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ALMOST EVERYWHERE CONVERGENCE OF SEQUENCES OF TWO-DIMENSIONAL VILENKIN–FEJÉR MEANS OF INTEGRABLE FUNCTIONS

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Dedicated to Professor Karl-Heinz Indlekofer on the occasion of his 70*th birthday*

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Abstract. The aim of this paper is to prove the a.e. convergence of sequences of the Fejér means of the Vilenkin–Fourier series of two variable integrable functions on two dimensional bounded Vilenkin groups. That is, let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ such that $a_j(n+1) \geq \delta \sup_{k \leq n} a_j(n)$ $(j = 1, 2,$ $n \in \mathbb{N})$ for some $\delta > 0$ and $a_1(+\infty) = a_2(+\infty) = +\infty$. Then for each integrable function $f \in L^1(G_m^2)$ we have the a.e. relation $\lim_{n\to\infty} \sigma_{a_1(n),a_2(n)} f = f$. It will be a straightforward and easy consequence of this result the cone restricted a.e. convergence of the two-dimensional Vilenkin–Fejér means of integrable functions which was proved earlier by Weisz [13] and Blahota and the author [2] independently. The trigonometric and Walsh's analogue of the main result see Gát [6], [5]

1. Introduction

First, we give a brief introduction to the theory of the Vilenkin–Fourier series. Denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{P}$ the set of positive integers,

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respectively. Denote $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and Z_{m_k} the discrete cyclic group of order m_k (Z_{m_k} can be identified by the set $\{0, 1, ..., m_k - 1\}$, the group operation by the $\mod m_k$ addition). Suppose that each (coordinate) group Z_{m_k} has the discrete topology and measure μ_k which maps every singleton of Z_{m_k} to $\frac{1}{m_k}$ ($\mu_k(Z_{m_k}) = 1$), $k \in \mathbb{N}$. Let G_m be the compact Abelian group formed by the complete direct product of Z_{m_k} with the product of the topologies and measures (μ). Thus, each $x \in G_m$ is a sequence $x = (x_0, x_1, ...)$, where $x_k \in Z_{m_k}$, $k \in \mathbb{N}$. The group operation on G_m is the coordinate-wise addition, i.e. for $x, y \in G_m$ we have $x + y = (x_k + y_k(\mod m_k) : k \in \mathbb{N})$. The inverse operation is denoted by $-$. G_m is called a Vilenkin group. G_m is a compact totally disconnected group, with normalized Haar measure μ, $\mu(G_m) = 1$. The Vilenkin group G_m is said to be bounded if the generating system m is a bounded one. This property of sequence m is supposed. In this paper c denote absolute constants which may not be the same at different occurences.

A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P}$.

$$\mathcal{I} := \{I_n(x) : n \in \mathbb{N}, x \in G_m\}$$

is the set of intervals on G_m. Denote by $e_n \in G_m$ the sequence the n-th coordinate of which is 1 the rest are zeros ($n \in \mathbb{N}$).

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ denote the corresponding norms) ($1 \leq p \leq \infty$), $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ ($x \in G_m$) and let E_n be the conditional expectation operator with respect to $\mathcal{A}_n$ ($n \in \mathbb{N}$).

Let $M_0 := 1$, $M_{n+1} := m_n M_n, (n \in \mathbb{N})$ be the generalized powers with respect to m. Then each $n \in \mathbb{N}$ can uniquely be expressed as $n = \sum_{i=0}^{\infty} n_i M_i$ ($n_i \in \{0, 1, ..., m_i - 1\}$). For $t = (t^1, t^2) \in G_m \times G_m$, $b = (b_1, b_2) \in \mathbb{N}^2$ set the two-dimensional dyadic rectangle

$$I_b^2(t) := I_{b_1}(t^1) \times I_{b_2}(t^2).$$

For $n = (n_1, n_2) \in \mathbb{N}^2$ denote by $E_n = E_{(n_1,n_2)}$ the two-dimensional expectation operator with respect to the to $\mathcal{A}_n = \mathcal{A}_{(n_1,n_2)} = \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$. For $n \in \mathbb{N}$ denote by $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $M_{|n|} \leq n < M_{|n|+1}$. The generalized Rademacher functions are defined as:

$$r_n(x) := \exp(2\pi \imath x_n / m_n) \quad (x \in G_m,\, n \in \mathbb{N}, \imath = \sqrt{-1}).$$

The Vilenkin system is defined as the sequence of the Vilenkin functions:

$$\psi_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = \exp(2\pi\imath \sum_{k=0}^{|n|} n_k x_k / m_k) \quad (x \in G_m,\, n \in \mathbb{N}).$$

That is, $\psi := (\psi_n, n \in \mathbb{N})$. Let us consider the Dirichlet and Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \psi_k,\ K_n := \frac{1}{n}\sum_{k=1}^{n} D_k,\ D_0, K_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C,1)$ mean of $f \in L^1(G_m)$:

$$\hat{f}(n) := \int\limits_{G_m} f(x)\bar{\psi}_n(x)d\mu(x)\ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k(y) = \int\limits_{G_m} f(x)D_n(y-x)d\mu(x)\ (n \in \mathbb{P},\ S_0 f = 0),$$

$$\sigma_n f(y) := \frac{1}{n}\sum_{k=1}^{n} S_k f(y) = \int\limits_{G_m} f(x)K_n(y-x)d\mu(x)\ (n \in \mathbb{P}, \sigma_0 f = 0).$$

