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A REMARK ON THE ISOMORPHIC CLASSIFICATION OF WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS ON THE UPPER HALF PLANE

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Abstract. We complete the isomorphic classification for weighted spaces $H_v(\mathbb{G})$ of holomorphic functions on the upper half plane $\mathbb{G}$ with respect to standard weights v which are of at most moderate growth. We show that there are only two isomorphism classes for the corresponding Banach spaces $H_v(\mathbb{G})$, namely l_∞ and H_∞. We prove that $H_v(\mathbb{G})$ is isomorphic to H_∞ if and only if v grows slowly. In particular $H_v(\mathbb{G})$ is isomorphic to H_∞ if v is bounded.

1. Introduction

Let $O \subset \mathbb{C}$ be an open subset and $v : O \to [0, \infty[$ a given function. Then we consider, for $f : O \to \mathbb{C}$, the weighted sup-norm

$$||f||_v = \sup_{z \in O} |f(z)|v(z)$$

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and the space

$$H_v(O) = \{f : O \to \mathbb{C} \text{ holomorphic } : ||f||_v < \infty\}$$

In our paper we are concerned with the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the upper half plane

$$\mathbb{G} = \{w \in \mathbb{C} : \text{ Im } w > 0\}.$$

We want to investigate how far the spaces $H_v(\mathbb{D})$ and $H_v(\mathbb{G})$ are related to classical Banach spaces.

Actually the isomorphic classification of $H_v(\mathbb{D})$ is well-known for weights v on $\mathbb{D}$ which satisfy $v(z) = v(|z|)$, $z \in \mathbb{D}$, $v(t) \leq v(s)$ if $0 \leq s \leq t < 1$ and $\lim_{t\to 1} v(t) = 0$. Then, depending on v, $H_v(\mathbb{D})$ is either isomorphic to l_∞ or to H_∞, the space of all bounded holomorphic functions on $\mathbb{D}$ endowed with the sup-norm ([7]).

In particular, if v is of moderate decay, i.e. if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty$$

then $H_v(\mathbb{D})$ is isomorphic to l_∞ if and only if v is 'normal', i.e. if

$$\inf_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \frac{v(1-2^{-n-m})}{v(1-2^{-n})} < 1 \quad ([5,\ 6]).$$

On $\mathbb{G}$ we study now the following class of weights.

Definition 1.1. (i) Let $v : \mathbb{G} \to]0, \infty[$ be continuous such that $v(w) = v(i \text{ Im } w)$, $w \in \mathbb{G}$, $v(is) \leq v(it)$ if $0 < s \leq t$ and $\lim_{t\to 0} v(it) = 0$. Then v is called a *standard weight*.

(ii) A standard weight v on $\mathbb{G}$ satisfies *condition* $(\star)$ if

$$\sup_{k\in\mathbb{Z}} \frac{v(2^{k+1}i)}{v(2^k i)} < \infty.$$

(iii) A standard weight v satisfies *condition* $(\star\star)$ if

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{v(2^k i)}{v(2^{k+n} i)} < 1.$$

Examples. $v_1(it) = t^\alpha$, $t > 0$, for some $\alpha > 0$, $v_2(it) = \min(t^\alpha, 1)$, $t > 0$,

$$v_3(it) = \left\{ \begin{array}{ll} (1-\ln(t))^{-\alpha}, & 0 < t \leq 1, \\ t, & t > 1, \end{array} \right.$$

define standard weights on $\mathbb{G}$ which satisfy $(\star)$. Only v_1 satisfies $(\star\star)$.

Condition $(\star)$ means that $v(it)$ is at most moderately growing while a weight with $(\star\star)$ grows at least moderately. The following lemma is easily seen ([1]).

