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RATIONAL ORTHOGONAL SYSTEMS ON THE PLANE

Dedicated
to Professor János Fehér on his 75th birthday
and
to Professor Karl-Heinz Indlekofer on his 70th birthday

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Abstract. In this paper we construct rational orthogonal and biorthogonal systems on the unit disc with respect to the area measure. These orthogonal systems can be considered as the planar version of the Malmquist–Takenaka systems defined on the unit circle. Similarly to the one dimensional case the starting functions are elementary rational functions and the Gram–Schmidt orthogonalization process is applied to them. Unfortunately, unlike the Malmquist–Takenaka systems there exist no explicit representation for the generated orthogonal system in the two dimensional case. We show that if the poles of the starting elementary rational functions have multiplicity 2 then the orthogonalization process reduces to an interpolation problem. Moreover the orthogonal projection can be directly calculated by means of the values of the function taken only at the so called inverse poles. At the end of the paper we present results of numerical tests that were performed for several test functions and parameters.

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1. Introduction

Rational orthogonal and biorthogonal systems have been successfully employed in several areas including signal processing ([5]), in particular mathematical modeling of ECG signals ([3], [4]), system and control theories ([7]), and theory of Hardy- and Bergman-spaces ([1], [6], [8], [9], [10]). The members of these systems are rational functions with poles located outside the closed unit disc $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ of the complex plane $\mathbb{C}$. The set of analytic functions on $\overline{\mathbb{D}}$ will be denoted by $\mathfrak{A}$, and $\mathfrak{R}$ will stand for the set of rational functions belonging to $\mathfrak{A}$. Then $\mathfrak{R}$ can be decomposed as

$$\mathfrak{R} = \text{span}\, \mathfrak{P} \cup \mathfrak{R}_0 \,,$$

where $\mathfrak{P}$ is the set of polynomials, and $\mathfrak{R}_0$ is the set of proper rational functions in $\mathfrak{A}$.

Clearly, $\mathfrak{P}$ is the linear span of the power functions $h^m(z) = z^m$ $(z \in \mathbb{C}, m \in \mathbb{N})$. On the other hand the *elementary rational functions* of the form

$$r_{a,m}(z) := \frac{1}{(1 - \overline{a}z)^m}$$
$$(a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},\ z \in \overline{\mathbb{D}},\ m \in \mathbb{N}^* := \{1, 2, \dots\})$$

play the same role with respect to $\mathfrak{R}_0$:

$$\mathfrak{R}_0 = \text{span}\{r_{a,m} : a \in \mathbb{D},\ a \neq 0,\ m \in \mathbb{N}^*\} \,.$$

The mirror image of a with respect to the unit circle is $a^* := 1/\overline{a} \notin \overline{\mathbb{D}}$. It is the pole of $r_{a,m}$, of order m. Therefore the parameter a itself will be called the *inverse pole* of order m of the elementary rational function $r_{a,m}$.

Taking the boundary values of $f, g \in \mathfrak{A}$ on the torus we introduce the notation

$$[f, g] := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{g}(e^{it}) \, dt \qquad (f,\ g \in \mathfrak{A}) \,, \tag{1.1}$$

which is in fact the scalar product in $L^2 := L^2(\mathbb{T})$ $(\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\})$. If one of the functions is $h^m r_{a,m+1}$ then this scalar product can be directly calculated (see Theorem 2.1.):

$$[f, h^m r_{a,m+1}] = \frac{1}{m!} f^{(m)}(a) \qquad (a \in \mathbb{D},\ m \in \mathbb{N},\ f \in \mathfrak{A}) \,. \tag{1.2}$$

Based on this formula an explicit form can be given for the so called *Malmquist–Takenaka* systems (MT systems). The MT systems, which are used for approximation of periodic functions, are orthogonal systems generated from elementary rational functions by means of Gram–Schmidt orthogonalization. In this paper we introduce the planar version, i.e. when L^2 replaced by the Bergman space, of the MT systems in order to develop an effective method for approximating surfaces. In optometry, in particular in the mathematical representation of cornea the Zernike functions play a special role (see e.g [2]). The Zernike functions form an orthonormal system on the unit disc with respect to the scalar product

$$\langle f, g\rangle := \int_{\mathbb{D}} f(z)\overline{g}(z)\, d\sigma(z) \qquad (f,\, g \in \mathfrak{A})\,, \tag{1.3}$$

where σ is the area measure $d\sigma(z) = dxdy/\pi$ $(z = x + iy \in \mathbb{D})$. Taking elementary rational functions and applying Gram–Schmidt orthogonalization with this scalar product for them one can generate orthonormal systems of functions which are in $\mathfrak{R}_0$. Such systems can be regarded as the planar analogues of the MT systems. We will show, see Theorem 2.1., that the scalar product (1.3) enjoys a property similar to (1.2)

$$\langle f, r_{a,m}\rangle = \frac{1}{(m-1)!}(h^{m-2}f^{(-1)})^{(m-1)}(a) \qquad (a \in \mathbb{D},\, m \in \mathbb{N}^*,\, f \in \mathfrak{A})\,, \tag{1.4}$$

where

$$f^{(-1)}(z) := \int_0^z f(\zeta)\, d\zeta \qquad (z \in \mathbb{D})$$

is the antiderivative of f.