Define the two-dimensional Dirichlet and Fejér kernel functions as

$$D_n(x) := \sum_{\substack{0\le j_1<n_1\\ 0\le j_2<n_2}} \psi_{j_1}(x^1)\psi_{j_2}(x^2) = D_{n_1}(x^1)D_{n_2}(x^2),$$

$$K_n(x) := \frac{1}{n_1 n_2}\sum_{\substack{1\le j_1\le n_1\\ 1\le j_2\le n_2}} D_n(x) = K_{n_1}(x^1)K_{n_2}(x^2),$$

where $x = (x^1, x^2) \in G_m \times G_m$, $n = (n_1, n_2) \in \mathbb{P}^2$ (for $n_1 n_2 = 0$ set $D_n = = K_n = 0$). The Fourier coefficients, the $n \in \mathbb{N}^2$-th partial sum of the Fourier series, the $n \in \mathbb{N}^2$-th $(C,1)$ mean of $f \in L^1(G_m^2)$:

$$\hat{f}(n_1, n_2) := \int\limits_{G_m^2} f(x^1, x^2)\bar{\psi}_{n_1}(x^1)\bar{\psi}_{n_2}(x^2)d\mu(x^1, x^2)\ (n \in \mathbb{N}^2),$$

$$S_{n_1,n_2} f(y) :=\sum_{\substack{k_1<n_1\\ k_2<n_2}} \hat{f}(k_1, k_2)\psi_{k_1}(y^1)\psi_{k_2}(y^2) = \int\limits_{G_m^2} f(x)D_n(y-x)d\mu(x)$$

$(n \in \mathbb{P}^2$, $S_n f = 0$ for $n_1 n_2 = 0)$,

$$\sigma_{n_1,n_2} f(y) := \frac{1}{n_1 n_2}\sum_{\substack{1\le k_1\le n_1\\ 1\le k_2\le n_2}} S_k f(y) = \int\limits_{G_m^2} f(x)K_n(y-x)d\mu(x)$$

($n \in \mathbb{P}^2, \sigma_n f = 0$ for $n_1 n_2 = 0$). If m is bounded then $G_m \times G_m = G_m^2$ is said to be a two-dimensional bounded Vilenkin group. In this paper we discuss two-dimensional bounded Vilenkin groups only. Set $p := \sup_i m_i$.

For double trigonometric Fourier series Marcinkiewicz and Zygmund [7] proved the a.e. convergence of Fejér means of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \le n_1/n_2 \le \beta$ is provided with some fixed parameter $\beta > 1$. We mention that Jessen, Marcinkiewicz and Zygmund [1] also proved the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices (other than $\min\{n_1, n_2\} \to \infty$), but for functions in $L \log^+ L$. For double Walsh-Fourier series ($m_k = 2$ for all k), Móricz, Schipp and Wade [8] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min\{n_1, n_2\} \to \infty$ for all functions $f \in L \log^+ L$. In [4] Gát proved that the theorem of Móricz, Schipp and Wade can not be improved. Namely, the following was proved. Let $\gamma : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\gamma(+\infty) = 0$, then there exists a function $f \in L \log^+ L\gamma(L)$ such that $\sigma_n f$ does not converge to f a.e. as $\min\{n_1, n_2\} \to +\infty$. For double Walsh system the result of Marcinkiewicz and Zygmund was proved by Gát [3] and Weisz [12] independently. The (bounded) Vilenkin version of this result of Gát and Weisz is due to Weisz [13] and for more general systems to Blahota and Gát [2]. This result on bounded Vilenkin groups will be a straightforward consequence of the main theorem of this paper below.

2. The results

Theorem 2.1. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$, i.e. $a_j(n) \to +\infty$ $(n \to \infty)$. Suppose that there exists a $\delta > 0$ such that $a_j(n+1) \ge \delta \sup_{k \le n} a_j(n)$ $(j = 1, 2,\ n \in \mathbb{N})$. Then for each integrable function $f \in L^1(G_m^2)$ we have the a.e. relation*

$$\lim_{n \to \infty} \sigma_{a(n)} f = f.$$

This Theorem, which is the main result of this paper is an easy consequence of the following lemma.

Lemma 2.1. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that $\lfloor \log_p a_j \rfloor$ is monotone increasing $(j = 1, 2)$. Then for each integrable function $f \in L^1(G_m^2)$ we have the a.e. relation*

$$\lim_{n \to \infty} \sigma_{a(n)} f = f.$$

A straightforward and easy consequence of Lemma 2.1 is the result of Blahota, Gát and Weisz ([2, 13]) with respect to the ,,cone restricted" almost everywhere convergence of two-dimensional Vilenkin–Fejér means of integrable functions.

