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ON THE MODELS OF INDLEKOFER AND KUBILIUS IN PROBABILISTIC NUMBER THEORY

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Dedicated to Professor Karl-Heinz Indlekofer on his 70th birthday

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Abstract. In his book [7] J. Kubilius constructed a finite probability space adapted for the investigation of additive functions. K.-H. Indlekofer presented in his articles [3], [4] a new method to investigate number theoretical questions with the help of probability theory. In this paper we describe the (finite) probability model of Kubilius and the model of Indlekofer which is based on the Stone–Čech compactification of $\mathbb{N}$.

1. Introduction

The book [7] of J. Kubilius mainly deals with the application of probability theory to the distribution of additive and multiplicative functions. In order to give a probability-theoretic interpretation of additive functions, say, he reduced questions on the distribution of the values of these functions to the corresponding problems of the theory of series of independent random variables.

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He constructed a finite probability space on which independent random variables could be defined so as to mimic the (infrequency) behaviour of truncated additive functions f_r, defined by

$$f_r(n) := \sum_{\substack{p|n \\ p \le r}} f(p).$$

A direct application of this model gives, for example, the celebrated theorem of Erdős and Kac.

Refinements and further applications can be found in Elliott's monograph [2].

Whereas the construction of Kubilius is adapted to the investigation of additive functions the probabilistic model of Indlekofer is more general and motivated by the following question:

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$ and let δ be a content on $\mathbb{N}$, i.e. $\delta : \mathcal{A} \to \mathbb{R}$ is finitely additive, How can one "extend" δ uniquely to some measure $\overline{\delta}$?

In this paper we describe the underlying ideas of the two models and give some typical applications.

2. Model of Kubilius

Suppose we are given an arbitrary real-valued arithmetical function $f : \mathbb{N} \to \to \mathbb{R}$. Then

$$F_n(x) := \frac{1}{n} \#\{m \le n : f(m) \le x\}, \quad -\infty < x < \infty \tag{2.1}$$

represents a distribution function. Obviously, the characteristic function for F_n is

$$\varphi_n(t) := \int_{-\infty}^{\infty} e^{itx} \, dF_n(x) = \frac{1}{n} \sum_{m=1}^{n} e^{itf(m)}. \tag{2.2}$$

The known properties of distribution functions and characteristic functions lead to the following criterion (Levy):

In order that the distribution functions (2.1) converge to some limiting distribution $F(x)$ at each of its points of continuity, it is necessary and sufficient that the $\varphi_n(t)$ converge for all real t to some function $\varphi(t)$, continuous at the point $t = 0$. The function $\varphi(t)$ is then the characterisitc function of the law $F(x)$.

Thus the question of the existence of a limiting distribution function for (2.1) reduces to the study of trigonometric sums like (2.2). In the case in which Kubilius was interested, that of additive arithmetical functions, he could show that, properly interpreted, additive arithmetical functions behave like sums of certain (simple) random variables. He could ascertain the probability-theoretic nature of the distribution of the values of these functions and reduced the questions of their asymptotic distribution to well-known limit theorems of probability theory.

We shall describe here several variants of such an interpretation due to Kubilius and follow the lines of the presentation in Chapter 2, pp 25-29 of his monograph [7]. In order to ease notational difficulties and to make the basic ideas clearer we restrict ourselves to *strongly* additive functions f, i.e.

$$f = \sum_p f(p)\varepsilon_p \tag{2.3}$$

where

$$\varepsilon_p(n) = \begin{cases} 1, & \text{if } p|n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

First variant

Let $E = \{1, 2, \ldots, n\}$ denote the set of positive integers not exceeding n and let $\mathcal{A}_1$ denote the algebra of all subsets of E. If we define a probability measure ν_n for all $A \in \mathcal{A}_1$ by

$$\nu_n(A) := \nu_n(m \in A) := \frac{1}{n}\#\{m \le n : m \in A\}, \tag{2.5}$$

then, obviously, the triple $\Omega_1 := (E, \mathcal{A}_1, \nu_n)$ becomes a probability space.

