TWO–SERIES THEOREM IN ADDITIVE ARITHMETICAL SEMIGROUPS

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Communicated by I. Kátai

(Received December 30, 2012)

Abstract. In this paper, we embed the additive arithmetical semigroup in a probability space $\Omega := (\beta G, \sigma(\overline{\mathcal{A}}), \overline{\delta})$ where βG denotes the Stone-Čech compactification of G. We show that every additive function $\tilde{g}$ on G, $\tilde{g}(a) = \sum_{p^k || a} \tilde{g}(p^k)$ $(a \in G)$, can be identified with a sum $X := \sum_p X_p$ of independent random variables on Ω. Further, we characterize the class of essentially distributed additive functions.

1. Introduction

Let (G, ∂) be an additive arithmetical semigroup. By definition G is a free commutative semigroup with identity element 1, generated by a countable subset $\mathcal{P}$ of primes and admitting an integer valued degree mapping $\partial : G \to \mathbb{N} \cup \{0\}$ which satisfies

(i) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in \mathcal{P}$,

(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \geq 0$.

https://doi.org/10.71352/ac.38.309

Obviously, $G(0)=1$ and G is countable. Let

$$\pi(n) := \#\{p \in \mathcal{P} : \partial(p) = n\}$$

denote the total number of primes of degree n in G. We obtain the identity, at least in the formal sense,

$$Z(y) := \sum_{n=0}^{\infty} G(n) y^n = \prod_{n=1}^{\infty} (1-y^n)^{-\pi(n)}.$$

Z can be considered as the zeta-function associated with the semigroup (G, ∂). In this paper, we assume that $\pi(n) = O(q^n/n)$ and the generating function of (G, ∂) has the form

$$Z(y) = \sum_{n=0}^{\infty} G(n) y^n = \frac{H(y)}{(1-qy)^{\tau}} \qquad (|y| < q^{-1}), \tag{1.1}$$

where $\tau > 0$ and $H(y) = O(1)$ for $|y| < q^{-1}$ and $\lim\limits_{y \to q^{-1}} H(y)$ exists and is positive. By a paper of K.-H. Indlekofer (see [4]) $\lim\limits_{y \to q^{-1}} H(y) = H(q^{-1})$, and

$$G(n) \sim \frac{H(q^{-1})}{\Gamma(\tau)} q^n n^{\tau - 1} \tag{1.2}$$

holds.

Here, as in the classical case, an arithmetical function $\tilde{f} : G \to \mathbb{R}$ is called *multiplicative* if $\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b)$ whenever $a, b \in G$ are coprime and an arithmetical function $\tilde{g}$ on G is called *additive* if $\tilde{g}(ab) = \tilde{g}(a) + \tilde{g}(b)$ for all coprime $a, b \in G$. $\tilde{g}$ is said to be *strongly additive* if $\tilde{g}(p^k) = \tilde{g}(p)$ hold for all $k \geq 2$.

Let $\tilde{f} : G \to \mathbb{C}$. We define the average value of $\tilde{f}$ by

$$M(n, \tilde{f}) := \begin{cases} \frac{1}{G(n)} \sum\limits_{\substack{a \in G \\ \partial(a) = n}} \tilde{f}(a), & \text{if } G(n) \neq 0, \\ 0, & \text{if } G(n) = 0. \end{cases}$$

We say that the function $\tilde{f}$ possesses an (arithmetical) *mean-value* $M(\tilde{f})$, if the limit

$$M(\tilde{f}) := \lim_{n \to \infty} M(n, \tilde{f})$$

exists.

Now, we embed the additive arithmetical semigroup in a probability space $\Omega := (\beta G, \sigma(\overline{\mathcal{A}}), \overline{\delta})$ where βG denotes the Stone-Čech compactification of G. For this suppose that $\mathcal{A}$ is an *algebra of subsets of* G, i.e.

(i) $G \in \mathcal{A}$,
(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,
(iii) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

We embed G, endowed with the discrete topology, in the compact space βG, the Stone-Čech compactification of G. This implies

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in βG, where $\bar{A} := clos_{\beta G} A$ (for details see K.-H. Indlekofer [2], [3] and K.-H. Indlekofer, E. Kaya [6].

Let $\delta(A)$ be a content on $\mathcal{A}$ and define $\bar{\delta}$ on $\bar{\mathcal{A}}$ by

$$\bar{\delta}(\bar{A}) = \delta(A), \quad \bar{A} \in \bar{\mathcal{A}}.$$

Then $\bar{\delta}$ is a pseudo-measure in $\bar{\mathcal{A}}$ and can be extended to a measure in $\sigma(\bar{\mathcal{A}})$. This leads to the measure space $(\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta})$.