Lemma 1.2. *Let v be a standard weight on $\mathbb{G}$. Then*

(i) *v satisfies $(\star)$ if and only if there are $c > 0$ and $\beta > 0$ with*

$$\frac{v(it)}{v(is)} \leq c\left(\frac{t}{s}\right)^{\beta} \quad \text{whenever} \quad 0 < s \leq t.$$

In this case we can take $c = a^2$ and $\beta = \ln a / \ln 2$ where

$$a = \sup_{k \in \mathbb{Z}} v(2^{k+1}i)/v(2^k i).$$

(ii) *v satisfies $(\star\star)$ if and only if there are $d > 0$ and $\gamma > 0$ with*

$$\frac{v(it)}{v(is)} \geq d\left(\frac{t}{s}\right)^{\gamma} \quad \text{whenever} \quad 0 < s \leq t.$$

The aim of this note is to complete the isomorphic classification of $H_v(\mathbb{G})$ for standard weights satisfying $(\star)$. We show that there are only two isomorphism classes for $H_v(\mathbb{G})$. We obtain

Theorem 1.3. *Let v be a standard weight on $\mathbb{G}$ with $(\star)$. Then*

(i) *$H_v(\mathbb{G})$ is isomorphic to l_∞ if and only if v satisfies $(\star\star)$.*

(ii) *$H_v(\mathbb{G})$ is isomorphic to H_∞ if and only if v does not satisfy $(\star\star)$.*

(i) was shown in [1]. We prove (ii) in the following sections.

Corollary 1.4. *Let v be a bounded standard weight on $\mathbb{G}$ with $(\star)$. Then $H_v(\mathbb{G})$ is isomorphic to H_∞.*

Proof. If v is bounded then it cannot satisfy $(\star\star)$. Hence 1.4. follows from Theorem 1.3. ■

The preceding theorem for standard weights v on $\mathbb{G}$ cannot be inferred directly from the corresponding result for radial weights on $\mathbb{D}$. If v is a standard weight on $\mathbb{G}$ and we consider a conformal map $\alpha : \mathbb{D} \to \mathbb{G}$ then $u(z) = v(\alpha(z))$ is a weight on $\mathbb{D}$. Moreover $H_u(\mathbb{D})$ is isometrically isomorphic to $H_v(\mathbb{G})$. But $v \circ \alpha$ is not radial and does not satisfy the other requirements for u. For weights on $\mathbb{D}$ of the form $v \circ \alpha$, v a standard weight on $\mathbb{G}$, nothing is known about $H_{(v\circ\alpha)}(\mathbb{D})$.

2. $H_v(\mathbb{G})$ is complemented in H_∞

Theorem 2.1. *Let v be a standard weight on $\mathbb{G}$ satisfying $(\star)$. Then $H_v(\mathbb{G})$ is isomorphic to a complemented subspace of H_∞.*

To prove Theorem 2.1. we need to recall some facts from [1]. At first, consider a holomorphic function f on $\mathbb{D}$, say $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$. Put

$$(R_n f)(z) = \sum_{k=0}^{2^n} \alpha_k z^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1}-k}{2^n} \alpha_k z^k. \tag{2.1}$$

Moreover, for $r > 0$, put $M_\infty(f, r) = \sup_{|z|=r} |f(z)|$. It is well-known (see e.g. [6]) that

$$M_\infty(R_n f, r) \le 3M_\infty(f, r) \quad \text{for all} \quad r \text{ and } n. \tag{2.2}$$

Clearly, we have

$$R_n R_m = R_{\min(m,n)} \quad \text{if} \quad m \ne n. \tag{2.3}$$

Consider a radial weight u on $\mathbb{D}$, i.e. u satisfies $u(z) = u(|z|)$, $z \in \mathbb{D}$. Moreover, assume $u(s) \ge u(t)$ if $0 \le s \le t < 1$ and $\lim_{t\to 1} u(t) = 0$. Finally suppose

$$a := \sup_{n\in\mathbb{N}} \frac{u(1-2^{-n})}{u(1-2^{-n-1})} < \infty. \tag{2.4}$$

Then we use induction to find integers $m_0 = 0 < m_1 < m_2 < \dots$ such that

$$\frac{1}{2a} \le \frac{u(1-2^{-m_{k+1}})}{u(1-2^{-m_k})} \le \frac{1}{2}. \tag{2.5}$$

We have (see [1])

Proposition 2.2. *Put $|||f||| = \sup_k M_\infty((R_{m_k} - R_{m_{k-1}})f, 1)u(1-2^{-m_k})$. Then there is a universal constant $b > 0$ depending only on a such that*

$$\frac{1}{96}|||f||| \le ||f||_u \le b|||f||| \quad \textit{for all} \quad f \in H_u(\mathbb{D}).$$

(Actually, $b = 32a + 4a\sum_{j=1}^{\infty} 2^j a^j \exp(-2^{j-1})$.)

