ON THE VALUES OF ARITHMETIC FUNCTIONS IN SHORT INTERVALS

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Dedicated to Professor Antal Iványi on his seventieth anniversary

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Abstract. In this short paper the following assertion is proved. For positive integer d and $c > 0$ let $J_c(N) = [N,\ N + c\sqrt{N}]$ and $\mathcal{K}_d = \{n \in \in \mathbb{N} \mid (n, d) = 1\}$. Let $1 < N_1 < N_2 < \cdots$ be an infinite sequence of integers and $\ell_1, \ell_2, \cdots$ be integers coprime to d. Assume that f and g are completely additive functions defined on $\mathcal{K}_d$, for which $f(n) = g(n)$ if $n \equiv \equiv \ell_j \pmod{d}$, $n \in J_c(N_j)$ $(j = 1, 2, \cdots)$. If $c > 2d$, then $f(n) = g(n)$ identically on $\mathcal{K}_d$.

1. Introduction

1.1. Notations

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of positive integers, real and complex numbers, respectively. Let $(\mathbb{G}, +)$ and $(\mathbb{H}, \cdot)$ be commutative semigroups. We shall denote by $\mathcal{A}_{\mathbb{G}}$ $(\mathcal{A}^*_{\mathbb{G}})$ the set of additive (completely additive) arithmetical functions

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taking values on $\mathbb{G}$. Similarly, let $\mathcal{M}_{\mathbb{H}}$ ($\mathcal{M}^*_{\mathbb{H}}$) be the set of multiplicative (completely multiplicative) arithmetical functions taking values on $\mathbb{H}$. For $\mathbb{G} = \mathbb{R}$ we write $\mathcal{A}$ ($\mathcal{A}^*$) instead of $\mathcal{A}_{\mathbb{R}}$ ($\mathcal{A}^*_{\mathbb{R}}$) and for $\mathbb{H} = \mathbb{C}$ we write $\mathcal{M}$ ($\mathcal{M}^*$) instead of $\mathcal{M}_{\mathbb{C}}$ ($\mathcal{M}^*_{\mathbb{C}}$).

1.2. Known results

We state some of the known results in this direction.

Theorem A. ([1]) *Let $f \in \mathcal{A}^*$, for which*

$$f(n) = 0 \quad holds\ for \quad n \in [N_j, N_j + 4\sqrt{N_j}]$$

$(j = 1, 2, \cdots)$, $1 < N_1 < N_2 < \cdots$ is an arbitrary infinite sequence of integers. Then $f(n) = 0$ identically.

Theorem B. ([3]) *Let $f \in \mathcal{A}^*$. Let $\lambda(N) = (2 + \epsilon)\sqrt{N}$ for an arbitrary constant $\epsilon > 0$. Assume that $1 < N_1 < N_2 < \cdots$ is an infinite sequence of integers such that*

$$f(n) \le f(n+1) \quad holds\ for \quad n \in [N_j, N_j + \lambda(N_j)]$$

$(j = 1, 2, \cdots)$. Then $f(n) = c \log n$, where c is a constant.

Theorem C. ([3]) *There exists an $f \in \mathcal{A}^*$ which is not identically zero, and for $1 < N_1 < N_2 < \cdots$ satisfies*

$$f(n) = A_j \quad if \quad n \in [N_j, N_j + \varrho(N_j)]$$

for $j = 1, 2, \cdots$ and

$$\varrho(N) = \exp\left(c\sqrt{(\log N)\ (\log\log\log N)}\right),$$

where c is a suitable positive constant and A_j are arbitrary complex or reals.

Theorem D. [6]) *Let $\Phi_j(z) \in \mathbb{C}[z]$ be a sequence of polynomials with $\deg \Phi_j \le h$ and $\alpha_h = \frac{h+1}{h+2}$. Assume that $b_j \to \infty$ is an infinite sequence of reals, $1 < N_1 < N_2 < \cdots$ is an infinite sequence of integers and $a_1, a_2, \cdots$ is a sequence of arbitrary complex numbers. For $f \in \mathcal{A}^*$ assume that*

$$\Phi_j(E)f(n) = a_j \quad holds\ for \quad n \in [N_j, N_j + b_j N_j^{\alpha_h}]$$

$(j = 1, 2, \cdots)$. Then $f(n) = 0$, identically.