Corollary 2.1. *Let $\beta > 1$ and $f \in L^1(G_m^2)$. Then we have the a.e. relation*

$$\lim_{\substack{n_1,n_2\to\infty \\ 1/\beta\le n_1/n_2\le\beta}} \sigma_{n_1,n_2} f = f.$$

Proof. The proof of this corollary comes directly from Lemma 2.1. So, let $\gamma := \lceil \log_p \beta \rceil$. For $k, l \in \mathbb{N}$ set

$$N_{\gamma,l,k} := \left\{(n_1,n_2) \in \mathbb{N}^2 : p^k \le n_1 < p^{k+1}, p^{k-\gamma+l} \le n_2 < p^{k-\gamma+l+1}\right\}.$$

Let $N_{\gamma,l}$ be the union of the disjoint sets $N_{\gamma,l,k}$. It is easy to give a sequence $a : \mathbb{N} \to \mathbb{N}^2$ such that $\lfloor \log_p a_1 \rfloor, \lfloor \log_p a_2 \rfloor$ are monotone increasing (for $n \in N_{\gamma,l,k}$ we have $\lfloor \log_p n_1 \rfloor = k, \lfloor \log_p n_2 \rfloor = k - \gamma + l$) and $a(\mathbb{N}) = N_{\gamma,l}$. This by Lemma 2.1 gives that for each integrable function f

$$\sigma_{n_1,n_2} f \to f$$

a.e. provided by $n \in N_{\gamma,l}$ and $n_1, n_2 \to \infty$. Hence, we also have this a.e. relation for $n \in \bigcup_{l=0}^{2\gamma} N_{\gamma,l} =: N_\gamma$ and $n_1, n_2 \to \infty$. After then, let $n \in \mathbb{N}^2$ be such that $1/\beta \le n_1/n_2 \le \beta$. Denote by k the natural number for which $p^k \le n_1 < p^{k+1}$. Then, $p^{k-\gamma} \le p^k/\beta \le n_2 < p^{k+1}\beta \le p^{k+\gamma+1}$. Consequently, $n \in N_\gamma$. This completes the proof of this corollary. ■

Let $A = (A_1, A_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence of pairs of natural numbers, such that monotone increasing with respect to both indices, and they do not increase too fast. More precisely, suppose that there exists a constant $C > 0$ depending only on A and p such that

$$A_j(n) \le A_j(n+1) \le A_j(n) + C$$

for $n \in \mathbb{N}$ and $j = 1, 2$. (More precisely $C = C_{A,p}$.)

In order to prove Lemma 2.1 we need some lemmas. The first is the following Calderon-Zygmund type decomposition lemma. For the Calderon-Zygmund decomposition lemma on bounded Vilenkin groups see e.g. the paper of Simon [9]. Let $A_j(+\infty) = +\infty$ $(j = 1, 2)$ and

$$C_{A,p} = p^{\sup_n\{A_1(n+1)-A_1(n)+A_2(n+1)-A_2(n)\}}.$$

Lemma 2.2. *Let $f \in L^1(G_m^2), \lambda > 0$ and $A = (A_1, A_2) : \mathbb{N} \to \mathbb{N}^2$ as above. Then there exists a sequence of natural numbers (k_n) and $x_n = (x_{n,1}, x_{n,2}) \in \in G_m^2 \quad (n \in \mathbb{P})$ such that for the disjoint two-dimensional dyadic rectangles $J_n = I_{A(k_n)}(x_n)$ we have functions $f_n \in L^1(G_m^2) \quad (n \in \mathbb{N})$ satisfying the a.e. relation $f = \sum_{n=0}^{\infty} f_n$ and besides,*

$$\|f_0\|_\infty \leq C_{A,p}\lambda, \quad \|f_0\|_1 \leq \|f\|_1, \quad \operatorname{supp} f_n \subset J_n, \quad \int\limits_{J_n} f_n d\mu = 0 \ (n \in \mathbb{P}).$$

Moreover, for the set $F = \bigcup_{n=1}^{\infty} J_n$ we have $\mu(F) \leq \|f\|_1/\lambda$.

Proof. Set the following sets of two-dimensional dyadic rectangles. (Recall that $I_{A(k_n)}(x_n) = I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2})$.)

$$\Omega_0 := \left\{ I_{A(0)}(x) : M_{A_1(0)} M_{A_2(0)} \int\limits_{I_{A(0)}(x)} |f(y)| d\mu(y) > \lambda, x \in G_m^2 \right\},$$

$$\Omega_1 := \left\{ I_{A(1)}(x) : M_{A_1(1)} M_{A_2(1)} \int\limits_{I_{A(1)}(x)} |f(y)| d\mu(y) > \lambda, \right.$$
$$\left. \nexists J \in \Omega_0 : I_{A(1)}(x) \subset J, x \in G_m^2 \right\},$$

$$\vdots$$

$$\Omega_n := \left\{ I_{A(n)}(x) : M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x)} |f(y)| d\mu(y) > \lambda, \right.$$
$$\left. \nexists J \in \bigcup_{i=0}^{n-1} \Omega_i : I_{A(n)}(x) \subset J, x \in G_m^2 \right\}.$$

Then the elements of Ω_n are disjoint rectangles of measure $1/(M_{A_1(n)} M_{A_2(n)})$ $(n \in \mathbb{N})$. Moreover, if $i \neq j$, then for all $J \in \Omega_i, K \in \Omega_j$ we have $J \cap K = \emptyset$.