The paper is organized as follows. In Section 2. we show the two formulas (1.2), and (1.4) with respect to the scalar products (1.1), and (1.3). Then using these formulas we show that the construction of MT systems can be reduced to a problem of interpolation type. In Section 3. we introduce the PMT systems, the planar analogues of MT systems. Then the cases when the inverse poles are of second degree are discussed in detail. Two constructions will be given for the projections onto the subspaces generated by them. One is based on the bases of elementary rational functions, the other on the PMT orthogonal bases. In both cases discrete formulas will be derived for the coefficients. They are given by the values of the corresponding function in $\mathfrak{A}$ taken at the inverse poles that generate the system. By means of these formulas numerical methods can be constructed for interpolation and extrapolation of surfaces. In Section 4 we summarize our test results and experiences concerning the related numerical calculations.

2. Projection onto rational subspaces

First we deal with the connection between the scalar products (1.1) and (1.3). For $f, g \in \mathfrak{A}$ let us take the power series representations

$$f(z) = \sum_{n=0}^{\infty} u_n z^n\,, \qquad g(z) = \sum_{n=0}^{\infty} v_n z^n$$

($z \in \overline{\mathbb{D}}$, $u_n = f^{(n)}(0)$, $v_n = g^{(n)}(0)$, $n \in \mathbb{N}$). Since these series are uniformly and absolute convergent we have

$$[f, g] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=0}^{\infty} u_k e^{ikt}\Big)\Big(\sum_{k=0}^{\infty} \overline{v_k} e^{-ikt}\Big)\, dt = \sum_{k=0}^{\infty} u_k \overline{v_k}\,. \tag{2.1}$$

Taking the (1.3) scalar product of the same functions we obtain

$$\begin{aligned} \langle f, g\rangle &= \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} r\Big(\sum_{k=0}^{\infty} u_k r^k e^{ikt}\Big)\Big(\sum_{k=0}^{\infty} \overline{v_k} r^k e^{-ikt}\Big)\, dt\, dr = \\ &= 2\int_0^1 r\Big(\sum_{k=0}^{\infty} u_k \overline{v_k} r^{2k}\Big)\, dr = \sum_{k=0}^{\infty} \frac{u_k \overline{v_k}}{k+1}\,. \end{aligned} \tag{2.2}$$

Set $f^{(-1)}(z) = \int_0^z f(\zeta)\, d\zeta = \sum_{n=0}^{\infty} u_n \frac{z^{n+1}}{n+1}$, and define the operator J as follows

$$(Jf)(z) := \frac{1}{z} \int_0^z f(\zeta)\, d\zeta = \frac{f^{(-1)}(z)}{z} = \sum_{n=0}^{\infty} \frac{u_n z^n}{n+1} \qquad (z \in \overline{\mathbb{D}})\,. \tag{2.3}$$

Then by (2.1), and (2.2) the following relation holds true for the two scalar products

$$\langle f, g\rangle = [Jf, g] = [f, Jg] \qquad (f, g \in \mathfrak{A})\,. \tag{2.4}$$

In the next theorem we prove interpolation formulas for scalar products involving rational functions.

Theorem 2.1. *If* $f \in \mathfrak{A}$ *then*

(i) $[f, h^{m-1} r_{a,m}] = \frac{1}{(m-1)!} f^{(m-1)}(a)$,

(ii) $\langle f, r_{a,m}\rangle = \frac{1}{(m-1)!} (h^{m-2} f^{(-1)})^{(m-1)}(a) \qquad (a \in \mathbb{D},\ m \in \mathbb{N}^*)$.

Proof. i) By definition we have

$$[f, h^{m-1} r_{a,m}] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{it}) e^{-i(m-1)t}}{(1 - a e^{-it})^m} \, dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{it}) e^{it}}{(e^{it} - a)^m} \, dt = \int_{|\zeta|=1} \frac{f(\zeta)}{(\zeta - a)^m} \, d\zeta.$$

Then part i) follows by Cauchy's theorem.

ii) Using (2.1), (2.4) and part ii) of this theorem we obtain

$$\langle f, r_{a,m} \rangle = [Jf, r_{a,m}] = [h^{m-1} Jf, h^{m-1} r_{a,m}] = [h^{m-2} f^{(-1)}, h^{m-1} r_{a,m}] =$$
$$= \frac{1}{(m-1)!} \big(h^{m-2} f^{(-1)} \big)^{(m-1)} (a) \, . \qquad \blacksquare$$

It is appropriate to generate the MT systems by the sequence $\mathfrak{a} := (a_n \in \mathbb{D}, n \in \mathbb{N})$ of inverse poles where the terms do not need to be different from each other. Then the multiplicity of a_n is the number of indices $j \le n$ for which $a_j = a_n$. This number will be denoted by m_n. For the function series that generates the MT system we will use the notation

$$R_k^{\mathfrak{a}}(z) := (h^{m_k - 1} r_{a_k, m_k})(z) = \frac{z^{m_k - 1}}{(1 - \overline{a_k} z)^{m_k}} \qquad (z \in \overline{\mathbb{D}}, \, k \in \mathbb{N}) \, .$$