Corollary 2.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $|||f||| < \infty$. Then $f \in H_u(\mathbb{D})$.*

Proof. Fix n. Then (2.2) and (2.3) imply $\sup_n |||R_n f||| \le 3|||f|||$. Since $R_n f$ is a polynomial it is an element of $H_u(\mathbb{D})$. Hence (2.2) yields

$$\sup_n ||R_n f||_u \le 3b|||f|||.$$

On the other hand, $(R_n f)$ converges to f pointwise on $\mathbb{D}$. Hence we have $||f||_u \le \sup_n ||R_n f||_u < \infty$. This implies $f \in H_u(\mathbb{D})$. ■

Let, for $n \in \mathbb{N}$, A_n be the space of all complex polynomials of degree $\le n$ endowed with the norm $M_\infty(\cdot, 1)$. It is well-known that H_∞ is isomorphic to $(\sum_{n=1}^\infty \oplus A_n)_{(\infty)}$ (see [8]).

Proposition 2.4. *There is an isomorphism*

$$T : H_u(\mathbb{D}) \to (\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)}$$

and a projection

$$P : (\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)} \to TH_u(\mathbb{D})$$

with $||P|| \le 2 \cdot 10^4 ab$ and $||T|| \cdot ||T^{-1}|| \le 96b$.

Proof. This is essentially the argument of the proof of Lemma 3.3. of [6].

Let $B_k = A_{2^{m_k+2}}$ be endowed with the norm $M_\infty(\cdot, 1)u(1 - 2^{-m_k})$. Then $Y := (\sum_{k=1}^\infty \oplus B_k)_{(\infty)}$ is isometrically isomorphic to $(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)}$. We work with Y instead of $(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)}$. Define $T : H_u(\mathbb{D}) \to Y$ by $Tf = = (\ (R_{m_k} - R_{m_{k-1}})f\ )$. Then, by Proposition 2.2., $||T|| \cdot ||T^{-1}|| \le 96b$. Moreover, define $S : Y \to H_u(\mathbb{D})$ by $S(g_k) = \sum_{k=1}^\infty (R_{m_k+1} - R_{m_{k-1}-1})g_k$ where $g_k \in B_k$ for all k. (Put $R_{-1} = 0$). S makes sense, $S(g_k)$ is the Taylor series of a holomorphic function on $\mathbb{D}$. Indeed, we have

$$(R_{m_k+1} - R_{m_{k-1}-1})g_k = \sum_{j=2^{m_{k-1}-1}}^{2^{m_k+2}} \alpha_j z^j$$

where

$$|\alpha_j| \le \frac{|||f|||}{u(1-2^{-m_k})} \le u(0)2^k|||f|||.$$

Hence this Taylor series converges pointwise on $\mathbb{D}$. We obtain, with (2.3),

$$
\begin{aligned}
|||S(g_k)||| \le \sup_k M_\infty(\ (R_{m_k} - R_{m_{k-1}})((R_{m_k+1} - R_{m_{k-1}-1})g_k + \\
+(R_{m_{k-1}+1} - R_{m_{k-2}-1})g_{k-1} + (R_{m_{k+1}+1} - R_{m_k-1})g_{k+1} \ ), 1)u(1-2^{-m_k}) \\
\le \quad & 36 \sup_k (M_\infty(g_{k-1},1) + M_\infty(g_k,1) + M_\infty(g_{k+1},1))u(1-2^{-m_k}) \le \\
\le \quad & 36 \sup_k (M_\infty(g_{k-1},1)u(1-2^{-m_k+1}) + \\
& \quad +M_\infty(g_k,1)u(1-2^{-m_k}) + aM_\infty(g_{k+1},1)u(1-2^{-m_{k+1}}) \le \\
\le \quad & 108a||(g_k)||.
\end{aligned}
$$

Hence, in view of Corollary 2.3., $S(g_k) \in H_u(\mathbb{D})$. With (2.3) we see that $STf = f$ Then $P = TS$ is a projection with $||P|| \le 108a \cdot 96b \le 2 \cdot 10^4 ab$. ■