If $I_{A(n)}(x) \in \Omega_n$, then since there is no $J \in \bigcup_{i=0}^{n-1} \Omega_i$ such that $I_{A(n)}(x) \subset J$, then we have for $i = 0, 1, \ldots, n-1$

$$M_{A_1(i)} M_{A_2(i)} \int\limits_{I_{A(i)}(x)} |f(y)| d\mu(y) \leq \lambda \,.$$

Thus, since G_m is bounded, that is, $m_n \leq p$ we have

$$\lambda < M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x)} |f(y)| d\mu(y) \leq$$
$$\leq M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n-1)}(x)} |f(y)| d\mu(y) \leq$$
$$\leq C_{A,p} M_{A_1(n-1)} M_{A_2(n-1)} \int\limits_{I_{A(n-1)}(x)} |f(y)| d\mu(y) \leq C_{A,p} \lambda.$$

Since Ω_n has a finite number of elements, then we can set the notation:

$$\Omega_n = \left\{ I_{A(n)}(x_{n,i}) : i = 1, \dots, l_n \right\}, \quad F := \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{l_n} I_{A(n)}(x_{n,i}).$$

For the measure of set F we get:

$$\mu(F) = \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \mu(I_{A(n)}(x_{n,i})) =$$
$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \lambda \mu(I_{A(n)}(x_{n,i})) =$$
$$\leq \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \int\limits_{I_{A(n)}(x_{n,i})} |f| d\mu \leq \frac{1}{\lambda} \int\limits_{G_m^2} |f| d\mu = \|f\|_1 / \lambda.$$

Let function $1_B(x)$ be the characteristic function of the set $B \subset G_m^2$. Then,

$$f = \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} f 1_{I_{A(n)}(x_{n,i})} + f 1_{G_m^2 \setminus F} =$$
$$= \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \left(f - M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} f d\mu \right) 1_{I_{A(n)}(x_{n,i})} + g_{0,1} + f 1_{G_m^2 \setminus F} =:$$
$$=: \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} f_{n,i} + g_{0,1} + g_{0,2} =: \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} f_{n,i} + f_0,$$

where

$$g_{0,1} := \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \left(M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} f d\mu \right) 1_{I_{A(n)}(x_{n,i})} \text{ and } g_{0,2} := f 1_{G_m^2 \setminus F}.$$

Discuss the functions $f_{n,i}$. supp $f_{n,i} \subset I_{A(n)}(x_{n,i})$ and

$$\int\limits_{I_{A(n)}(x_{n,i})} f_{n,i} d\mu =$$

$$= \int\limits_{I_{A(n)}(x_{n,i})} \left(f(t) - M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} f(y) d\mu(y) \right) d\mu(t) = 0.$$

Moreover,

$$M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} |f_{n,i}| d\mu \leq M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} |f| d\mu +$$

$$+ M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} \left| M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} f d\mu \right| d\mu \leq 2C_{A,p}\lambda.$$

Since the rectangles $I_{A(n)}(x_{n,i})$ are disjoint, then we also have

$$\|f_0\|_1 \leq \int\limits_F |f| d\mu + \int\limits_{G_m^2 \setminus F} |f| d\mu = \|f\|_1.$$

The only relation left to prove is the inequality $\|f_0\|_\infty \leq C_{A,p}\lambda$. Recall that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \left(M_{A_1(n)} M_{A_2(n)} \int\limits_{I_{A(n)}(x_{n,i})} f d\mu \right) 1_{I_{A(n)}(x_{n,i})} + f 1_{G_m^2 \setminus F} =:$$

$$=: g_{0,1} + g_{0,2}.$$

First, discuss function $g_{0,1}$.

$$|g_{0,1}| \leq \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} C_{A,p} \lambda 1_{I_{A(n)}(x_{n,i})} = C_{A,p} \lambda 1_{\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{l_n} I_{A(n)}(x_{n,i})} \leq C_{A,p}\lambda.$$

Secondly, discuss function $g_{0,2}$. If $x \in F$, then $g_{0,2}(x) = f(x) 1_{G_m^2 \setminus F}(x) = 0$.

Finally, suppose that $x \notin F$. This gives

$$M_{A_1(i)} M_{A_2(i)} \int\limits_{I_{A(i)}(x)} |f(y)| d\mu(y) \leq \lambda \quad \text{for all} \quad i = 0, 1, \ldots .$$

This implies

$$S_{M_{A_1(i)},M_{A_2(i)}}|g_{0,2}(x)| = S_{M_{A_1(i)},M_{A_2(i)}}|f1_{G_m^2\setminus F}| \le S_{M_{A_1(i)},M_{A_2(i)}}|f(x)| \le \lambda.$$