Let us consider the orthogonal projection $P_n^{\mathfrak{a}} : \mathfrak{A} \to \mathfrak{R}_n^{\mathfrak{a}}$ onto the $(n+1)$ dimensional subspace

$$\mathfrak{R}_n^{\mathfrak{a}} := \operatorname{span}\{ R_k^{\mathfrak{a}} : 0 \le k \le n \} \qquad (n \in \mathbb{N}) \, .$$

By i) of Theorem 2.1. the orthogonal projection $P_n^{\mathfrak{a}} f$ of a function $f \in \mathfrak{A}$ can be characterized by means of the following interpolation condition:

$$(2.5) \quad (f - P_n^{\mathfrak{a}} f) \perp \mathfrak{R}_n^{\mathfrak{a}} \qquad \text{if and only if} \quad (P_n^{\mathfrak{a}} f)^{(m_j - 1)}(a_j) = f^{(m_j - 1)}(a_j)$$

$(f \in \mathfrak{A}, \, 0 \le j \le n, \, n \in \mathbb{N}^*)$.

The Gram–Schmidt orthogonalization process for $R_n^{\mathfrak{a}}$ $(n \in \mathbb{N})$ with the scalar product (1.1) results in the $M_n := M_n^{\mathfrak{a}}$ $(n \in \mathbb{N})$ *Malmquist–Takenaka system.* M_n is obviously a rational function of the form

$$M_n(z) = \frac{p_n(z)}{\prod_{k=0}^{n} (1 - \overline{a_k} z)} \qquad (z \in \mathbb{D}) \, ,$$

where p_n is a polynomial of degree n. By (2.5) we have

$$0 = [M_n, R_k] = \frac{1}{(m_k-1)!} M_n^{(m_k-1)}(a_k) \qquad (0 \le k < n)\,.$$

This implies that a_k is a root of p_n with multiplicity not less than m_k.

Let us introduce the Blaschke functions:

$$B_a(z) := \frac{z-a}{1-\overline{a}z} \qquad (z \in \overline{\mathbb{D}},\, a \in \mathbb{D})\,. \tag{2.6}$$

The Malmquist–Takenaka functions can be expressed as products of the Blaschke functions as follows (see e.g. [7])

$$M_n(z) = \frac{c_n}{1-\overline{a_n}z} \prod_{k=0}^{n-1} B_{a_k}(z) \qquad (z \in \mathbb{D})\,. \tag{2.7}$$

Since $|B_a(z)| = 1$ $(|z| = 1)$ ([7]) we have by (1.2) that

$$1 = [M_n, M_n] = |c_n|^2 [r_{a_n,1}, r_{a_n,1}] = |c_n|^2 r_{a_n,1}(a_n) = \frac{|c_n|^2}{(1-|a_n|^2)}\,.$$

Setting $c_n = \sqrt{1-|a_n|^2}$ an explicit form for the MT functions is obtained:

$$M_n^{\mathfrak{a}}(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z} \prod_{k=0}^{n-1} B_{a_k}(z) \qquad (n \in \mathbb{N},\, z \in \overline{\mathbb{D}})\,.$$

We note that in the special case $a_n = 0$ $(n \in \mathbb{N})$ the corresponding MT system is the sequence of power functions: $M_n^{\mathfrak{o}} = h^n$ $(n \in \mathbb{N})$. The MT systems can be suitable for representation of functions not only in $\mathfrak{A}$ but also in spaces that are wider than that. For instance in case of the $H^2(\mathbb{D})$ Hardy space the $(M_n^{\mathfrak{a}}, n \in \mathbb{N})$ system is complete, or equivalently an orthogonal basis, if and only if the so called Blaschke condition

$$\sum_{n=0}^{\infty} (1-|a_n|) = \infty \tag{2.8}$$

holds true. Another example is $L^2(\mathbb{T})$. Suppose that $a_0 = 0$ and the Blaschke condition holds. Let us extend the MT system to negative indices by $M_{-n} := := \overline{M}_n$ $(n \in \mathbb{N}^*)$. Then the extended $(M_n^{\mathfrak{a}},\, n \in \mathbb{Z})$ system forms an orthogonal basis in $L^2(\mathbb{T})$.

In this paper we construct orthogonal and biorthogonal rational systems, taking the scalar product (1.3) in $\mathfrak{A}$. In the construction we make use of the MT systems, the inverse

$$J^{-1}f = (hf)' \qquad (f \in \mathfrak{A})$$

of the operator J defined in (2.3), and the connection (2.4) between the scalar products on the torus and the disc. Indeed, the systems defined as

$$M_n := M_n^{\mathfrak{a}} , \qquad M_n^{\sim} := J^{-1} M_n^{\mathfrak{a}} \quad (n \in \mathbb{N}) \tag{2.9}$$

are biorthogonal to each other with respect to the scalar product (1.3). Then we can take the corresponding biorthogonal expansions

$$Q_n f := \sum_{k=0}^{n} \langle f, M_k^{\sim} \rangle M_k , \quad Q_n^{\sim} f := \sum_{k=0}^{n} \langle f, M_k \rangle M_k^{\sim} \qquad (f \in \mathfrak{A},\, n \in \mathbb{N}) .$$

The properties of these expansions are summarized in the following theorem.

