi_j(\sigma)\big) = \sum_{j\le n} k_j(\sigma) \sum_{0\le s\le j-1} f\Big(\frac{s}{j}\Big),$$

where $f : [0,1] \to \mathbb{R}$ is a sufficiently smooth function. The last relation, easily seen from (1.5), is present in [2]. A great portion of the newly announced results fall within the scope of the above formulated problem. Nevertheless, the authors seldom observe this and prefer to rediscover them for the particular statistics. In this regard, apart from papers [3], [11], [12], [13], [5], and others implicitly containing limit theorems for such statistics, the present note can be also of use.

In particular, the trace type statistics $\mathrm{Trf}(\sigma)$ is an additive function. The above mentioned its shift by λn naturally appears as an approximation by integral of the sum. To see this, it suffices to set

$$\sum_{0\le s\le j-1} f\Big(\frac{s}{j}\Big) =: j\int_0^1 f(u)\mathrm{d}u + a_j =: j\lambda + a_j.$$

If f is a fixed function, $\alpha(n) \in \mathbb{R}$, and $\beta(n) \to \infty$, chosen so that the conditions of Corollary 1.1 are satisfied with $a_{nj} := a_j/\beta(n)$ we obtain the weak law of large numbers for $\big(\mathrm{Trf}(\sigma) - \alpha(n)\big)/\beta(n)$. As we have stressed sequences of functions $f = f_n$ can also be involved.

The proof of Theorem 1.1 is based upon the number-theoretical ideas originated by I.Z.Ruzsa in [15] and already adopted in probabilistic combinatorics by the second author in the case $\theta = 1$ (see [11]). The obstacles arising for $\theta < 1$ will be discussed at the end of the paper.

2. Lemmata

We start with an estimate of the variance

$$\mathbf{V}\mathrm{ar}_n h(\sigma) = \mathbf{E}_n h(\sigma)^2 - \big(\mathbf{E}_n h(\sigma)\big)^2$$

with respect to the Ewens probability measure ν_n. Denote

$$x_{(m)} = x(x-1)\cdots(x-m+1) \qquad \text{and} \qquad x_{(0)} = 1.$$

Lemma 2.1. *For arbitrary $j, n \in \mathbb{N}$, $s_i \in \mathbb{Z}_+$, $i \le j$, and $q := 1s_1 + \dots + rs_r$, we have*

$$\mathbf{E}_n\Big(\prod_{i\le j} k_i(\sigma)_{(s_i)}\Big) = \psi_n(n-q)\prod_{i\le j}\Big(\frac{\theta}{i}\Big)^{s_i}.$$

Proof. See [1], p. 96. ■

Lemma 2.2. *If $\theta \ge 1$, then*

$$\mathbf{V}\mathrm{ar}_n h(\sigma) \le 2\theta \sum_{j\le n} \frac{a_j^2}{j}\psi_n(n-j) =: 2B_n^2(h). \tag{2.1}$$

Proof. Let x^+ denote the nonnegative part of $x \in \mathbb{R}$ and $x^- := x^+ - x$. The sequences $\{a_j^+\}$ and $\{a_j^-\}$, $1 \le j \le n$, give the splitting $h(\sigma) = h^+(\sigma) - h^-(\sigma)$, where $h^\pm(\sigma)$ are the completely additive functions defined via $a_j^\pm$ respectively. Thus, by virtue of $(x+y)^2 \le 2x^2 + 2y^2$, it suffices to prove that $\mathbf{V}\mathrm{ar}_n h(\sigma) \le$ $\le B_n^2(h)$ in the case $a_j \ge 0$ for all $j \le n$.