Since functions A_1 and A_2 are monotone increasing, then the partial sum operators $S_{M_{A_1(n)},M_{A_2(n)}}$ form a martingale with respect to the σ-algebras $\mathcal{A}_{A(n)} = \left\{I_{A(n)}(x) : x \in G_m^2\right\}$ $(n \in \mathbb{N})$. Therefore, by the well-known martingale convergence theorem we have that

$$|g_{0,2}(x)| = \lim S_{M_{A_1(n)},M_{A_2(n)}}|g_{0,2}(x)| \le \limsup S_{M_{A_1(n)},M_{A_2(n)}}|g_{0,2}(x)| \le \lambda$$

a.e. That is, $|g_{0,1}| \le C_{A,p}\lambda, |g_{0,2}| \le \lambda$ and consequently $\|f_0\|_\infty \le C_{A,p}\lambda$. (Recall that supp $g_{0,1} \subset F$ and supp $g_{0,2} \subset G_m^2 \setminus F$.) ∎

Define the following two-dimensional ,,shifted partial sum" or ,,shifted expectation operator" for $A = (A_1, A_2) : \mathbb{N} \to \mathbb{N}^2$ and $i = (i_1, i_2)$, $s = (s_1, s_2) \in \in \mathbb{N}^2$:

$$\begin{aligned} &E_{A(n),i,s}f(x) := \\ &= M_{A_1(n)}M_{A_2(n)} \int\limits_{I_{A_1(n)}(x_1+s_1e_{A_1(n)-i_1})\times I_{A_2(n)}(x_2+s_2e_{A_2(n)-i_2})} f(y)d\mu(y) = \\ &= S_{M_{A_1(n)},M_{A_2(n)}}f(x_1+s_1e_{A_1(n)-i_1}, x_2+s_2e_{A_2(n)-i_2}), \end{aligned}$$

where $e_j := (0,\dots,\underset{\smile}{\overset{j}{1}},0,0,\dots) (j \in \mathbb{N})$. However, $E_{A(n),i,s}f(x)$ periodic with respect to s_1 and s_2. That is,

$$E_{A(n),i,(s_1+m_{A_1(n)-i_1},s_2)}f(x) = E_{A(n),i,(s_1,s_2+m_{A_2(n)-i_2})}f(x) = E_{A(n),i,s}f(x).$$

Thus, we can suppose that $s_j \in \left\{0,1,\dots m_{A_j(n)-i_j} - 1\right\}$ $(j = 1,2)$. We can also say that $s_j \le p$ $(j = 1,2)$. Its maximal operator

$$E^*_{A,i}f := \sup\left\{|E_{A(n),i,s}f(x)| : n \in \mathbb{N}, A_j(n) \ge i_j, s \in \mathbb{N}^2, j = 1,2\right\}.$$

If $i_j > A_j(n)$ for $j = 1$ or $j = 2$, then let $E_{A(n),i,s}f(x) = 0$ (for all s). Besides, also for integers $k_1 \le 0$ or $k_2 \le 0$ with any s set $E_{k,i,s}f(x) = 0$ for every $i \in \mathbb{N}^2$ and $x \in G_m^2$. Also suppose $A_j(+\infty) = +\infty$ $(j = 1,2)$.

The following weak type inequality will play a fundamental role in the proof of the main theorem.

Lemma 2.3. *Let $A_1, A_2 : \mathbb{N} \to \mathbb{N}$ be monotone increasing functions such that $A_j(+\infty) = +\infty$ $(j = 1,2)$ and let $f \in L^1(G_m^2), \lambda > 0$. Then we have*

$$\mu\left\{x \in G_m^2 : E^*_{A,i}f(x) > \lambda\right\} \le \frac{C_{A,p}(i_1+1)(i_2+1)}{\lambda}\|f\|_1$$

for every $i_1, i_2 \in \mathbb{N}$.

Proof. In this lemma sequences A_1, A_2 are monotone increasing, but in other point of view they are arbitrary. That is, they can grow ,,very fast". Nevertheless, we can suppose that $A : \mathbb{N} \to \mathbb{N}^2$ satisfies the condition of Lemma 2.2, that is sequences $A_j(n+1) - A_j(n)$ are nonnegative and bounded (by some $C_{A,p}$) $(j = 1, 2)$. This can be supposed, that is, this is really not a restriction since we can insert members within the elements of the sequence $E_{A(n),i,s}f$. For instance this can be demonstrated in the following way:

$$\begin{aligned}&(A_1(n), A_2(n)), (A_1(n), A_2(n) + 1), \ldots, (A_1(n), A_2(n + 1)),\\&(A_1(n) + 1, A_2(n + 1)), (A_1(n) + 2, A_2(n + 1)), \ldots, (A_1(n + 1), A_2(n + 1)).\end{aligned}$$

If we denote by $\tilde{A}$ this modified sequence, then for $\tilde{A}$ we certainly have $0 \le \tilde{A}_j(n + 1) - \tilde{A}_j(n) \le 1$ $(n \in \mathbb{N}, j = 1, 2)$. Besides,

$$E^*_{\tilde{A},i}f \ge E^*_{A,i}f.$$

This means that if we prove the inequality

$$\mu\left\{E^*_{\tilde{A},i}f > \lambda\right\} \le \frac{C_{A,p}(i_1 + 1)(i_2 + 1)}{\lambda}\|f\|_1,$$

then we also have it for A. As a result of this assumption we can suppose that $0 \le A_j(n + 1) - A_j(n) \le \log_p C_{A,p}$ $(n \in \mathbb{N}, j = 1, 2)$. Apply Lemma 2.2.