Theorem 2.2.

(i) *The* $(M_n, n \in \mathbb{N})$ *and* $(M_n^{\sim}, n \in \mathbb{N})$ *systems are biorthogonal to each other with respect to the scalar product* (1.3)

$$\langle M_n, M_m^{\sim} \rangle = \delta_{mn} \qquad (m, n \in \mathbb{N}) .$$

(ii) *The operators* $Q_n : \mathfrak{A} \to \mathfrak{R}_n^{\mathfrak{a}}$ *and* $Q_n^{\sim} : \mathfrak{A} \to J^{-1}\mathfrak{R}_n^{\mathfrak{a}}$ *are projections, i.e.*

$$Q_n f = f \quad (f \in \mathfrak{R}_n^{\mathfrak{a}}) , \qquad Q_n^{\sim} g = g \qquad (g \in J^{-1}\mathfrak{R}_n^{\mathfrak{a}}) .$$

(iii) *The* Q_n, $Q_n^{\sim}$ *operators can be expressed by the orthogonal projections* $P_n^{\mathfrak{a}}$ *as follows*

$$Q_n f = P_n^{\mathfrak{a}} f , \qquad Q_n^{\sim} f = J^{-1}(P_n^{\mathfrak{a}}(Jf)) \qquad (f \in \mathfrak{A}) .$$

(iv) *If the Blaschke condition* (2.8) *holds for the sequence* $\mathfrak{a}$ *then the system* $(M_n^{\sim}, n \in \mathbb{N})$ *is complete in* $\mathfrak{A}$, *i.e.*

$$f \in \mathfrak{A} , \quad \langle f, M_n^{\sim} \rangle = 0 \quad (n \in \mathbb{N}) \qquad \text{imply} \qquad f = 0 .$$

Proof. i) By (2.4) we have

$$\langle M_n, M_m^{\sim} \rangle = \langle M_n, J^{-1} M_m \rangle = [M_n, J(J^{-1} M_m)] = [M_n, M_m] = \delta_{mn}$$

$(m, n \in \mathbb{N})$ which was to be proved.

ii) Every $f \in \mathfrak{R}_n^{\mathfrak{a}}$ can be uniquely decomposed as $f = \sum_{k=0}^{n} \lambda_k M_k$ $(\lambda_k \in \mathbb{C})$. Then it follows from i) that $\lambda_k = \langle f, M_k^{\sim} \rangle$ $(0 \le k \le n)$, and so $Q_n f = f$.

Similarly, every $g \in J^{-1}\mathfrak{R}_n^{\mathfrak{a}}$ can be written in the form $g = \sum_{k=0}^{n} \mu_k M_k^{\sim}$ $(\mu_k \in \mathbb{C})$. By i) again we have $\mu_k = \langle g, M_k \rangle$ $(0 \le k \le n)$. Consequently, $Q_n^{\sim} g = g$.

iii) We note that if f is replaced by $J^{-1}f$ in (2.4) then it takes the form

$$[f,g] = \langle J^{-1}f, g\rangle = \langle f, J^{-1}g\rangle \qquad (f,\, g \in \mathfrak{A}) .$$

Applying this we obtain

$$\begin{aligned} Q_n f &= \sum_{k=0}^{n} \langle f, M_k^{\sim}\rangle M_k = \sum_{k=0}^{n} \langle f, J^{-1}M_k\rangle M_k = \\ &= \sum_{k=0}^{n} [f, M_k] M_k = P_n^{\mathfrak{a}} f \qquad (f \in \mathfrak{A}) . \end{aligned}$$

The corresponding statement for $Q_n^{\sim} f$ can be deduced from the definition (2.9) and the relation (2.4)

$$J^{-1}P_n^{\mathfrak{a}}(Jf) = \sum_{k=0}^{n} [Jf, M_k] J^{-1}M_k = \sum_{k=0}^{n} \langle f, M_k\rangle M_k^{\sim} = Q_n^{\sim} f .$$

iv) Recall that $(M_n, n \in \mathbb{N})$ is complete in $\mathfrak{A}$. Thus the Parseval formula and (2.4) imply

$$\sum_{k=0}^{\infty} |\langle f, M_k^{\sim}\rangle|^2 = \sum_{k=0}^{\infty} \big|[f, M_k]\big|^2 = [f,f] \qquad (f \in \mathfrak{A}),$$