2.5. Proof of Theorem 2.1. Let

$$u_n(z) = v\left(n\frac{1-|z|}{1+|z|}i\right), \qquad z \in \mathbb{D}, \quad n \in \mathbb{N}.$$

Then $H_v(\mathbb{G})$ is isometrically isomorphic to a complemented subspace of $Z :=$ $:= (\sum_{n=1}^{\infty} \oplus H_{u_n}(\mathbb{D}))_{(\infty)}$ ([1], Corollary 1.5.). Put

$$a_n = \sup_{j\in\mathbb{N}} \frac{u_n(1-2^{-j})}{u_n(1-2^{-j-1})} \tag{2.6}$$

and find integers $0 = m_{n,0} < m_{n,1} < m_{n,2} < \ldots$ with

$$\frac{1}{2a_n} \le \frac{u_n(1-2^{-m_{n,k+1}})}{u_n(1-2^{-m_{n,k}})} \le \frac{1}{2}. \tag{2.7}$$

If v satisfies $(\star)$ then $\sup_n a_n < \infty$ ([1], Lemma 4.1.). According to Proposition 2.4., there are isomorphisms $T_n : H_{u_n}(\mathbb{D}) \to \left(\sum_k \oplus A_{2^{m_{n,k}+2}}\right)_{(\infty)} =: Y_n$ and projections $P_n : Y_n \to T_n H_{u_n}(\mathbb{D})$ where $||T_n|| \cdot ||T_n^{-1}||$ and $||P_n||$ depend only on a_n. Since $\sup_n a_n < \infty$ we see that there is an isomorphism $\tilde{T} : Z :=$ $:= (\sum_n \oplus H_{u_n}(\mathbb{D}))_{(\infty)} \to (\sum_n \oplus Y_n)_{(\infty)} =: Y$ and a bounded projection $\tilde{P} : Y \to Z$. Hence there is an isomorphism $T : H_v(\mathbb{G}) \to Y$ and a bounded projection $P : Y \to TH_v(\mathbb{G})$. We have $Y = \left(\sum_{n,k} \oplus A_{2^{m_{n,k}+2}}\right)_{(\infty)}$. According to [6], Lemma 2.2., Y is isomorphic to $(\sum_{n=1}^{\infty} \oplus A_n)_{(\infty)}$ and hence to H_∞. ■

3. H_∞ is complemented in $H_v(\mathbb{G})$

Theorem 3.1. *Let v be a standard weight on $\mathbb{G}$ satisfying $(\star)$ but not $(\star\star)$. Then $H_v(\mathbb{G})$ contains a complemented subspace which is isomorphic to H_∞.*

Theorem 3.1. together with Theorem 2.1. and Pelczynski's decomposition method prove that $H_v(\mathbb{G})$ is isomorphic to H_∞ if v satisfies $(\star)$ but not $(\star\star)$ since H_∞ is isomorphic to $(H_\infty \oplus H_\infty \oplus \ldots)_{(\infty)}$ ([8]). The converse of Theorem 1.3.(ii) follows from Theorem 1.3.(i).

To prove Theorem 3.1. we consider at first polynomials on $\mathbb{D}$.

Lemma 3.2. *Let $0 \leq n_1 < n_2 < n_2+2 < n_3 < n_4$ be integers and consider*

$$X = span\{z^{2^{n_1}+1}, z^{2^{n_1}+2}, \ldots, z^{2^{n_2+1}-1}\},$$
$$Y = span\{z^{2^{n_3}+1}, z^{2^{n_3}+2}, \ldots, z^{2^{n_4+1}-1}\}.$$

For any $r > 0$ and $f \in X$, $g \in Y$ we have

$$(R_{n_4} - R_{n_3})f = 0 = (R_{n_2} - R_{n_1})g \tag{3.1}$$

and

$$\frac{1}{6}\max\left(M_\infty(f,r), M_\infty(g,r)\right) \leq M_\infty(f+g,r) \leq 2\max\left(M_\infty(f,r), M_\infty(g,r)\right).$$

Proof. (3.1) follows from the definition of the operators R_n. The lower inequality follows from the fact that $(R_{n_2+1} - R_{n_1-1})(f+g) = f$ and $(R_{n_4+1} - R_{n_3-1})(f+g) = g$. The upper inequality is a consequence of the triangle inequality. ■

We need the following Lemma from [6].