$$F = \bigcup_{n=1}^{\infty} J_n = \bigcup_{n=1}^{\infty} I_{A(k_n)}(x_n) = \bigcup_{n=1}^{\infty} \left(I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2})\right).$$

Enlarge the rectangle J_n in the following way:

$$\begin{aligned}J_{n,i} := \bigcup_{l_2=0}^{i_2} \bigcup_{l_1=0}^{i_1} \bigcup_{s_1=0}^{m_{A_1(k_n)-l_1}-1} \bigcup_{s_2=0}^{m_{A_2(k_n)-l_2}-1} & I_{A_1(k_n)}(x_{n,1} + s_1 e_{A_1(k_n)-l_1}) \times \\ & \times I_{A_2(k_n)}(x_{n,2} + s_2 e_{A_2(k_n)-l_2}).\end{aligned}$$

Also set $F_i := \bigcup_{n=1}^{\infty} J_{n,i}$. The shift invariancy of measure μ gives

$$\mu(F_i) \le (i_1 + 1)(i_2 + 1)p^2\mu(F) \le (i_1 + 1)(i_2 + 1)p^2\|f\|_1/\lambda.$$

In the sequel, we prove that $E^*_{A,i}f_n = 0$ on the set $G^2_m \setminus J_{n,i}$. That is, let $z \in G^2_m \setminus J_{n,i}$.

If $u \le k_n$, then

$$\begin{aligned}
&E_{A(u),i,s} f_n(z) = S_{M_{A_1(u)}, M_{A_2(u)}} f_n(z_1 + s_1 e_{A_1(u)-i_1}, z_2 + s_2 e_{A_2(u)-i_2})\\
&= \int\limits_{G_m^2} f_n(y_1,y_2) D_{M_{A_1(u)}}(z_1 + s_1 e_{A_1(u)-i_1} - y_1)\times\\
&\qquad \times D_{M_{A_2(u)}}(z_2 + s_2 e_{A_2(u)-i_2} - y_2) d\mu(y)\\
&= \int\limits_{I_{A_1(k_n)}(x_{n,1})\times I_{A_2(k_n)}(x_{n,2})} f_n(y_1,y_2) D_{M_{A_1(u)}}(z_1 + s_1 e_{A_1(u)-i_1} - x_{n,1})\times\\
&\qquad \times D_{M_{A_2(u)}}(z_2 + s_2 e_{A_2(u)-i_2} - x_{n,2}) d\mu(y) = 0,
\end{aligned}$$

since

$$\int\limits_{I_{A_1(k_n)}(x_{n,1})\times I_{A_2(k_n)}(x_{n,2})} f_n(y_1,y_2) d\mu(y) = 0.$$

On the other hand, if $u > k_n$, then $y_j \in I_{A_j(k_n)}(x_{n,j})$ and $D_{M_{A_j(u)}}(z_j + {}+ s_j e_{A_j(u)-i_j} - y_j) \neq 0$ implies

$$\begin{aligned}
z_j &\in I_{A_j(u)}(y_j - s_j e_{A_j(u)-i_j}) \subset I_{A_j(k_n)}(y_j - s_j e_{A_j(u)-i_j}) =\\
&= I_{A_j(k_n)}(x_{n,j} - s_j e_{A_j(u)-i_j}) = I_{A_j(k_n)}(x_{n,j} + (m_{A_j(u)-i_j} - s_j) e_{A_j(u)-i_j}) \subset\\
&\subset \bigcup_{l_j=0}^{i_j} \bigcup_{s=0}^{m_{A_j(k_n)-l_j}-1} I_{A_j(k_n)}(x_{n,j} + s e_{A_j(k_n)-l_j}) \quad (j=1,2).
\end{aligned}$$

The relation $\subset$ in the last line above is implied by the fact that for $A_j(u) - i_j \ge {} \ge A_j(k_n)$ we have $I_{A_j(k_n)}(x_{n,j} - s_j e_{A_j(u)-i_j}) = I_{A_j(k_n)}(x_{n,j})$ and for $A_j(u) - {} - i_j < A_j(k_n)$ by $u > k_n$ we get $A_j(k_n) - i_j \le A_j(u) - i_j < A_j(k_n)$ and thus there exist $l_j \in \{1,\dots,i_j\}, s$ such that

$$I_{A_j(k_n)}(x_{n,j} - s_j e_{A_j(u)-i_j}) = I_{A_j(k_n)}(x_{n,j} + s e_{A_j(k_n)-l_j}).$$

That is, in every case $D_{M_{A_j(u)}}(z_j + s_j e_{A_j(u)-i_j} - y_j) \neq 0$ would give the relation

$$z_j \in \bigcup_{l_j=0}^{i_j} \bigcup_{s_j=0}^{m_{A_j(k_n)-l_j}-1} I_{A_j(k_n)}(x_{n,j} + s_j e_{A_j(k_n)-l_j}) \quad (j=1,2).$$

Thus, $z = (z_1, z_2)$ would be an element of $J_{n,i}$. This contradiction implies that $E_{A(n),i,s} f_n(z) = 0$ for all $n, s \in \mathbb{N}$ and $z \in G_m^2 \setminus J_{n,i}$. This gives $E^*_{A,i} f_n = 0$ on the set $z \in G_m^2 \setminus J_{n,i}$.