which was to be proved. ∎

We note that for the completeness in Bergman spaces there exist no such general necessary and sufficient condition as the Blaschke condition (2.8). Conditions under which the $(r_{a_k,2},\, k \in \mathbb{N})$ functions with special inverse poles become frames in the Bergman space are investigated in [9]. According to the identity ii) in *Theorem 2.1.* it will be suitable to take $\mathcal{R}_k^{\mathfrak{a}} := r_{a_k,m_k}$ $(k \in \mathbb{N})$ as the initial system and construct the PMT system that corresponds to it using the scalar product (1.3). In this case the orthogonal projection $\mathcal{S}_n^{\mathfrak{a}} : \mathfrak{A} \to \mathfrak{S}_n^{\mathfrak{a}}$ onto the subspace

$$\mathfrak{S}_n^{\mathfrak{a}} := \operatorname{span}\{r_{a_k,m_k} : 0 \le k \le n\} \qquad (n \in \mathbb{N}^*)$$

can be characterized by weighted interpolation condition. Namely,

$$\begin{gathered} (f - \mathcal{S}_n^{\mathfrak{a}} f) \perp \mathfrak{S}_n^{\mathfrak{a}} \\ \text{if and only if} \\ \Big(h^{m_j-2}(\mathcal{S}_n^{\mathfrak{a}} f)^{(-1)}\Big)^{(m_j-1)}(a_j) = \Big(h^{m_j-2} f^{(-1)}\Big)^{(m_j-1)}(a_j) \\ (f \in \mathfrak{A},\, 0 \le j \le n,\, n \in \mathbb{N}). \end{gathered} \tag{2.10}$$

The interpolation condition (2.10) leads to linear equations with respect to the projections $\mathcal{S}_n^{\mathfrak{a}} f$ which can be managed numerically. In what follows we deal with the case when the inverse poles have multiplicity two in detail. Then the condition (2.10) takes a simple form.

3. MT systems on the plane

Let $a_k \in \mathbb{D}$ $(k \in \mathbb{N})$ be all different, $a_0 = 0$, and set $\mathfrak{a} = (a_k, k \in \mathbb{N})$. In this section we will deal with the orthogonalization of the system of rational functions

$$\mathcal{R}_k(z) := \mathcal{R}_k^{\mathfrak{a}}(z) = r_{a_k,2}(z) = \frac{1}{(1-\overline{a_k}z)^2} \qquad (z \in \overline{\mathbb{D}}, k \in \mathbb{N})$$

with respect to the scalar product (1.3). The resulting orthonormal rational system called *planar Malmquist–Takenaka systems* (PMT systems) will be denoted by $\mathcal{M}_n = \mathcal{M}_n^{\mathfrak{a}}$ $(n \in \mathbb{N})$.

Let us start with the identity

$$\langle f, \mathcal{R}_n\rangle = \langle f, r_{a_n,2}\rangle = f(a_n) \qquad (n \in \mathbb{N}^*) \tag{3.1}$$

that follows from ii) of Theorem 2.1. Then

$$\gamma_{k\ell} := \langle \mathcal{R}_k, \mathcal{R}_\ell\rangle = \mathcal{R}_k(a_\ell) = s(\overline{a_k}a_\ell) \qquad (k, n \in \mathbb{N}^*)\,, \tag{3.2}$$

where $s(z) = \frac{1}{(1-z)^2}$ $(z \in \mathbb{D})$. Clearly, $\overline{s(z)} = s(\overline{z})$ $(z \in \mathbb{D})$ implies

$$\gamma_{k,\ell} = \overline{\gamma_{\ell,k}}\,.$$

Consequently, the Gram matrix

$$\mathbf{C}_n := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \cdots & \gamma_{0n} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \vdots & . & . & \cdots & \vdots \\ \gamma_{n0} & \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}$$

is selfadjoint and regular for every $n \in \mathbb{N}$. The later follows from the linear independence of the system $(\mathcal{R}_k,\, k \in \mathbb{N})$.

The projection $\mathcal{S}_n^{\mathfrak{a}} f$ of the function $f \in \mathfrak{A}$ onto the subspace $\mathfrak{S}_n^{\mathfrak{a}}$ can be described by the conditions

$$\mathcal{S}_n^{\mathfrak{a}} f \in \mathfrak{S}_n^{\mathfrak{a}}\,, \quad (\mathcal{S}_n^{\mathfrak{a}} f - f) \perp \mathfrak{S}_n^{\mathfrak{a}} \qquad (n \in \mathbb{N})\,.$$

Hence the linear system of equations

$$\sum_{k=0}^{n} \lambda_{nk} \langle \mathcal{R}_k, \mathcal{R}_\ell \rangle = \langle f, \mathcal{R}_\ell \rangle \qquad (0 \le \ell \le n) \tag{3.3}$$

is obtained for the projection

$$\mathcal{S}_n^{\mathfrak{a}} f = \sum_{k=0}^{n} \lambda_{nk} \mathcal{R}_k \,. \tag{3.4}$$

Both sides of this linear system can be written in an explicit form. Indeed, the explicit form for the right side comes from (3.1), while the values of the matrix elements are deduced in (3.2). Then the projection $\mathcal{S}_n^{\mathfrak{a}} f$ can be calculated by solving the linear equation (3.3).