Let us consider the following example.

For prime elements $p \in \mathcal{P}$ let

$$A_p := \{a \in G : p \mid a\}$$

be the set of all elements of G divisible by p. Let $\mathcal{A}$ be the algebra generated by the sets $\{A_p\}$. We assume that (see (1.1) and (1.2)

$$G(n) \sim \frac{H(q^{-1})}{\Gamma(\tau)} q^n n^{\tau-1} \qquad (\tau > 0) \tag{1.3}$$

and consider, for $A \in \mathcal{A}$, the means

$$M(n, 1_A) = \frac{\sum_{\substack{a \in A \\ \partial(a)=n}} 1}{\sum_{\substack{a \in G \\ \partial(a)=n}} 1},$$

where the indicator function 1_A of A is defined by

$$1_A(a) := \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the following relations of the indicator functions

$$\begin{aligned} 1_{A\cap B} &= 1_A \cdot 1_B, \\ 1_{A\setminus B} &= 1_A - 1_A \cdot 1_B, \\ 1_{A\cup B} &= 1_A + 1_B - 1_A \cdot 1_B \end{aligned}$$

imply that the characteristic function of an arbitrary set $A \in \mathcal{A}$ is a finite linear combination of products of $1_{A_{p_1}} \cdots 1_{A_{p_r}}$. Put $a' = p_1 \cdot \ldots \cdot p_r$. Then $1_{A_{p_1}} \cdot \ldots \cdot 1_{A_{p_r}} = 1_{A_{a'}}$ where $A_{a'} := \{a \in G : a' \mid a\}$. Obviously, by (1.3),

$$\begin{aligned} \lim_{n\to\infty} M(n, 1_{A_{a'}}) = \lim_{n\to\infty} \frac{G(n - \partial(a'))}{G(n)} = \\ = q^{-\partial(a')}. \end{aligned}$$

Putting, for every $A \in \mathcal{A}$

$$\delta(A) := \lim_{n\to\infty} M(n, 1_A),$$

we obtain a content on $\mathcal{A}$. The extension $\bar{\delta}$ of δ

$$\bar{\delta}(\bar{A}) := \delta(A) \quad (\bar{A} \in \bar{\mathcal{A}})$$

defines a premeasure on $\bar{\mathcal{A}}$ and leads to a measure P, induced by

$$\delta^*(A) := \overline{\lim_{n\to\infty}} M(n, 1_A) \qquad \text{for all } A \subset G,$$

and to a probability space $(\Omega, \sigma(\bar{\mathcal{A}}), P)$ with $\Omega = \beta G$ and $P(\bar{A}_p) = q^{-\partial(p)}$ (p prime).

To avoid notational difficulties we shall prove the results for strongly additive functions. The general case follows by standard arguments.

Let $\tilde{g} : G \to \mathbb{R}$ be a real-valued strongly additive function on G. Then $\tilde{g}$ can be written as

$$\tilde{g} = \sum_{p\in P} \tilde{g}(p)\varepsilon_p,$$

where ε_p denotes the indicator function of A_p.

For each $p \in \mathcal{P}$ the function ε_p can uniquely be extended to a function $\overline{\varepsilon_p}$ on βG. With this notation we write

$$\tilde{g} = \sum_p \tilde{g}(p)\varepsilon_p \quad \to X := \sum_p \tilde{g}(p)\bar{\varepsilon}_p = \sum_p X_p$$

with

$$P(X_p = \tilde{g}(p)) = q^{-\partial(p)} \text{ and } P(X_p = 0) = 1 - q^{-\partial(p)}.$$

The $\bar{\varepsilon_p}$ are independent, i.e.

$$X = \sum_p X_p$$

is a sum of independent random variables. In the paper [6] K.-H. Indlekofer and E. Kaya showed, that if

$$\mathcal{F}_n(x) := \frac{1}{G(n)} \#\{a \in G,\ \partial(a) = n :\ \tilde{g}(a) \le x\}$$

denotes the distribution function of the additive function $\tilde{g}$, then the convergence of $\mathcal{F}_n$ to a limit distribution is equivalent to the convergence of the following series (Three-Series Theorem)

$$\sum_{|\tilde{g}(p)|>1} q^{-\partial(p)}, \quad \sum_{|\tilde{g}(p)|\le 1} \tilde{g}(p) q^{-\partial(p)}, \quad \sum_{|\tilde{g}(p)|\le 1} \tilde{g}^2(p) q^{-\partial(p)}.$$

In this paper, we shall characterize the class of *essentially distributed* additive functions on G, which will be defined in the following section, and which corresponds to the case where the sum (1.4) is essentially convergent. For this we put $G_n := \{a \in G : \partial(a) = n\}$ and define finitely distributed functions on G (see A. Barát [1]).