Having chosen some integer $r \ge 2$, we consider the functions $f_p = f(p)\varepsilon_p$ for primes $p \le r$. The functions ε_p (and f_p) can be regarded as random variables on E with

(2.6)

$$\frac{1}{n}\#\{m \le n : \varepsilon_p(m) = \alpha\} = \begin{cases} \frac{1}{n}\left[\frac{n}{p}\right] = \frac{1}{p} + O\left(\frac{1}{n}\right), & \text{if } \alpha = 1, \\ 1 - \frac{1}{n}\left[\frac{n}{p}\right] = 1 - \frac{1}{p} + O\left(\frac{1}{n}\right), & \text{if } \alpha = 0. \end{cases}$$

Let $p \neq q$ be two different prime numbers with $p, q \leq r$. Since

$$\frac{1}{n}\#\{m \leq n : \varepsilon_p(m) = 1, \varepsilon_q(m) = 1\} = \frac{1}{n}\left[\frac{n}{pq}\right] = \frac{1}{pq} + O\left(\frac{1}{n}\right)$$

it is obvious that for $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$

$$\nu_n(\varepsilon_p(m) = \alpha, \varepsilon_q(m) = \beta) = \nu_n(\varepsilon_p(m) = \alpha)\nu_n(\varepsilon_q(m) = \beta) + O\left(\frac{1}{n}\right). \tag{2.7}$$

Therefore the random variables $\varepsilon_p, \varepsilon_q$ and f_p, f_q, respectively, are not, in general, independent.

However, as (2.7) shows, the dependence of f_p and f_q is in some sense weak and, in general, is weaker, the smaller p and q are in comparison with n. Moreover, if $r = r(n) \geq 2$ is a function of n with $\log r = o(\log n)$ as $n \to \infty$ and $\{p_1, \ldots, p_k\}$ is a set of primes, none of which exceeds r, then it can be shown (see [7], p.22), that as $n \to \infty$,

$$\nu_n(\varepsilon_p(m) = \alpha_1, \ldots, \varepsilon_{p_k}(m) = \alpha_r) = \prod_{i=1}^{k} \nu_n(\varepsilon_{p_i}(m) = \alpha_i) + o(1), \tag{2.8}$$

and this suggests that under some assumptions on f well-known limit theorems for sums of *almost* independent random variables could be applied. However, Kubilius did not dwell on this since more general results can be obtained by relating the limit laws for additive arithmetical functions to the well-developed theory of sums of independent random variables. This aspect runs through the next two variants as a common thread.

Second variant

Formulas (2.6), (2.7) and (2.8) suggest that the functions ε_p, $p \leq r$ can be regarded as independent random variables if we change the space constructed in the first variant, taking as E the set $\mathbb{N}$ of all natural numbers instead of a finite segment of it, and as the probability measure $\nu_n(A)$ the asymptotic density $\delta(A) := \lim \nu_n(m \in A)$ of the set A (if the limit exists). Thus we proceed as follows.

Let $r \geq 2$ be a constant. For $p \leq r$ let

$$A_p := \{m : p|m\} \tag{2.9}$$

be the set of natural numbers divisible by p, and define $\mathcal{A}_2$ as the smallest algebra containing all A_p, i.e. $\mathcal{A}_2$ is generated by sets A_p (and $\mathbb{N} \setminus A_p$). Then $\mathcal{A}_2$ can be described in the following way (cf. Elliott [2], Chapter 3).

Let D denote the product of the primes not exceeding r,

$$D = \prod_{p \le r} p.$$

For each k which divides D let

$$E_k = \bigcap_{p|k} A_p \bigcap_{p|(D/k)} (\mathbb{N} \setminus A_p). \tag{2.10}$$

Obviously

$$\lim_{n\to\infty} \nu_n(m \in E_k) = \frac{1}{k} \prod_{p|(D/k)} \left(1 - \frac{1}{p}\right).$$

For differing values of k these sets are disjoint and every $A \in \mathcal{A}_2$ is a union of finitely many of them.

If

$$A = \bigcup_{j=1}^{m} E_{k_i}$$

then

$$\lim_{n\to\infty} \nu_n(m \in A) = \sum_{j=1}^{m} \lim_{n\to\infty} \nu_n(m \in E_{k_i})$$

and therefore all sets $A \in \mathcal{A}_2$ have asymptotic density $\delta(A)$.