After then, we prove that operator $E^*_{A,i}$ is of type (∞,∞). More precisely, we prove $\|E^*_{A,i}g\|_\infty \le \|g\|_\infty$ for each $g\in L^\infty(G_m^2)$. This is quite simple to verify. This property of $E^*_{A,i}$ by the inequality $\|f_0\|_\infty \le C_{A,p}\lambda$ gives

$$\mu\left\{E^*_{A,i}f_0 > C_{A,p}\lambda\right\} = 0.$$

Consequently, by the σ-sublinearity of operator $E^*_{A,i}$ we have

$$\mu\left\{E^*_{A,i}f > 2C_{A,p}\lambda\right\} \le \mu\left\{E^*_{A,i}f_0 > C_{A,p}\lambda\right\} + \mu\left\{E^*_{A,i}\left(\sum_{n=1}^\infty f_n\right) > C_{A,p}\lambda\right\} \le$$

$$\le \mu(F_i) + \mu\left\{x\in G_m^2\setminus F_i : E^*_{A,i}\left(\sum_{n=1}^\infty f_n\right)(x) > C_{A,p}\lambda\right\} \le$$

$$\le \frac{(i_1+1)(i_2+1)}{\lambda}p^2\|f\|_1 + \frac{1}{C_{A,p}\lambda}\int\limits_{G_m^2\setminus F_i} E^*_{A,i}\left(\sum_{n=1}^\infty f_n\right)d\mu \le$$

$$\le \frac{(i_1+1)(i_2+1)}{\lambda}p^2\|f\|_1 + \frac{1}{C_{A,p}\lambda}\sum_{n=1}^\infty\int\limits_{G_m^2\setminus F_i} E^*_{A,i}f_n d\mu \le$$

$$\le \frac{(i_1+1)(i_2+1)}{\lambda}p^2\|f\|_1 + \frac{1}{C_{A,p}\lambda}\sum_{n=1}^\infty\int\limits_{G_m^2\setminus J_{n,i}} E^*_{A,i}f_n d\mu =$$

$$= \frac{(i_1+1)(i_2+1)}{\lambda}p^2\|f\|_1.$$

This completes the proof of Lemma 2.3. ∎

Proof of Lemma 2.1. Let $A_j(n) := \lfloor\log_p a_j(n)\rfloor$ $(n\in\mathbb{N}, j=1,2)$. That is, $p^{A_j(n)} \le a_j(n) < p^{A_j(n)+1}$. Since sequences $\lfloor\log_p a_j\rfloor$ are monotone increasing for both $j=1,2$, then the condition of Lemma 2.3 are satisfied.

In the paper of Simon and Pál [11] one can find the the following estimation for the one-dimensional Vilenkin Fejér kernel functions for $M_A \le n < M_{A+1}$:

$$|nK_n(x)| \le C_p\sum_{\nu=0}^A M_\nu\sum_{i=\nu}^A\left(D_{M_i}(x) + \sum_{s=0}^{m_\nu-1}D_{M_i}(x+se_\nu)\right) \le$$

$$\le C_p\sum_{\nu=0}^A M_\nu\sum_{i=\nu}^A\sum_{s=0}^{m_\nu-1}D_{M_i}(x+se_\nu).$$

Since $M_\nu/n \le M_\nu/M_A \le 2^{\nu-A}$, the we have

$$|K_n(x)| \le C_p \sum_{j=0}^{A}\sum_{\nu=0}^{j} 2^{\nu-A} \sum_{s=0}^{m_\nu-1} D_{M_j}(x+se_\nu) =$$

$$= C_p \sum_{k=0}^{A}\sum_{i=0}^{A-k} 2^{-k-i} \sum_{s=0}^{m_{A-k-i}-1} D_{M_{A-k}}(x+se_{A-k-i})$$

as we used the variable change $k = A-j, i = j-\nu = A-k-\nu$. This implies for the two dimensional Vilenkin–Fejér means $\sigma_{a(n)}f$

$$|\sigma_{a_1(n),a_1(n)}f| \le$$

$$\le C_p \sum_{k_1=0}^{A_1(n)}\sum_{k_2=0}^{A_2(n)}\sum_{i_1=0}^{A_1(n)-k_1}\sum_{i_2=0}^{A_2(n)-k_2}\sum_{s_1=0}^{m_{A_1(n)-k_1-i_1}-1}\sum_{s_2=0}^{m_{A_2(n)-k_2-i_2}-1} 2^{-k_1-k_2-i_1-i_2}\times$$

$$\times E_{(A_1(n)-k_1,A_2(n)-k_2),(i_1,i_2),(s_1,s_2)}|f|,$$

where sequence $A-k$ is $(A_1(n)-k_1, A_2(n)-k_2)$.