Lemma 3.3. *Let X and Y be as in Lemma 3.2. Fix some constant $c > 0$. Consider the norm $M_\infty(\cdot, 1)c$ on X and on Y. Let*

$$m = \min(2^{n_2-n_1-1}, 2^{n_4-n_3-1}).$$

Then there is an isometry $i : A_m \to (X \oplus Y)_\infty$ and a projection $Q : (X \oplus \oplus Y)_{(\infty)} \to i(A_m)$ with $||Q|| \leq 2$ such that

$$(\,(R_{n_2} - R_{n_1})f, (R_{n_4} - R_{n_3})g\,) = (f,g) \tag{3.2}$$

whenever $(f,g) \in i(A_m) \subset (X \oplus Y)_\infty$. (We regard $(X \oplus Y)_\infty$ as the space of all pairs (f,g) with $f \in X$ and $g \in Y$).

Then we obtain

Proposition 3.4. *Let v be a standard weight on $\mathbb{G}$ satisfying $(\star)$ but not $(\star\star)$. Then there is a universal constant $d > 0$ such that for every $j > 0$ there exists an integer $m > j$, an isomorphism $T : A_m \to H_v(\mathbb{G})$ and a projection $P : H_v(\mathbb{G}) \to TA_m$ with $||P|| \leq d$ and $||T|| \cdot ||T^{-1}|| \leq d$.*

Proof. Put $v_n(w) = v(nw)$, $w \in \mathbb{G}$, $n \in \mathbb{N}$. Then $(S_n f)(w) = f(nw)$, $w \in \mathbb{G}$, defines an isometry between $H_v(\mathbb{G})$ and $H_{v_n}(\mathbb{G})$. Moreover, put

$$u_n(z) = v\left(n\frac{1-|z|}{1+|z|}i\right), \qquad z \in \mathbb{D}.$$

Then u_n is a radial weight on $\mathbb{D}$. We consider again a_n with (2.6) and $m_{n,k}$ with (2.7). $(\star)$ implies $a := \sup_n a_n < \infty$. Since v does not satisfy $(\star\star)$ we have

$$\sup_n \sup_k (m_{n,k} - m_{n,k-1}) = \infty \tag{3.3}$$

([1], Lemma 4.1.)

By [1], Propositions 3.1 and 3.2., there is a universal constant $c > 0$, depending only on a, an (into-)isomorphism $T_n : H_{u_n}(\mathbb{D}) \to H_{v_n}(\mathbb{G})$ and a projection $P_n : H_{v_n}(\mathbb{G}) \to T_n H_{u_n}(\mathbb{D})$ with $||T_n|| \cdot ||T_n^{-1}|| \leq c$ and $||P_n|| \leq c$.

Now let $j > 0$. By (3.3) we find n and k and integers $m_{n,k-1} < n_1 < n_2 < n_2 + 2 < n_3 < n_4 < m_{n,k}$ such that $m := \min(2^{n_2-n_1-1}, 2^{n_4-n_3-1}) > j$. Using Proposition 2.2. with u_n and Lemmas 3.2. and 3.3. we see that there is an isomorphism $i : A_m \to H_{u_n}(\mathbb{D})$ and a projection $Q : (R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1})H_{u_n}(\mathbb{D}) \to i(A_m)$ with $||Q|| \leq 2$ satisfying (3.2). We have $||i|| \cdot ||i^{-1}|| \leq 96b$. (3.1) and (3.2) imply that $Q(R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1})$ is a projection from $H_{u_n}(\mathbb{D})$ onto $i(A_m)$. Put $T = S_n^{-1}T_n i$ and $P = S_n^{-1}T_n Q(R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1})T_n^{-1}P_n S_n$. Then P is a projection from $H_v(\mathbb{G})$ onto TA_m and we have $||T|| \cdot ||T^{-1}|| \leq 96bc$, $||P|| \leq 24 \cdot 96bc^2$. ∎

Now, for a standard weight v on $\mathbb{G}$ we introduce

$$(H_v)_0(\mathbb{G}) = \{f \in H_v(\mathbb{G}) : |f(w)|v(w) \text{ vanishes at infinity}\}.$$

(Here $|f(w)|v(w)$ vanishes at infinity if for every $\epsilon > 0$ there is a compact subset $K \subset \mathbb{G}$ such that $|f(w)|v(w) \leq \epsilon$ for $w \in \mathbb{G} \setminus K$.)