Thus

$$\Omega_2 := (\mathbb{N}, \mathcal{A}_2, P) \tag{2.11}$$

with

$$P(A) := \delta(A) \quad \text{for } A \in \mathcal{A}_2$$

is a finite probability space.

With respect to this space the random variables f_p for $p \le r$ can be regarded as random variables assuming the values $f(p)$ and 0 with probabilities $1/p$ and $1 - 1/p$, respectively. Furthermore, the random variables f_p $(p \le r)$ are independent.

Thus, with respect to the probabilty space Ω_2 the "truncated function"

$$f_r = \sum_{p \le r} f_p \tag{2.12}$$

is a sum of independent random variables.

Combining the construction of the spaces Ω_1 and Ω_2 we arrive at the

Third variant

As in the first variant we take the finite segment $E = \{1, \dots, n\}$ of $\mathbb{N}$ as the set of elementary events. Similarly as in the second variant we define, for each integer k which divides D, the set

$$E_k := \bigcap_{p|k} E(p) \bigcap_{p|(D/k)} (E \setminus E(p)) \tag{2.13}$$

where

$$E(p) := A_p \cap E. \tag{2.14}$$

For differing values of k the sets E_k are disjoint. By taking unions of finitely many of them we form $\mathcal{A}_3$, the least σ-algebra which contains the $E(p)$ for $p \le r$.

In terms of the frequency measure ν_n we have: If

$$A = \bigcup_{j=1}^{m} E_{k_j}$$

then

$$\nu_m(A) = \sum_{j=1}^{m} \nu_n(E_{k_j}).$$

The triple $(E, \mathcal{A}_3, \nu_n)$ becomes a finite probability space Ω_3. Motivated by the construction in the second variant we define another probability measure for sets $A \in \mathcal{A}_3$. We put

$$P(E_k) = \frac{1}{k} \prod_{p|(D/k)} \left(1 - \frac{1}{p}\right)$$

and obtain

$$P(A) = \sum_{j=1}^{m} P(E_{k_j})$$

if

$$A = \bigcup_{j=1}^{m} E_{k_j}.$$

Thus we arrive at the probability space ${\Omega_3}' := (E, \mathcal{A}_3, P)$.

By sieve methods it can be shown (cf. Elliott [2], Chapter 3) that the estimate

$$\nu_n(A) = P(A) + O(L) \tag{2.15}$$

where

$$L = \exp\left(-\frac{1}{8}\frac{\log n}{\log r}\log\left(\frac{\log n}{\log r}\right)\right) \tag{2.16}$$

holds uniformly for all sets A in the algebra $\mathcal{A}_3$.

With respect to the measure P define the random variables X_p by putting $X_p(m) = f_p(m)$. Then the random variable assumes the values $f(p)$ and 0 with probabilities $1/p$ and $1-1/p$ respectively. It is easy to see that the joint distribution of random variables X_p $(p \le r)$ is equal to the product over $p \le r$ of the one-dimensional distributions of the random variables X_p. Hence it follows that the distribution with respect to the measure ν_n of the random variable

$$f_r = \sum_{p \le r} f_p \tag{2.17}$$

differs only by an amount $O(L)$ from the distribution with respect to P of the sum $\sum_{p\le x} X_p$ of independent random variables.

An immediate consequence of the above construction is as follows.

Proposition 2.1. (See Elliott [2], Lemma 3.2.) *Let r and n be natural numbers, $2 \le r \le n$. Define the strongly additive function*

$$g(m) = \sum_{\substack{p|m \\ p\le r}} f(p)$$

where the $f(p)$ assume real values. Define the independent random variables X_p on the probability space ${\Omega_3}' := (E, \mathcal{A}_3, P)$, one for each prime not exceeding r, by

$$X_p = \begin{cases} f(p), & \textit{with probability } \frac{1}{p}, \\ 0, & \textit{with probability } 1-\frac{1}{p}. \end{cases}$$

Then, the estimate

$$n^{-1}\#\{m \le n : g(n) \le z\} = P\left(\sum_{p\le r} X_p \le z\right) + \\ + O\left(exp\left(-\frac{1}{8}\frac{\log n}{\log r}\log\left(\frac{\log n}{\log r}\right)\right)\right) + O\left(n^{-\frac{1}{15}}\right)$$

holds uniformly for all numbers $f(p), z, n$ $(n \ge 2)$ and r $(2 \le r \le n)$.