The Schmidt orthogonalization process with the scalar product (1.3) transforms the linearly independent system $(\mathcal{R}_k, k \in \mathbb{N})$ into the PMT system $(\mathcal{M}_n, n \in \mathbb{N})$. This planar Malmquist–Takenaka system, however, doesn't admit such an explicit representation like (2.7) for the one dimensional MT functions. On the other hand it can be characterized by the following two properties:

i) $\operatorname{span}\{\mathcal{R}_i : 0 \le i \le n\} = \operatorname{span}\{\mathcal{M}_i : 0 \le i \le n\} \qquad (n \in \mathbb{N})$,

ii) $\langle \mathcal{M}_i, \mathcal{M}_j \rangle = \delta_{ij} \qquad (i,\, j \in \mathbb{N})$.

It follows from condition i) that there exist unique numbers α_{nk}, β_{nk}, $(0 \le k \le$ $\le n,\ \alpha_{nn} > 0,\ \beta_{nn} > 0)$ for which

$$\mathcal{M}_n = \sum_{k=0}^{n} \alpha_{nk} \mathcal{R}_k \,, \quad \mathcal{R}_n = \sum_{k=0}^{n} \beta_{nk} \mathcal{M}_k \qquad (n \in \mathbb{N}) \tag{3.5}$$

hold. Hence by (3.2) we have

$$\overline{\beta_{nk}} = \langle \mathcal{M}_k, \mathcal{R}_n \rangle = \sum_{j=0}^{k} \alpha_{kj} \gamma_{jn} \qquad (0 \le k \le n)\,. \tag{3.6}$$

Set the triangular matrices in $\mathbb{C}^{(n+1)\times(n+1)}$ as

$$\mathbf{A}_n := \begin{pmatrix} \alpha_{00} & 0 & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & 0 & \cdots & 0 \\ \vdots & . & . & \cdots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix},$$

$$\mathbf{B}_n := \begin{pmatrix} \beta_{00} & 0 & 0 & \cdots & 0 \\ \beta_{10} & \beta_{11} & 0 & \cdots & 0 \\ \vdots & . & . & \cdots & \vdots \\ \beta_{n0} & \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}.$$

By definition $\mathbf{A}_n^{-1} = \mathbf{B}_n$. We claim that the two sequences of matrices can be determined in a recursive way. Indeed, $\mathcal{R}_0 = \mathcal{M}_0 = 1$ implies $\alpha_{00} = \beta_{00} = 1$. Suppose that $\mathbf{A}_{n-1}, \mathbf{B}_{n-1}$ have already been determined. Then use (3.6) to calculate β_{nk} for $k < n$. It follows from (3.5) that

$$|\beta_{nn}|^2 = \langle \mathcal{R}_n, \mathcal{R}_n \rangle - \sum_{k=0}^{n-1} |\beta_{nk}|^2 > 0 \,.$$

Hence we can calculate $\beta_{nn} > 0$. Then the nth row of $\mathbf{B}_n$ is determined. $\mathbf{A}_n$ can be generated by inversion.

We note that the nth row of $\mathbf{A}_n$ can be deduced from the equations

$$\alpha_{nn} = 1/\beta_{nn} \,, \quad \sum_{j=k}^{n} \alpha_{nj}\beta_{jk} = 0 \qquad (k = n-1, n-2, \ldots, 1, 0)$$

which follow from the matrix condition $\mathbf{A}_n \mathbf{B}_n = \mathbf{E}_n$, where $\mathbf{E}_n$ is the unit matrix in $\mathbb{C}^{(n+1)\times(n+1)}$.

Clearly, (3.6) is equivalent to $\mathbf{B}_n^* = \mathbf{A}_n \mathbf{C}_n$. Consequently, $\mathbf{C}_n = \mathbf{B}_n \mathbf{B}_n^*$ $(n \in \mathbb{N})$. Recall that $\mathbf{A}_n = \mathbf{B}_n^{-1}$. Then for the inverse of $\mathbf{C}_n$ we have:

$$D_n := \mathbf{C}_n^{-1} = \mathbf{A}_n^* \mathbf{A}_n \,. \tag{3.7}$$

We conclude that the solution of (3.3) can be obtained from the matrix A_n. Consequently, see (3.4), a representation of the projection $S_n^{\mathfrak{a}} f$ in the basis $\mathcal{R}_k^{\mathfrak{a}}$ $(k \in \mathbb{N})$ is obtained. We note that the scalar products on the right side of (3.3) can be expressed by the values of the function f taken at the points a_j $(j = 1, 2, \ldots, N)$. Therefore, no integration is needed to produce the orthogonal projection $S_n^{\mathfrak{a}} f$.

On the other hand the orthogonal projection $S_n^{\mathfrak{a}} f$ can be represented in the PMT basis as the Fourier partial sum

$$S_n^{\mathfrak{a}} f = \sum_{k=0}^{n} \langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle \mathcal{M}_k^{\mathfrak{a}} \qquad (n \in \mathbb{N}) \,.$$

Using the decomposition in (3.5) and (3.1) the Fourier coefficients with respect to the PMT system can be written in the form

$$\langle f, \mathcal{M}_k^{\mathfrak{a}} \rangle = \sum_{j=0}^{k} \overline{\alpha_{kj}} \langle f, \mathcal{R}_j \rangle = \sum_{j=0}^{k} \overline{\alpha_{kj}} f(a_j) \,. \tag{3.8}$$

We conclude that the Fourier coefficients can be calculated by the $\mathbf{A}_n$ matrix, and the values of the function f at the inverse poles.