It is well known ([2, 3]) that $H_v(\mathbb{G})$ is isometrically isomorphic to $(H_v)_0(\mathbb{G})^{**}$ and the canonical embedding of $(H_v)_0(\mathbb{G})$ into $(H_v)_0(\mathbb{G})^{**}$ corresponds to the embedding of $(H_v)_0(\mathbb{G})$ into $H_v(\mathbb{G})$.

Lemma 3.5. *Let $E \subset H_v(\mathbb{G})$ be a finite dimensional subspace and $P : H_v(\mathbb{G}) \to E$ a projection. Then, for every $\epsilon > 0$, there is an isomorphism $T : E \to (H_v)_0(\mathbb{G})$ and a projection $Q : (H_v)_0(\mathbb{G}) \to TE$ with $||Q|| \leq (1+\epsilon)||P||$ and $||T|| \cdot ||T^{-1}|| \leq 1 + \epsilon$.*

Proof. Put $X = (H_v)_0(\mathbb{G})$ and identify X^{**} with $H_v(\mathbb{G})$. Let $i : X \to \to X^{**}$ be the canonical embedding. Then $i^* : X^{***} \to X^*$ is the map with $(i^* x^{***})(x) = x^{***}(ix)$, $x^{***} \in X^{***}$, $x \in X$. Put $F = i^* P^* X^{***}$. By the principle of local reflexivity ([4], p.53,) we find $T : E \to X$ with $||T|| \cdot ||T^{-1}|| \leq \leq 1 + \epsilon$ and $e(f) = f(Te)$ for $e \in E$ and $f \in F$. Put $Q = TPi$. Let $x^* \in X^*$ and $x \in X$. Then we obtain $(i^* P^* T^*)(x^*) \in F$ and hence $x^*(TPiTPix) = = x^*(TPix)$. This follows since we have $P^{**} i^{**} P = P$ taking into account that dim $E < \infty$. Therefore, Q is the desired projection. ■

Lemma 3.6. *Let $B_j \subset (H_v)_0(\mathbb{G})$, $j = 1, 2$, be two finite dimensional subspaces and assume that $P_j : (H_v)_0(\mathbb{G}) \to B_j$, $j = 1, 2$, are bounded and linear and P_2 is a projection. Then, for every $\epsilon > 0$, there is an isometry $T : B_2 \to (H_v)_0(\mathbb{G})$ and a linear map $Q : (H_v)_0(\mathbb{G}) \to B_1 + TB_2$ such that*

$$(1-\epsilon) \max(||f||_v, ||g||_v) \leq ||f+Tg||_v \leq (1+\epsilon) \max(||f||_v, ||g||_v), \;\; f \in B_1, g \in B_2,$$

$$||Q|| \leq (1 + \epsilon) \max(||P_1||, ||P_2||) \text{ and}$$

$$||(Q-id)|_{(B_1+TB_2)}|| \leq (1+\epsilon)||(P_1-id)|_{B_1}||+\epsilon \max(||P_1||, ||P_2||), \; f \in B_1, g \in B_2.$$

Proof. Fix $0 < \epsilon' < 1$. Since $B_j \subset (H_v)_0(\mathbb{G})$ are finite dimensional we find compact subsets $K_j \subset \mathbb{G}$ such that $|f(w)|v(w) \leq \epsilon' ||f||_v$ for all $w \in \mathbb{G} \setminus K_j$ and $f \in B_j$, $j = 1, 2$. For any $x \in \mathbb{R}$ and $f \in H_v(\mathbb{G})$ put $(T_x f)(w) = f(x + w)$, $w \in \mathbb{G}$. Since v is a standard weight, T_x is an isometry $(H_v)_0(\mathbb{G}) \to (H_v)_0(\mathbb{G})$. Let $\varphi \in (H_v)_0(\mathbb{G})^*$. In view of the Riesz representation theorem there is a regular Borel measure μ on $\mathbb{G}$ with $|\mu|(\mathbb{G}) < \infty$ such that