The Kubilius model can directly be applied to obtain the theorem of Erdős and Kac. For this we confine our attention to real-valued strongly additive functions f. Let f be a (real-valued) strongly additive function. For positive numbers x define the functions

$$A(x) = \sum_{p \le x} f(p)p^{-1},$$

$$B(x) = \left(\sum_{p \le x} f^2(p)p^{-1} \right)^{\frac{1}{2}} \ge 0.$$

Following Kubilius, we shall say that f belongs to the class H if there exists a function $r = r(x)$ so that as $x \to \infty$,

$$\frac{\log r}{\log x} \to 0, \quad \frac{B(r)}{B(x)} \to 1, \quad B(x) \to \infty.$$

The following result of Kubilius is archetypal.

Proposition 2.2. (See Elliott [2], Theorem 12.1.) *Let f be a strongly additive function of class H. Then the frequencies*

$$x^{-1}\#\{n \le x : f(n) - A(x) \le zB(x)\} \tag{2.18}$$

converge weakly to a limiting distribution as $x \to \infty$, if and only if there is a distribution function $K(u)$, so that almost surely in u

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ f(p) \le uB(x)}} \frac{f^2(p)}{p} \to K(u) \quad (as\, x \to \infty).$$

When this condition is satisfied the characteristic function $\varphi(t)$ of the limit law will be given by Kolmogorov's formula

$$\log \varphi(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu)u^{-2}dK(u)$$

and the limit law will have mean zero, and variance 1.

Whether the frequencies (2.18) *converge or not,*

$$\frac{1}{B(x)} \sum_{n \le x} (f(n) - A(x)) \to 0, \frac{1}{xB^2(x)} \sum_{n \le x} (f(n) - A(x))^2 \to 1,$$

holds as $x \to \infty$.

Bearing in mind that in the Kolmogorov representation of the characteristic function of the normal law with variance 1 we have

$$K(u) = \begin{cases} 1, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases}$$

Then we arrive at the following result.

Proposition 2.3. (See Elliott [2], Theorem 12.3.) *Let f be a real valued strongly additive function which satisfies $|f(p)| \leq 1$ for every prime p. Let $B(x) \to \infty$ as $x \to \infty$. Then*

$$x^{-1}\#\{n \leq x : f(n) - A(x) \leq zB(x)\} \Rightarrow \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} e^{-\omega^2/2} d\omega.$$

Kubilius model may also be applied directly to the study of multiplicative functions. The following result can be given.

Proposition 2.4. (See Elliott [2], Lemma 3.10.) *Let r and x be real numbers, $2 \leq r \leq x$. Define the (strongly) multiplicative function*

$$g(n) = \prod_{\substack{p|n \\ p \leq r}} f(p)$$

where the $f(p)$ assume real values. Define independent random variables X_p, one for each prime p not exceeding r, by

$$X_p = \begin{cases} f(p), & \textit{with probability } \frac{1}{p}, \\ 1, & \textit{with probability } 1 - \frac{1}{p}. \end{cases}$$

Then the estimate

$$x^{-1}\#\{n \leq x : g(n) \leq z\} = P\left(\prod_{p \leq r} X_p \leq z\right) + O(L),$$

where

$$L = \exp\left(-\frac{1}{8}\frac{\log x}{\log r}\log\left(\frac{\log x}{\log r}\right)\right) + x^{-\frac{1}{15}}$$

holds uniformly for all real numbers $f(p), z, x$ $(x \geq 2)$ and r $(2 \leq r \leq x)$.

Remark 2.5. The above considerations are also applicable when the additive arithmetical function is studied on some sequence of positive integers instead of on the entire set $\mathbb{N}$, provided only that it is possible to apply Selberg's (or another) sieve method to this sequence.

3. The model of Indlekofer

The limitation in the second variant in Section 2 is that the construction cannot be extended to the algebra generated by the sets A_p for all primes p. The crux of this matter is that not every algebra of sets of positive integers which have asymptotic density forms a probability space or can be extended to a probability space.