Lemma 2.1 yields

$$\mathbf{E}_n k_j(\sigma) = \frac{\theta a_j}{j}\psi_n(n-j), \qquad \mathbf{E}_n h(\sigma) = \theta\sum_{j\le n}\frac{a_j}{j}\psi_n(n-j), \tag{2.2}$$

and

$$\begin{aligned}
\mathbf{E}_n h(\sigma)^2 &= \sum_{i,j\le n} a_i a_j \mathbf{E}_n\big(k_i(\sigma)k_j(\sigma)\big) = \\
&= \sum_{j\le n} a_j^2\Big(\mathbf{E}_n k_j(\sigma) + \mathbf{E}_n k_j(\sigma)_{(2)}\Big) + \theta^2 \sum_{\substack{i+j\le n\\ i\ne j}} \frac{a_i a_j}{ij}\psi_n(n-i-j) = \\
&= B_n^2(h) + \sum_{i+j\le n} \frac{a_i a_j}{ij}\psi_n(n-i-j).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{V}\mathrm{ar}_n h(\sigma) &= B_n^2(h) + \sum_{i+j\le n} \frac{a_i a_j}{ij}\Big(\psi_n(n-i-j) - \psi_n(n-i)\psi_n(n-j)\Big) - \\
&\quad - \sum_{\substack{i+j> n\\ i,j\le n}} \frac{a_i a_j}{ij}\psi_n(n-i)\psi_n(n-j)
\end{aligned}$$

for $\theta > 0$.

If $\theta \ge 1$, we have $\psi_n(n-i-j) \le \psi_n(n-i)\psi_n(n-j)$. Recalling that $a_j \ge 0$, $j \le n$, we can omit negative terms and obtain the desired claim $\mathbf{V}\mathrm{ar}_n h(\sigma) \le B_n^2(h)$.

The lemma is proved. ■

Remark 2.1. It is worth to recall two results showing the quality of the constant in (2.1). Denote

$$\tau_n(\theta) = \sup\big\{\mathbf{V}\mathrm{ar}_n h(\sigma)\big/B_n^2(h):\ h(\sigma)\not\equiv 0\big\}.$$

We have that $\tau_n(1) = 3/2 + \mathrm{O}(n^{-1})$ and $\tau_n(2) = 4/3 + \mathrm{O}(n^{-1})$ (see [10] and [14].

In the sequel $J \subset \{j : j \le n\}$ will be an arbitrary nonempty set, maybe, depending on n, and $\overline{J} = \{j : j \le n\} \setminus J$.

Lemma 2.3. *Let $\theta \ge 1$, $K > 0$, and J be such that*

$$\sum_{j\in J} \frac{1}{j} \le K < \infty. \tag{2.3}$$

Denote

$$\mu_n(K) = \inf_J\ \nu_n\big(k_j(\sigma) = 0\ \forall\, j \in J\big),$$

where the infimum is taken over J satisfying (2.3). For a sufficiently large $n_0(K)$, there exists a positive constant $c(K)$, depending at most on θ and K, such that $\mu_n(K) \ge c(K)$ if $n \ge n_0(K)$.

Moreover, for any $I \subset J \cap [1, n - n_0(K)]$ *and*

$$\widetilde{S}_n := \bigcup_{j\in I} S_n^j := \bigcup_{j\in I} \Big\{\sigma \in \mathbb{S}_n : k_j(\sigma) = 1,\ k_i(\sigma) = 0\ \forall i \in J \setminus \{j\}\Big\},$$

we have that

$$\nu_n(\widetilde{S}_n) \geq c(K) \sum_{j\in I} \frac{1}{j}\psi_n(n-j) \tag{2.4}$$

provided that $n \geq 2n_0(K)$.

Proof. The first claim is Corollary 1.3 of Theorem 1.2 (see in [9]).

To prove the second one, we apply (1.2) and obtain

$$\nu_n(S_n^j) = P\Big(\xi_j = 1,\, \xi_i = 0\ \forall i \in J \setminus \{j\} \Big|\, \ell(\bar\xi) = n\Big).$$

Set $p(m) = P\big(\ell(\bar\xi) = m\big)$ for $0 \leq m \leq n$. Then

$$p(n)\nu_n(S_n^j) = \frac{\theta}{j} P\Bigg(\xi_i = 0\ \forall i \in J \mid \sum_{\substack{i\notin J\\ i\leq n}} i\xi_i = n - j\Bigg).$$