$$\varphi(f) = \int_{\mathbb{G}} f(w)v(w)d\mu(w).$$

Hence $\lim_{x \to \pm\infty} \varphi(T_x f) = 0$. Since $P_j^*(H_v)_0(\mathbb{G})^*$ are finite dimensional we find $x \in \mathbb{R}$ so large that $K_1 \cap (x + K_2) = \emptyset = K_2 \cap (-x + K_1)$ and $|\varphi_1(T_x g)| \leq \epsilon' ||g||_v$ for $g \in B_2$, $\varphi_1 \in P_1^*(H_v)_0(\mathbb{G})^*$ and $|\varphi_2(T_{-x} f)| \leq \epsilon' ||f||_v$ for $f \in B_1$, $\varphi_2 \in P_2^*(H_v)_0(\mathbb{G})^*$. Hence $||P_1 T_x g||_v \leq \epsilon' ||g||_v$ for $g \in B_2$ and $||P_2 T_{-x} f||_v \leq \leq \epsilon' ||f||_v$ for $f \in B_1$.

Put $T = T_x|_{B_2}$ and $Q = P_1 + T_x P_2 T_{-x}$. Then we obtain

$$(1 - \epsilon') \max(||f||_v, ||g||_v) \leq ||f + Tg||_v \leq (1 + \epsilon') \max(||f||_v, ||g||_v)$$

for $f \in B_1$ and $g \in B_2$. Using this we see that

$$||Q|| \leq \left(\frac{1 + \epsilon'}{1 - \epsilon'}\right) \max(||P_1||, ||P_2||).$$

Finally, we have, for $f \in B_1$ and $g \in B_2$,

$$\begin{aligned} ||Q(f+T_xg)-(f+T_xg)||_v &= ||P_1f - f + P_1T_xg + T_xP_2T_{-x}f||_v \leq \\ &\leq (||(P_1 - id)|_{B_1}|| + 2\epsilon' \max(||P_1||, ||P_2||)) \cdot \\ &\qquad \cdot \max(||f||_v, ||g||_v) \leq \\ &\leq (||(P_1 - id)|_{B_1}|| + 2\epsilon' \max(||P_1||, ||P_2||)) \cdot \\ &\qquad \cdot \frac{||f+T_xg||_v}{1-\epsilon'}. \end{aligned}$$

If ϵ' is small enough we obtain the estimates of the assertion of Lemma 3.6. ■

Proof of Theorem 3.1. Use Proposition 3.4. and Lemma 3.5. to find integers $0 < k_1 < k_2 < \ldots$, isomorphisms $T_n : A_{k_n} \to (H_v)_0(\mathbb{G})$ and projections $P_n : (H_v)_0(\mathbb{G}) \to T_nA_{k_n}$ with $||T_n|| \cdot ||T_n^{-1}|| \leq d$ and $||P_n|| \leq d$ where $d > 0$ is a universal constant.

Then use Lemma 3.6. and induction to find an isomorphic copy $X \subset \subset (H_v)_0(\mathbb{G})$ of $(\sum_{n=1}^{\infty} \oplus A_{k_n})_{(0)}$ and a linear bounded map $Q : (H_v)_0(\mathbb{G}) \to X$ with $||(Q - id)|_X|| < 1$. (Apply Lemma 3.6. successively with ϵ_n small enough such that in particular $0 < \prod_{n=1}^{\infty}(1-\epsilon_n) < \prod_{n=1}^{\infty}(1+\epsilon_n) < \infty$.)

Hence $S := (Q|_X)^{-1}$ exists and is bounded. Put $P = SQ$. Then P is a bounded projection from $(H_v)_0(\mathbb{G})$ onto X. With biduality we see that $P^{**} : H_v(\mathbb{G}) \to X^{**}$ is a bounded projection from $H_v(\mathbb{G})$ onto an isomorphic copy of $(\sum_{n=1}^{\infty} \oplus A_{k_n})_{(\infty)}$. This space is isomorphic to $(\sum_{n=1}^{\infty} \oplus A_n)(\infty)$ and hence to H_∞ ([6]). ■

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