For example, let $\mathcal{A}$ be generated by the finite sets of natural numbers. Then all sets have an asymptotic density δ (0 or 1). Within $\mathcal{A}$ the asymptotic density is finitely additive but not countably additive. Indeed, let $A(n) := \{n\}$. Then

$$\delta\left(\bigcup_{n=1}^{\infty} A(n)\right) = \delta(\mathbb{N}) = 1$$

while

$$\sum_{n=1}^{\infty} \delta(A(n)) = 0.$$

As a further example let $\mathcal{A}_2^{'}$ be generated by the sets A_p defined in (2.9) for all primes p. Then all sets A in $\mathcal{A}_2^{'}$ have asymptotic density $\delta(A)$. Within $\mathcal{A}_2^{'}$ the asymptotic density δ is finitely additive but cannot be extended to a countably additive measure $\overline{\delta}$ on the σ-algebra $\sigma(\mathcal{A}_2^{'})$ generated by $\mathcal{A}_2^{'}$.

For the proof assume that $\overline{\delta}$ is a measure on $\sigma(\mathcal{A}_2^{'})$ extending δ.

Put

$$A'(n) := \bigcap_{p \nmid n} (\mathbb{N} \setminus A_p).$$

Then

$$A'(n) = \{m : p|m \Rightarrow p|n\} \in \sigma(\mathcal{A}_2^{'})$$

and $\overline{\delta}(A'(n))$ exists for every $n \in \mathbb{N}$.

Since

$$\sum_{\substack{m \\ m \in A'(n)}} m^{-1/2} = \prod_{p|n} \left(1 - p^{-1/2}\right)^{-1} < \infty$$

we obviously have

$$\frac{1}{N} \sum_{\substack{m \leq N \\ m \in A'(n)}} 1 \leq N^{-\frac{1}{2}} \sum_{\substack{m \\ m \in A'(n)}} m^{-\frac{1}{2}} = O(N^{-\frac{1}{2}}),$$

and thus $\overline{\delta}(A'(n)) = 0$. Then

$$\mathbb{N} = \bigcup_n A'(n)$$

and

$$1 = \overline{\delta}\left(\bigcup_n A'(n)\right) \leq \sum_n \overline{\delta}(A'(n)) = 0$$

which contradicts the assumption.

More generally, Indlekofer considers the following problem:

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$, and let δ be finitely additive on $\mathcal{A}$. How can one build up an "integration theory" on $\mathbb{N}$?

In his papers [3], [4] Indlekofer developed such an integration theory. The underlying idea is as follows: Embedding $\mathbb{N}$ (and $\mathcal{A}$) into the Stone–Čech compactification $\beta\mathbb{N}$ and extending δ in a natural way, i.e.

$$\mathbb{N} \hookrightarrow \beta\mathbb{N}$$
$$\mathcal{A} \mapsto \overline{\mathcal{A}}$$
$$\delta \mapsto \overline{\delta}$$

leads in an unique way to an algebra $\overline{\mathcal{A}}$ together with a premeasure $\overline{\delta}$.

In this environment, for example, additive arithmetical functions can be seen as a sum of independent random variables.

In the following we describe the model of Indlekofer and give some applications regarding spaces of arithmetical functions.

Suppose that $\mathcal{A}$ is an *algebra of subsets of* $\mathbb{N}$, i.e.

(i) $\mathbb{N} \in \mathcal{A}$,

(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,

(iii) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

We embed $\mathbb{N}$, endowed with the discrete topology, in the compact space $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$. This implies

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in $\beta\mathbb{N}$, where $\bar{A} := cl_{\beta\mathbb{N}} A$ (for details see K.-H. Indlekofer [3], [4]).

Let δ be a content on $\mathcal{A}$, i.e. $\delta : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is finitely additive, and define $\bar{\delta}$ on $\bar{\mathcal{A}}$ by

$$\bar{\delta}(\bar{A}) = \delta(A), \quad \bar{A} \in \bar{\mathcal{A}}.$$

Then $\bar{\delta}$ is a pseudo-measure on $\bar{\mathcal{A}}$ and can be extended to a measure on $\sigma(\bar{\mathcal{A}})$ which we denote by $\bar{\delta}$, too. This leads to the measure space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$.