Definition 1.1. A function $\tilde{h} : G \mapsto \mathbb{R}$ is called *finitely distributed* if there exists a sequence of integers $(n_1, n_2, ...)$ and a subset $H \subseteq G$ such that for every n_l, $\#(H \cap G_{n_l}) \ge cG(n_l)$ and $|\tilde{h}(a_1) - \tilde{h}(a_2)| < C$ for all $a_1, a_2 \in H \cap G_{n_l}$ with some parameters $c > 0$, $C > 0$.

Although this definition appears unwieldly, functions of this kind are convenient to use, because of the characterization described in the next section.

2. Finitely distributed functions on G

Here we shall characterize all additive functions which, after a suitable translation, possess a limiting distribution.

Theorem 2.1. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$Z(y) = \sum_{n=0}^{\infty} G(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} y^m\right) = \frac{H(y)}{(1-qy)^{\tau}},$$

where $H(y) = O(1)$ for $|y| < q^{-1}$, $\lim_{y \to \frac{1}{q}^-} H(y)$ exists and is positive and $\tau > 0$.

Assume that $\Lambda(m) = O(q^m)$.

Let $\tilde{g}$ be a real-valued additive function on G. Then the following assertions hold:

(i) If, for some $\alpha(n)$ the frequencies

$$\frac{1}{G(n)} \#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \le x\}$$

converge to a weak limit as $n \to \infty$, then $\tilde{g}$ is finitely distributed.

(ii) If $\tilde{g}$ is finitely distributed, then it has a decomposition $\tilde{g}(a) = c\partial(a) + \tilde{h}(a)$ with a real constant c and an additive function $\tilde{h}$ where both the series

$$(2.1) \qquad \sum_{\substack{p \\ |\tilde{h}(p)|>1}} \frac{1}{q^{\partial(p)}} \qquad\qquad \sum_{\substack{p \\ |\tilde{h}(p)|\le 1}} \frac{\tilde{h}(p)^2}{q^{\partial(p)}}$$

converge.

(iii) If $\tilde{g}$ has a representation $c\partial + \tilde{h}$, where the series (2.1) both converge, and if we define

$$\alpha(n) = cn + \sum_{\partial(p) \le n, |\tilde{h}(p)| \le 1} \frac{\tilde{h}(p)}{q^{\partial(p)}} \qquad (n \ge 1),$$

then the frequencies

$$\mathcal{G}_n(x) := \frac{1}{G(n)} \#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \le x\}$$

converge to a weak limit as $n \to \infty$.

Remark. The first author proved Theorem 2.1 in her PhD thesis under the additional conditions

$$(2.2) \qquad G(n) \asymp q^n n^{\tau-1} \quad \text{and} \quad \frac{G(n-1)}{G(n)} = q^{-1} + o(1) \quad n \to \infty,$$

(for details see A. Barát [1]). By a paper of the second author [4] the assumptions of Theorem 2.1 imply that

$$G(n) \sim \frac{H(q^{-1})}{\Gamma(\tau)} q^n n^{\tau-1}$$

holds, and therefore the conditions in (2.2) are satisfied.

All finitely distributed functions on G have a representation $\tilde{g} = c\partial + \tilde{h}$ with convergent series (2.1). We shall study the case $c = 0$.

Definition 2.2. The additive function $\tilde{g}$ is called *essentially distributed* iff the series

$$\sum_{\substack{p \\ |\tilde{g}(p)|>1}} \frac{1}{q^{\partial(p)}}, \qquad \sum_{\substack{p \\ |\tilde{g}(p)|\le 1}} \frac{\tilde{g}(p)^2}{q^{\partial(p)}}$$

converge.

An easy but interesting consequence of Theorem 2.1 is formulated in

Corollary 2.3. *Let an additive arithmetical semigroup G satisfy the assumptions of Theorem 2.1, and let $\tilde{g}$ be an additive function with only one sign defined on G. There exist two numbers $x_1 < x_2$ such that*

$$\limsup_{n\to\infty} (\mathcal{F}_n(x_2) - \mathcal{F}_n(x_1)) > 0 \tag{2.3}$$

if and only if $\{\mathcal{F}_n\}$ converges weakly.