Denote $J_m := J \cap [1; m]$ and $\overline{J}_m := \{j : j \leq m\} \setminus J_m$ for $0 \leq m \leq n$. Observing that $\ell(\bar\xi) = n - j$ implies $\xi_i = 0$ for each $n - j < i \leq n$, we obtain

$$\begin{aligned} p(n)\nu_n(S_n^j) &= \frac{\theta}{j} \exp\Big\{-\theta \sum_{n-j<i\leq n} \frac{1}{i}\Big\} \times \\ &\qquad \times P\Bigg(\xi_i = 0\ \forall i \in J_{n-j},\ \sum_{i\leq n-j} i\xi_i = n - j\Bigg) = \\ &= \frac{\theta}{j} \exp\Big\{-\theta \sum_{n-j<i\leq n} \frac{1}{i}\Big\} p(n-j)\nu_{n-j}\big(k_i(\sigma) = 0\ \forall i \in J_{n-j}\big). \end{aligned}$$

Here we again applied (1.2) for the symmetric group $\mathbb{S}_{n-j}$. By Cauchy's equality,

$$\begin{aligned} p(n) &= P\big(\ell(\bar\xi) = n\big) = \sum_{\substack{s_1,\dots,s_n\geq 0\\ \ell(\bar s)=n}} \prod_{i\leq n} e^{-\theta/i}\Big(\frac{\theta}{i}\Big)^{s_i} \frac{1}{s_i!} = \\ &= \frac{\theta^{(n)}}{n!} \exp\Big\{-\sum_{i\leq n} \frac{\theta}{i}\Big\}. \end{aligned}$$

Inserting this into the previous equality and using Lemma 2.3, we obtain

$$\nu_n(S_n^j) \geq c(K)\frac{\theta}{j}\psi_n(n-j)$$

provided that $n - j \geq n_0(K)$.

The sets S_n^j for $j \in I$ are pairwise disjoint, therefore summing up over $j \in I$ the latter inequalities, we complete the proof of the lemma. ■

The next lemma concerns the concentration function

$$Q_n(u) := \sup\big\{\nu_n\big(|h(\sigma) - x| < u\big) : \ x \in \mathbb{R}\big\}, \quad u \geq 0.$$

Denote

$$D_n(u) = \min\bigg\{\sum_{j\leq n}\frac{u^2 \wedge a_j^2(\lambda)}{j} : \ \lambda \in \mathbb{R}\bigg\}.$$

Lemma 2.4. *For arbitrary $\theta > 0$, we have*

$$Q_n(u) \ll u\big(D_n(u)\big)^{-1/2}.$$

Proof. This is Lemma 4 in [8]. ■

3. Proof of Theorem 1.1

The upper estimate. Recall that $\ell(\bar{k}(\sigma)) = n$ for each $\sigma \in \mathbb{S}_n$. Hence $L(h;\nu_n) = L(h - \lambda n;\nu_n)$ for every $\lambda \in \mathbb{R}$. Without loss of generality, we further assume that $\lambda = 0$ and set $U_n(h, 0) = U_n(h) =: \delta$. If $\delta = 0$, then $a_j = 0$ for each $j \leq n$ and $L(h, \nu_n) = 0$. If $\delta \geq 1/2$, the trivial upper bound in Theorem 1.1 holds. It remains the case with $0 < \delta < 1/2$.

Define

$$h'(\sigma) = \sum_{j\leq n} a_j \mathbf{1}\{|a_j| < 1\}k_j(\sigma), \qquad h''(\sigma) = h(\sigma) - h'(\sigma).$$

Observe that, by virtue of (2.2),

$$\begin{aligned}
\nu_n\big(h''(\sigma) \neq 0\big) \quad &\leq \quad \sum_{j\leq n}\mathbf{1}\{|a_j| \geq 1\}\nu_n\big(k_j(\sigma) \geq 1\big) \leq \\
&\leq \quad \sum_{j\leq n}\mathbf{1}\{|a_j| \geq 1\}\mathbf{E}_n k_j(\sigma) \leq \\
&\leq \quad \sum_{j\leq n}\mathbf{1}\{|a_j| \geq 1\}\frac{\theta}{j}\psi_n(n-j).
\end{aligned}$$

Lemma 2.2 implies

$$\nu_n\big(\big|h'(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)\leq 2\varepsilon^{-2}B_n^2(h').$$

Now,

$$\nu_n\big(\big|h(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)\leq \nu_n\big(\big|h'(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)+\nu_n\big(h''(\sigma)\neq 0\big)\leq$$
$$\leq 2\varepsilon^{-2}B_n^2(h')+\sum_{j\leq n}\mathbf{1}\{|a_j|\geq 1\}\frac{\theta}{j}\psi_n(n-j)\leq 2\varepsilon^{-2}U_n(h).$$

For $\varepsilon=(2\delta)^{1/3}$ we achieve the minimum of $\varepsilon+2\varepsilon^{-2}\delta$. Thus, $L(h;\nu_n)\leq 2(2\delta)^{1/3}$ in the case $0<\delta<1/2$. Recalling the previous trivial bound we complete the upper estimation in Theorem 1.1.