Remark 3.1. It is worthwhile to notice the following characterizations of the Stone–Čech compactification of $\mathbb{N}$ which are contained in Propositions 9 and 10 of [4].

Proposition. *There exists a compactification $\beta\mathbb{N}$ of $\mathbb{N}$ with the following equivalent properties.*

(i) *Every mapping f from $\mathbb{N}$ into any compact Hausdorff space Y has a continuous extension $\bar{f}$ from $\beta\mathbb{N}$ into Y.*

(ii) *Every bounded real-valued function on $\mathbb{N}$ has an extension to a function in $C(\beta\mathbb{N})$.*

(iii) *For any two subsets A and B of $\mathbb{N}$,*

$$\overline{A \cap B} = \bar{A} \cap \bar{B},$$

where $\bar{A} = cl_{\beta\mathbb{N}}A$ and $\bar{B} = cl_{\beta\mathbb{N}}B$ are the closures of A and B in $\beta\mathbb{N}$, respectively.

(iv) *Any two disjoint subsets of $\mathbb{N}$ have disjoint closures in $\beta\mathbb{N}$.*

(v) *For any algebra $\mathcal{A}$ in $\mathbb{N}$ the family*

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in $\beta\mathbb{N}$.

This ends Remark 3.1.

Here we restrict ourselves to applications of this model to the investigation of spaces of arithmetical functions.

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$. Then, if $\mathcal{E}$ denotes the family of bounded functions on $\mathbb{N}$, the set

$$\mathcal{E}(\mathcal{A}) := \left\{ s \in \mathcal{E}, s = \sum_{j=1}^{m} \alpha_j 1_{A_j};\ \alpha_j \in \mathbb{C},\ A_j \in \mathcal{A},\ j = 1, \ldots, m \right\}$$

of simple functions on $\mathcal{A}$ is a vector space. We introduce the following spaces.

For a function $f : \mathbb{N} \to \mathbb{C}$, we define the seminorm $||\cdot||_\alpha$ for $1 \le \alpha < \infty$ by

$$\|f\|_\alpha := \left\{ \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)|^\alpha \right\}^{\frac{1}{\alpha}} .$$

Let

$$\mathcal{L}^\alpha := \{f : \mathbb{N} \to \mathbb{C} : \|f\|_\alpha < \infty\}$$

denote the linear space of functions on $\mathbb{N}$ with bounded seminorm $\|f\|_\alpha$. By L^α we denote the quotient space $\mathcal{L}^\alpha$ modulo null-functions (i.e functions f with $\|f\|_\alpha = 0$).

Definition 3.2. For a given algebra $\mathcal{A}$ and for $1 \le \alpha < \infty$ the space $\mathcal{L}^{*\alpha}(\mathcal{A})$ is defined as the $\|\cdot\|_\alpha$-closure of $\mathcal{E}(\mathcal{A})$. A function $f \in \mathcal{L}^{*\alpha}(\mathcal{A})$ is called *uniformly* $(\mathcal{A})$*-summable.* By $L^{*\alpha}(\mathcal{A})$ we denote the quotient space $\mathcal{L}^{*\alpha}(\mathcal{A})$ modulo null functions.

Definition 3.3.

(i) A nonnegative arithmetical function f is called $\mathcal{A}$-measurable in case each truncation function $f_K = \min\{K, f\}$ lies in $\mathcal{L}^{*1}(\mathcal{A})$ and f is tight, i.e. for every $\varepsilon > 0$ the estimate

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\le n \\ |f(k)|>K}} 1 < \varepsilon$$

holds for some K.

(ii) A real-valued arithmetical function f is called $\mathcal{A}$-*measurable* in case its positive and negative parts f^+ and f^- are $\mathcal{A}$-measurable.

(iii) A complex-valued arithmetical function f is called $\mathcal{A}$-measurable in case Ref, Imf are $\mathcal{A}$-measurable. The space of all $\mathcal{A}$-measurable functions is denoted by $\mathcal{L}^*(\mathcal{A})$. Further we define $L^*(\mathcal{A})$ as $\mathcal{L}^*(\mathcal{A})$ modulo null functions, i.e. functions f for $\delta(\{m : f(m) \ne 0\}) = 0$.