Proof. If (2.3) holds, then $\tilde{g}$ is finitely distributed, $\tilde{g}$ has the representation $\tilde{g}(a) = c\partial(a) + \tilde{h}(a)$ where the series (2.1) converge, and the frequencies $\mathcal{G}_n$ converge to a distribution function. Further, there exists a subsequence $\{n_k\}$ and a real constant c_1 such that

$$\lim_{k\to\infty} (\mathcal{F}_{n_k}(x_2) - \mathcal{F}_{n_k}(x_1)) = c_1 > 0. \tag{2.4}$$

Clearly

$$\mathcal{F}_{n_k}(x) = \mathcal{G}_{n_k}(x - \alpha(n_k)). \tag{2.5}$$

Assume that $|\alpha(n_k)| \to \infty$ as $k \to \infty$. Then, since $\mathcal{G}_n \Rightarrow \mathcal{G}$

$$\lim_{k\to\infty} (\mathcal{F}_{n_k}(x_2) - \mathcal{F}_{n_k}(x_1)) = \lim_{k\to\infty} (\mathcal{G}_{n_k}(x_2 - \alpha(n_k)) - \mathcal{G}_{n_k}(x_1 - \alpha(n_k))) = 0$$

which contradicts (2.4). Since

$$\begin{aligned}\alpha(n) =& cn + \sum_{\substack{\partial(p)\le n\\ |\tilde h(p)|\le 1}} q^{-\partial(p)}\tilde h(p) =\\ =& cn + O\left(\sum_{m\le n} q^{-m}\pi(m)\right) =\\ =& cn + O\left(\sum_{m\le n}\frac{1}{m}\right) =\\ =& cn + O(\log n)\end{aligned}$$

we conclude $c=0$ and, because $\tilde g(a)$ has only one sign,

$$\sum_{\substack{p\in\mathcal{P}\\ |\tilde h(p)|\le 1}} \frac{\tilde h(p)}{q^{\partial(p)}} \quad \text{converges.} \tag{2.6}$$

Thus $\tilde g(a) = \tilde h(a)$ with convergent series (2.1) and (2.6). Then (Three-Series Theorem) $\{\mathcal{F}_n\}$ converges weakly.

In the other direction, the weak convergence of $\{\mathcal{F}_n\}$ trivially implies (2.3), and Corollary 2.3 is proven.

3. Two-Series Theorem

Putting

$$\alpha(n) = \sum_{\substack{\partial(p)\le n\\ |\tilde g(p)|\le 1}} q^{-\partial(p)}\tilde g(p)$$

and

$$a_p = \mathbb{E}(X_p^s), \quad Y_p = X_p - a_p, \quad S_n := \sum_{\partial(p)\le n} X_p, \quad T_n := \sum_{\partial(p)\le n} Y_p$$

where X_p^s denotes the truncation of X_p at $s>0$, we prove

Theorem 3.1. *Let an additive arithmetical semigroup G satisfy the assumptions of Theorem 2.1, and let $\tilde{g}$ be an additive function defined on G. Then the following assertions are equivalent.*

(i) *$\tilde{g}$ is essentially distributed.*

(ii) *The distribution functions*

$$\mathcal{G}_n(x) := \frac{1}{G(n)} \#\{a \in G, \quad \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

converge weakly as $n \to \infty$.

(iii) *The series*

$$\sum_{\substack{p \\ |\tilde{g}(p)|>1}} q^{-\partial(p)}, \qquad \sum_{\substack{p \\ |\tilde{g}(p)|\leq 1}} q^{-\partial(p)} \tilde{g}(p)^2$$

are convergent.

(iv) *The series $\sum_p X_p$ is essentially convergent, i.e. the series $Y = \sum_p (X_p - a_p)$ converges a.s..*

(v) *For some $s > 0$ the two series*

$$\sum_p P(|X_p| > s) \quad \textit{and} \quad \sum_p \sigma^2(X_p)$$

converge.

(vi) *The limit*

$$\lim_{n\to\infty} \prod_{\partial(p)\leq n} |1 + q^{-\partial(p)}(e^{it\tilde{g}(p)} - 1)|$$

exists on a set of positive Lebesgue measure.

Proof. The equivalence of (i), (ii) and (iii) follows from Theorem 2.1. The equivalence of (iii) and (v) is obvious, whereas the remaining assertions of Theorem 3.1 are well-known (cf. P. Loeve [7]).

If all the values $\tilde{g}(a)$ of an additive function $\tilde{g}$ are ≥ 0 (or ≤ 0) we have seen (Corollary 2.3) that the condition (2.3) holds if and only if the frequencies $\mathcal{F}_n$ converge weakly. We shall deal with an analogue characterization of *real-valued* additive functions in a later joint paper of the authors, and prove a result, which corresponds to a conjecture of Erdős about additive functions on $\mathbb{N}$.

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