The lower estimate. If $L(h;\nu_n)\geq c>0$ for some constant c, the task is trivial. Now, let $\delta:=2L(h;\nu_n)<c$ for a constant $c<1/2$ to be chosen later. We have that

$$\nu_n(|h(\sigma)-a|\geq\delta)\leq\delta$$

for some $a\in\mathbf{R}$ and

$$Q_n(\delta)\geq\nu_n(|h(\sigma)-a|<\delta)\geq 1-\delta\geq 1/2.$$

Hence, by Lemma 2.4,

$$\sum_{j\leq n}\frac{a_j^2}{j}\mathbf{1}\{|a_j|<\delta\}\leq C\delta^2,\qquad \sum_{j\leq n}\frac{\mathbf{1}\{|a_j|\geq\delta\}}{j}\leq C. \tag{3.1}$$

Here we would have used $a_j(\lambda)$ instead of $a_j=a_j(0)$ for some $\lambda\in\mathbb{R}$. Justifying this simplification, we recall that $L(h;\nu_n)=L(h-\lambda n;\nu_n)=\delta/2$ for every λ; therefore, we could further deal with the shifted function $h(\sigma,\lambda)$. Thus, taking $\lambda=0$ had no effect on the generality. Afterwards, having in mind that $\psi_n(n-j)\leq 1$ if $\theta\geq 1$, we will include this quantity as a factor of the summands in (3.1).

Set $\hat a_j=a_j$ if $|a_j|<\delta$ and $\hat a_j=0$ otherwise, and denote $\check a_j=a_j-\hat a_j$ for $j\leq n$. Further, define, as in (1.4), two completely additive functions $\hat h(\sigma)$ and $\check h(\sigma)$ via $\hat a_j$ and $\check a_j$ respectively.

We now use Lemma 2.3 with $J=\{j\leq n:|\check a_j|\geq\delta\}$, $K=C$, and

$$I=\big\{1\leq j\leq n-n_0(C),|\check a_j|\geq\sqrt{\delta}\big\},$$

where $n>2n_0(C)$. If $\widetilde S_n$ is defined as in Lemma 2.3, then

$$\nu_n(\widetilde S_n)\geq c_1\sum_{j\leq n-n_0(C)}\frac{1}{j}\psi_n(n-j)\mathbf{1}\{|a_j|\geq\sqrt{\delta}\}=:c_1\alpha.$$

The completion of this sum over $n-n_0(C)<j\le n$ would contribute not more than C_2/n with some $C_2>1$ for $n\ge 2n_0(C)$. Hence if $\alpha\le M\delta$, where $M\ge 1$ is arbitrary, then taking into account the first estimate in (3.1) with $\sqrt{\delta}$ instead of δ, we had the desired claim in the form

$$U_n(h)\le \theta(C\delta+M\delta+C_2n^{-1})\ll n^{-1}\vee\delta.$$

Since now we assume that $\alpha\ge M\delta$, where $M>c_1^{-1}$ is a constant to be chosen later. This gives $\nu_n(\widetilde{S}_n)\ge c_1M\delta$. Further we examine the values of the additive functions when $\sigma\in\widetilde{S}_n$. If $\sigma\in S_n^j$, then $\check{h}(\sigma)=a_j$, where $|a_j|\ge\sqrt{\delta}$. So, $|\check{h}(\sigma)|\ge\sqrt{\delta}$ for each $\sigma\in\widetilde{S}_n$. Hence, if $\sigma\in\widetilde{S}_n$ and $|h(\sigma)-a|<\delta$, then $|\hat{h}(\sigma)-a|\ge\sqrt{\delta}-\delta$ and

$$\begin{aligned}\nu_n\big(|\hat{h}(\sigma)-a|\ge\sqrt{\delta}-\delta\big) &\ge \nu_n\big(\sigma\in\widetilde{S}_n\big)-\nu_n\big(|h(\sigma)-a|\ge\delta\big)\ge\\ &\ge (c_1M-1)\delta. \end{aligned}\tag{3.2}$$