Recall the definition of $E_{A,i,s}$. More precisely, the fact that for $A_1(n)-k_1 \le \le 0$ or $A_2(n)-k_2 \le 0$ or $A_1(n)-k_1 < i_1$ or $A_2(n)-k_2 < i_2$ we have $E_{A-k,i,s}f(x) = 0$ for each $x \in G_m^2$. Also recall that

$$C_{A,p} = p^{\sup_n\{A_1(n+1)-A_1(n)+A_2(n+1)-A_2(n)\}}.$$

($A_1(0) = A_2(0) = 0$ may be supposed.) That is, for every fixed $k \in \mathbb{N}^2$ the equality $C_{A-k,p} = C_{A,p}$ can be supposed and by Lemma 2.3

$$\mu\left\{E^*_{A-k,i}|f| > \lambda\right\} \le C_{A,p}(i_1+1)(i_2+1)\||f|\|_1/\lambda = C_{A,p}(i_1+1)(i_2+1)\|f\|_1/\lambda.$$

Set

$$\sigma^*_A f := \sup_n |\sigma_{a_1(n),a_2(n)}f|.$$

Then

$$\sigma^*_A f \le C_p \sum_{k_1,k_2\in\mathbb{N}}\sum_{i_1,i_2\in\mathbb{N}} 2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f|.$$

We get that the operator σ^*_A is of weak type $(1,1)$ in the following way:

$$\mu\{\sigma^*_A f > \lambda\} \le \mu\left\{C_p \sum_{k_1,k_2\in\mathbb{N}}\sum_{i_1,i_2\in\mathbb{N}} 2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f| > \lambda\right\} \le$$

$$\le \mu\left(\bigcup_{k_1,k_2\in\mathbb{N}}\bigcup_{i_1,i_2\in\mathbb{N}}\left\{2^{-k_1-k_2-i_1-i_2}E^*_{A-k,i}|f| > \frac{C_p\lambda}{(k_1k_2i_1i_2+1)^2}\right\}\right) \le$$

$$\leq \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} \mu\left\{2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f| > \frac{C_p\lambda}{(k_1k_2i_1i_2+1)^2}\right\} \leq$$

$$\leq C_{A,p} \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} \frac{(k_1k_2i_1i_2+1)^2(i_1+1)(i_2+1)}{\lambda 2^{k_1+k_2+i_1+i_2}} \|f\|_1 \leq$$

$$\leq C_{A,p}\|f\|_1/\lambda.$$

That is, we proved that the maximal operator σ^*_A is of weak type $(1,1)$. Since for each Vilenkin polynomial P we have the everywhere relation

$$\lim_{n\to\infty} \sigma_{a_1(n),a_2(n)} P = P,$$

then by the standard density argument (see this principal for instance [10]) the proof of Lemma 2.1 is complete. ■

Finally, we have to prove Theorem 2.1. That is, the main result of this paper. The proof comes from Lemma 2.1 with some easy calculations.

Proof of Theorem 2.1. Let L be a positive integer discussed later. For $l,\nu = 0,1,\dots,L-1$ let some disjoint subsets of $\mathbb{N}$ be defined as:

$$B_{l,\nu} = \left\{n\in\mathbb{N} : (a_1(n),a_2(n)) \in \bigcup_{s,t=0}^{\infty} [p^{sL+l}, p^{sL+l+1}) \times [p^{tL+\nu}, p^{tL+\nu+1})\right\}.$$

It is clear that these sets are pairwise disjoint and their union is $\mathbb{N}$. Denote the elements of $B_{l,\nu}$ by $n_1^{l,\nu} < n_2^{l,\nu} < \dots$. We prove that $\lfloor \log_p a_j(n_k^{l,\nu})\rfloor \leq$ $\leq \lfloor \log_p a_j(n_{k+1}^{l,\nu})\rfloor$ for every $k\in\mathbb{N}, l,\nu \in \{0,1,\dots L-1\}$ and $j=1,2$. On the contrary, suppose that $\lfloor \log_p a_j(n_{k+1}^{l,\nu})\rfloor < \lfloor \log_p a_j(n_k^{l,\nu})\rfloor$ for some k,l,ν and j. Then the definition of $B_{l,\nu}$ gives that $\lfloor \log_p a_j(n_{k+1}^{l,\nu})\rfloor \leq \lfloor \log_p a_j(n_k^{l,\nu})\rfloor - L$. Thus,

$$\frac{1}{p} a_j(n_{k+1}^{l,\nu}) \leq p^{\lfloor \log_p a_j(n_{k+1}^{l,\nu})\rfloor} \leq p^{\lfloor \log_p a_j(n_k^{l,\nu})\rfloor - L} \leq \frac{1}{p^L} a_j(n_k^{l,\nu}).$$

Since, $n_{k+1}^{l,\nu} > n_k^{l,\nu}$, then we have $a_j(n_{k+1}^{l,\nu}) \geq \delta a_j(n_k^{l,\nu})$ and consequently, also have $\delta \leq p^{1-L}$. This is obviously not possible for an L large enough. That is, we proved that $\lfloor \log_p a_j(n_k^{l,\nu})\rfloor$ is monotone increasing with respect to $k\in\mathbb{N}$. Lemma 2.1 gives the a.e. convergence

$$\lim_{k\to\infty} \sigma_{a(n_k^{l,\nu})} f = f$$

for each integrable function f and $l,\nu = 0,1,\dots,L-1$. Merging the L^2 pieces of subsequences of $\sigma_{a(n)}f$ the proof of Theorem 2.1 is complete. ■

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