We remark that one can define a discrete version of the scalar product (1.3) on $\mathfrak{S}_n$ in which only the values at the inverse poles are taken as follows

$$\langle f, g\rangle = \sum_{i,j=0}^{n} f(a_i)\overline{g}(a_j)\overline{d_{ij}^{n}} \qquad (f, g \in \mathfrak{S}_n^{\mathfrak{a}}) . \tag{3.9}$$

Then (3.9) and (1.3) coincide on $\mathfrak{S}_n$. Indeed, if $f, g \in \mathfrak{S}_n$ then taking the PMT representation of both f, and g it follows by (3.8) and (3.7) that

$$\begin{aligned}\langle f, g\rangle &= \sum_{k=0}^{n}\langle f, \mathcal{M}_k^{\mathfrak{a}}\rangle\overline{\langle g, \mathcal{M}_k^{\mathfrak{a}}\rangle} = \sum_{i,j=0}^{n} f(a_i)\overline{g}(a_j)\sum_{k=0}^{n}\alpha_{ki}\overline{\alpha_{kj}} = \\ &= \sum_{i,j=0}^{n} f(a_i)\overline{g}(a_j)\overline{(A_n^* A_n)}_{ij} = \sum_{i,j=0}^{n} f(a_i)\overline{g}(a_j)\overline{d_{ij}^{n}} \qquad (f, g \in \mathfrak{S}_N^{\mathfrak{a}}) .\end{aligned}$$

4. Numerical tests

In Section 3. we introduced the PMT system and examined the orthogonal projection (3.4) with respect to this system. In this section we present the results of the numerical tests performed to evaluate the approximation properties of the projection operator. To this order we solved the linear system of equations (3.3) numerically to obtain the $\lambda_{k,n}$ $(0 \le k \le n)$ coefficients in (3.4).

In the test we used the following test functions:

$$f_1(z) = (1 - \overline{b}z)^2 , \qquad f_2(z) = \frac{z - b}{1 - \overline{b}z} = B_b(z) ,$$

$$f_3(z) = \log(1 - \overline{b}z) , \qquad f_4(z) = \cos z ,$$

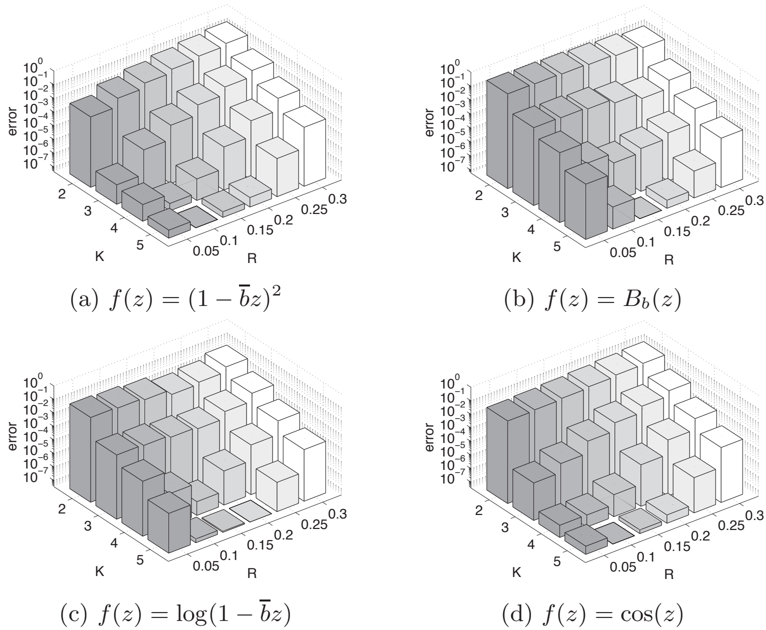
where $b = \frac{1}{2}$, and B_b is the Blaschke-function (2.6).

The inverse poles $0 = a_0, \dots a_N$ were arranged on concentric circles around the origin in groups of powers of 2 in the following way. Set $N = 2^K - 2$ $(K \in \mathbb{N}^*)$, and $0 < R < 1$. Taking the *pseudo hyperbolic metric* (see e.g. [5]) ρ defined as

$$\rho(z, w) = |B_z(w)| \qquad (z, w \in \mathbb{D})$$

let the radii $0 = r_0 < r_1 < \cdots < r_{K-1} < 1$ be uniformly distributed according to it. More precisely, let the pseudo hyperbolic distance between the consecutive values all be equal to R :

$$\rho(r_k, r_{k+1}) = \frac{r_{k+1} - r_k}{1 - r_{k+1}r_k} = R \qquad (k = 0, \dots, K - 1) .$$



(a) $f(z) = (1 - \overline{b}z)^2$ (b) $f(z) = B_b(z)$

(c) $f(z) = \log(1 - \overline{b}z)$ (d) $f(z) = \cos(z)$

Figure 1: Overview of $\|.\|_\infty$ approximation errors.