In the following we assume that $\mathcal{A}$ is an algebra such that the asymptotic density $\delta(A)$ exists for every $A \in \mathcal{A}$.

We say that an arithmetical function f possesses an (*arithmetical*) *mean-value* $M(f)$ if

$$M(f) := \lim_{n\to\infty} \frac{1}{n} \sum_{k\le n} f(k)$$

exists.

If we put, for every subset $A \subset \mathbb{N}$,

$$1_A(n) = \begin{cases} 1, & n \in A, \\ 0, & \text{otherwise} \end{cases}$$

then

$$\delta(A) = M(1_A)$$

for every $A \in \mathcal{A}$.

A first consequence of the above construction is that, for all $s \in \mathcal{E}(\mathcal{A})$,

$$M(s) = \int\limits_{\beta\mathbb{N}} \bar{s} d\bar{\delta},$$

where $\bar{s} : \beta\mathbb{N} \to \mathbb{C}$ denotes the extension of s.

Starting from this we consider measurable and integrable functions on the probability space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ and relate these to the functions from $\mathcal{L}^*(\mathcal{A})$.

The probability space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ leads to the well-known space

$$L(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta}) := \{\bar{f} : \beta\mathbb{N} \to \mathbb{C}, \sigma(\bar{\mathcal{A}}) - \text{measurable}\} \quad \text{modulo null-functions}$$

and

$$L^1((\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})) := \{\bar{f} : \beta\mathbb{N} \to \mathbb{C}, \|\bar{f}\| < \infty\} \quad \text{modulo null-functions}$$

where

$$\|\bar{f}\| := \int\limits_{\beta\mathbb{N}} |\bar{f}| d\bar{\delta}.$$

There exists a vector-space isomorphism between the spaces $L^*(\mathcal{A})$ and $L^{*1}(\mathcal{A})$ and there exists a norm-preserving vector-space isomorphism between L and L^1. For details see [3] and [4].

Let us consider the following applications.

Almost even functions

For primes p and $k = 0, 1, \ldots$ let

$$A_{p^k} := \{n \in \mathbb{N} : p^k || n\}.$$

Let $\mathcal{A}_4$ be the algebra generated by the sets A_{p^k}. Then the asymptotic density of A_{p^k} equals

$$\delta(A_{p^k}) = \frac{1}{p^k}\left(1 - \frac{1}{p}\right).$$

Because of the following relation of the characteristic functions

$$\begin{aligned} 1_{A\cap B} &= 1_A \cdot 1_B, \\ 1_{A\setminus B} &= 1_A - 1_A \cdot 1_B \\ 1_{A\cup B} &= 1_A + 1_B - 1_A \cdot 1_B \end{aligned}$$

we obtain that the characteristic function of a set $A \in \mathcal{A}_4$ is a finite linear combination of products of $1_{A_{p_1^{k_1}}} \cdots 1_{A_{p_r^{k_r}}}$. Then $\delta(A)$ exists for every $A \in \mathcal{A}_4$.

By the above construction we obtain that $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}_4), \bar{\delta})$ is a probability space and we arrive at the space $L(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}_4), \bar{\delta})$ which corresponds to the space of *almost-even* functions.

Distribution of additive functions

If f is a real valued additive function, we can put

$$f = \sum_p f_p$$

where f_p is defined by

$$f_p(n) = \begin{cases} f(p^k), & \text{if } n \in A_{p^k}, \\ 0, & \text{otherwise,} \end{cases}$$

Obviously, every f_p is uniformly $\mathcal{A}_4$ summable, and we denote $\bar{f}_p$ its unique extension to an integrable function on $\beta\mathbb{N}$. Then $\{\bar{f}_p\}_{p\ prime}$ is a set of independent random variables and $\sum_p \bar{f}_p$ converges a.s. if and only if f possesses a limit distribution.

This result can be seen as another *a posteriori* justification of the mentioned idea of Kac connected with the role of independence in probabilistic number theory. By this model we obtain the result of Erdős and Kac which we constituted in Proposition 2.3. (For details see Indlekofer [4])

Remark 3.4. In the case of multiplicative functions we proceed in a similar manner. If a real-valued multiplicative function g is given we put

$$g = \prod_p g_p$$

where

$$g_p(n) := \begin{cases} g(p^k), & \text{if } p^k || n, \\ 1, & \text{otherwise.} \end{cases}$$

The unique extension $\bar{g}_p$ of g_p build a set $\bar{g}_p$ of independent random variables, and an application of Zolotarev's result [8] concerning the characteristic function of products of random variables gives necessary and sufficient conditions for the convergence of the product $\prod_p \bar{g}_p$ which turns out to be equivalent to the existence of the limit distribution of g.