Denote

$$\widehat{S}_n=\big\{\sigma\in\mathbb{S}_n:\ k_j(\sigma)=0\ \forall\, j\in J\big\}.$$

By Lemma 2.3, we also have $\nu_n(\widehat{S}_n)\ge c_2>0$ if $n>n_0(C)$.

Hence and the fact that $h(\sigma)=\hat{h}(\sigma)$ if $\sigma\in\widehat{S}_n$ we obtain

$$\begin{aligned}\nu_n(|\hat{h}(\sigma)-a|<\delta) &\ge \nu_n(\sigma\in\widehat{S}_n:\ |h(\sigma)-a|<\delta)\ge\\ &\ge c_2-\nu_n(|h(\sigma)-a|\ge\delta)\ge c_2-\delta\ge c_2/2\end{aligned}\tag{3.3}$$

if $\delta<c\le c_2/2$, where, as we have agreed, the choice of c is at our disposition.

It is known (see, e.g. [11]) that, for a real random variable X, we have that $\mathbf{V}\mathrm{ar}X\ge 1/2p_1p_2d^2$ if $P(X\in A)\ge p_1$, $P(X\in B)\ge p_2$, and

$$d=\inf\{|x-y|:\ x\in A, y\in B\},$$

where $A,B\subset\mathbb{R}$. This, (3.2), and (3.3) yield

$$\mathbf{V}\mathrm{ar}_n\hat{h}\ge(1/4)(c_1M-1)c_2\delta(\sqrt{\delta}-2\delta)^2\ge(1/16)(c_1M-1)c_2\delta^2$$

if $\delta<c<1/16$ and $n\ge 2n_0(C)$.

On the other hand, by Lemma 2.2 and (3.1), we have

$$\mathbf{V}\mathrm{ar}_n\hat{h}\le 2B_n^2(\hat{h})\le 2\theta C\delta^2$$

which contradicts to the previous inequality if M and n are sufficiently large. Consequently, the estimate $U_n(h)\ll n^{-1}\vee\delta$ is proved for $n>2n_0(C)$. For $1\le n\le 2n_0(C)$, it is trivial.

Theorem 1.1 is proved. ■

Proof of Corollary 1.1. It suffices to apply the well known equality $\psi_n(n-j) = (1-j/n)^{\theta-1}\big(1+O((n-j)^{-1})\big)$ if $0 \le j \le n-1$ and the fact that, in the weak law of large numbers, the centralizing sequence $\alpha(n)$ is uniquely determined up to an error o(1). ■

Remark 3.1. The first claim of Theorem 1.1 can be extended to general additive functions if $\theta > 0$. For this, one needs an extension of Lemma 2.2; thus, it suffices to adopt technical ideas going back also to a number-theoretic paper by A. Biró and T. Szamuely [4]. There exists an indirect possibility to obtain the upper estimates based upon the inequality

$$\nu_{n,\theta}\big(|h(\sigma - a| \ge u\big) \ll P^{1\wedge\theta} P\big(|X_1 + \cdots + X_n \ge u/3\big) + \mathbf{1}\{\theta < 1\},$$

where $a \in \mathbb{R}$ and $u \ge 0$ are arbitrary, proved jointly with G.J. Babu by the second author [3]. In the case $\theta < 1$, it and an appropriate estimate for these independent r.vs yields

$$L(h, \nu_{n,\theta}) \ll \left(\min_{\lambda} \sum_{j \le n} \frac{a_j^{*2}(\lambda)}{j}\right)^{\theta/(2\theta+1)} + n^{-\theta}.$$

The lower estimate, if $\theta < 1$, raises much more difficulties. To overcome them, one needs effective lower estimates for the mean values of multiplicative functions defined on the symmetric group. The approach applied by the authors of the present note in [9] is of little help.

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