Then the inverse poles are defined as follows

$$a_{2^k-1+j} = r_k e^{i2\pi \frac{j}{2^k}} \qquad (0 \leq k < K,\, 0 \leq j < 2^k)\,.$$

The tests were performed using $K = 2, 3, 4, 5$ and $R = 0.05, 0.1, \ldots, 0.3$. We note that the case $K = 1$ involves only constant functions and is of less interest.

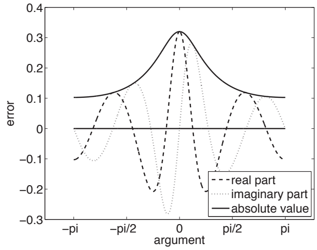
In Figures 1, 2 the dependence of the error

$$\|f_\ell - \mathcal{S}_N^{\mathfrak{a}} f_\ell\|_\infty \qquad (\ell = 1, 2, 3) \tag{4.1}$$

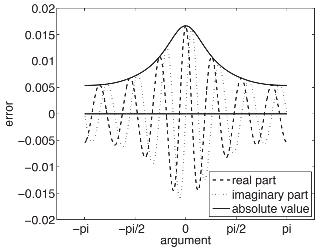
on the values of the parameters K, and R is visualized. As a consequence of the maximum principle, it is enough to consider values taken on the torus. For the estimation of the error on the torus we used uniform sampling.

In Figure 1 the approximation errors with respect to the inverse pole configuration parameters are displayed using a logarithmic scale. As an example, Figure 2 shows the approximation error on $\mathbb{T}$ in case of the Blaschke test function with $R = 0.15$.

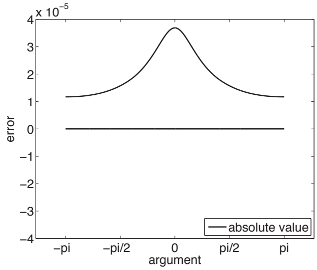
Finally, we conclude that the analytic functions on the unit disc can be well approximated by means of our method.



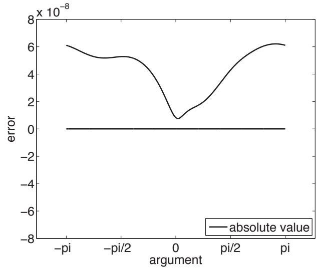
(a) $K = 2$, error < 0.33



(b) $K = 3$, error < 0.017



(c) $K = 4$, error $< 3.7 \times 10^{-5}$



(d) $K = 5$, error $< 6.3 \times 10^{-8}$

Figure 2: $\|.\|_\infty$ approximation errors of the Blaschke function, $R = 0.15$.

References

[1] **Duren, P. and A. Schuster,** *Bergman Spaces*, American Mathematical Society, Providence, RI (2004).

[2] **Fazekas, Z., F. Schipp and A. Soumelidis,** Utilizing the discrete orthogonality of Zernike functions in corneal measurements, in: S.I. Ao et al. (Eds.) *WCE 2009, London,* Lecture Notes in Engineering and Computer Science, 2009, 795–800.

[3] **Fridli, S., P. Kovács, L. Lócsi and F. Schipp,** Rational modeling of multi-lead QRS complexes in ECG signals, *Annales Univ. Sci, Budapest, Sect. Comp.*, **36** (2012), 145–155.

[4] **Fridli, S., L. Lócsi and F. Schipp,** Rational function systems in ECG processing, in: R. Moreno-Díaz et al. (Eds.) *EUROCAST 2011, Part I,* Lecture Notes in Computer Science **6927**, Springer-Verlag, Berlin, Heidelberg, 2011, 88–95.

[5] **Fridli, S. and F. Schipp,** Biorthogonal systems to rational functions, *Annales Univ. Sci, Budapest, Sect. Comp.*, **35** (2011), 95–105.

[6] **Hedenmalm, H.B. and K. Zhu Korenblum,** *Theory of Bergman Spaces*, Graduate Text in Mathematics, **199**, Springer Verlag, New York, 2000.

[7] **Heuberger, P.S.C., P.M.J. Van den Hof and Bo Wahlberg,** *Modelling and Identification with Rational Orthogonal Basis Function*, Springer Verlag, London, 2005.

[8] **Pap, M.,** Hyperbolic wavelets and multiresolution in $H^2(\mathbb{T})$, *J. Fourier Anal. Appl.*, **17(5)** (2011), 755–766.

[9] **Pap, M.,** Multiresolution in the Bergman space, *Annales Univ. Sci. Budapest., Sect. Comp.*, **39** (2013), to appear.

[10] **Pap, M. and F. Schipp,** The Voice transform on the Blaschke group II., *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 157–173.

[11] **Zhu, K.,** Interpolating and recapturating in reproducing Hilbert Spaces, *Bull. Hong Kong Math. Soc.*, **1** (1997), 21–33.

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