Erdős–Wintner Theorem

For primes p let
$$A_p := \{n \in \mathbb{N} : p|n\}$$
be the set of all natural numbers divisible by p. Let $\mathcal{A}_2'$ be the algebra generated by the sets A_p. Then obviously each $A \in \mathcal{A}_2'$ possesses an asymptotic density $\delta(A)$ and $\delta(A_p) = \frac{1}{p}$ (p prime). The above construction leads to the probability space $(\beta\mathbb{N}, \sigma(\mathcal{A}_2'), P)$ where $P := \bar{\delta}$, i.e. $P(\bar{A}_p) = \frac{1}{p}$ (p prime).

Let f be strongly additive function. Then f can be written in the form
$$f = \sum_p f(p)\varepsilon_p$$
where ε_p denote the characteristic function of A_p. There is a unique extension of ε_p to a function $\bar{\varepsilon}_p$ on $\beta\mathbb{N}$, and
$$f = \sum_p f(p)\varepsilon_p \quad \rightarrow X := \sum_p f(p)\bar{\varepsilon}_p = \sum_p X_p$$
with
$$P(X_p = f(p)) = \frac{1}{p} \;\text{ and }\; P(X_p = 0) = 1 - \frac{1}{p}.$$
The $\bar{\varepsilon_p}$ are independent, i.e. $X = \sum\limits_p X_p$ is a sum of independent random variables.

An immediate consequence of the above construction is as follows

Theorem 3.5. *Let f be real-valued strongly additive function. Then the following assertions are equivalent:*

(i) $f = \sum\limits_p f_p$ *possesses a limit distribution,*

(ii) $\bar{f} = \sum\limits_p X_p$ *converges P-almost everywhere,*

(iii) *the series*
$$\sum_{\substack{p \\ |f(p)|>1}} P(\bar{A}_p), \quad \sum_{\substack{p \\ |f(p)|\le 1}} \mathbb{E}[X_p], \quad \sum_{\substack{p \\ |f(p)|\le 1}} Var[X_p]$$
converge (*Three series theorem*),

(iv) *the series*

$$\sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}$$

converge.

More applications can be found in Indlekofer's articles [3] and [4]. In the recent papers [5] by E. Kaya and K.-H. Indlekofer and [1] by A. Barát, E. Kaya and K.-H. Indlekofer the model has been adapted to additive arithmetical semigroups. In [6] Indlekofer successfully applied the model to prove a conjecture of P. Erdös about the distribution of additive functions.

References

[1] **Barát, A., K.-H. Indlekofer and E. Kaya,** Two series theorem in additive arithmetical semigroups, *Annales Univ. Sci. Budapest., Sect. Comp.*, **38** (2012), 323–332.

[2] **Elliott, P.D.T.A.,** *Probabilistic Number Theory I-II*, Springer-Verlag, New York, 1979.

[3] **Indlekofer, K.-H.,** A new method in probabilistic number theory, *Probability Theory and Applications, Math. Appl.*, **80** (1992), 299–308.

[4] **Indlekofer, K.-H.,** New approach to probabilistic number theory-compactifications and integration, *Advenced Studies in Pure Mathematics*, **49** (2005), 133–170.

[5] **Indlekofer, K.-H. and E. Kaya,** Three series theorem in additive arithmetical semigroups. *Annales Univ. Sci. Budapest., Sect. Comp.* **38** (2012), to appear.

[6] **Indlekofer, K.-H.,** On a conjecture of Erdős about additive functions (manuscript)

[7] **Kubilius, J.,** *Probabilistic methods in the Theory of Numbers*, American Mathematical Society, Providence, 1964.

[8] **Zolotarev, V. M.,** On a general theory of multiplication of independent random variables, *Izv. Akad. Nauk*, **142** (1962), 166–170.

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