Here E is the shift operator. If $P(Z) = a_0 + a_1 z + \cdots + a_k z^k$, then

$$P(E)f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k).$$

In [5] P. Erdős and I. Kátai showed the existence of a completely additive function which vanishes in particular short intervals but takes the value 1 in one interval. Here is their result.

Theorem E. ([5]) *Let* $x > x_0(\epsilon)$ *for* $\epsilon > 0$. *Then there exists a function* $f \in \mathcal{A}^*$ *for which*

$$f(n) = 0 \quad for \quad n \in [N+1, N+\lambda(x)],$$

where $\frac{x}{2} \leq N \leq x$ *and*

$$\lambda(x) = \exp\left(\left(\frac{1}{2} - \epsilon\right) \frac{(\log x)(\log\log\log x)}{\log\log x}\right)$$

and which takes on a non-zero value in $[1, \sqrt{x}]$.

Remark 1. *Existence of an* $f(n)$ *with infinitely many such intervals is yet to be established.*

In [2] I. Kátai, examined arbitrary complex valued multiplicative functions which remain constant on square-free numbers in short intervals and proved the following theorem.

Theorem F. ([2]) *Let* $\theta = 0{,}6108$ *and* $J(N) = [N, N+N^\theta]$. *Let* f *be a multiplicative function defined on the set of square-free numbers and* $f(n) \neq$ $\neq 0$, $(n \in \mathbb{N})$. *Assume that there exists a sequence of complex numbers* $a_1, a_2, \cdots$ *and a sequence of positive integers* $1 < N_1 < N_2 < \cdots$ *such that*

$$f(n) = a_j \quad if \quad n \in J(N_j) \quad (n \quad is \ square-free).$$

Then $f(n) = 1$ *for every square-free* n.

Remark 2. *Theorem F remains valid with* $\theta = 0{,}6$.

This comes from the following result of M. Filaseta:

Let $g(x)$ be a function, $1 \leq g(x) \leq \log x$ for x sufficiently large, and set $h(x) = x^{\frac{1}{5}} g(x)^3$. Then the number of square-free integers belonging to the interval $[x, x+h(x)]$ is

$$\frac{h(x)}{\xi(2)} + O\Big(\frac{h(x)\log x}{g(x)^3}\Big) + O\Big(\frac{h(x)}{g(x)}\Big).$$

The interested reader can look at [7] for a complete proof.

2. Results

Let $J_c(N) = [N, N + c\sqrt{N}]$, where c is a fixed constant. For each positive integer d, let $\mathcal{K}_d = \{n \in \mathbb{N} \mid (n,d) = 1\}$. Here we prove the following results which are variants of the results quoted in the previous section.

Theorem 1. *Let $f \in \mathcal{M}^*$ be defined on $\mathcal{K}_d$, where $d \in \mathbb{N}$ is given. Assume that there exists an infinite sequence of integers $1 < N_1 < N_2 < \cdots$, an infinite sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ and a sequence of nonzero complex numbers $a_1, a_2, \cdots$ such that*

$$f(n) = a_{\ell_\nu} \quad \textit{if} \quad n \in J_c(N_\nu) \quad \textit{and} \quad n \equiv \ell_\nu \pmod d \tag{2.1}$$

$(\nu = 1, 2, \cdots)$. If $c > 2d$, then $f(n) = \chi(n)$ for a Dirichlet character $\chi \pmod d$.

Theorem 2. *Let d, the sequences N_ν, ℓ_ν be as in Theorem 1. Let $g \in \mathcal{A}^*$ be defined on $\mathcal{K}_d$. Assume that*

$$g\big(n\big) \;\le\; g\big(n+d\big) \quad \textit{if} \quad n \in J_c(N_\nu) \quad \textit{and} \quad n \equiv \ell_\nu \pmod d \tag{2.2}$$

$(\nu = 1, 2, \cdots)$. If $c > 2d$, then there exists a constant A such that

$$g(n) = A \log n \quad \textit{for} \quad n \in \mathcal{K}_d.$$

Now for the Abelian group $\mathbb{G}$ let

$$\mathcal{X}_{\mathbb{G}} = \{\ g \in \mathcal{A}^*_{gg} \ \mid\ g(n) = g(m) \quad \text{all} \ \ n,\ m \in \mathcal{K}_d,\ n \equiv m \ \ (mod\ d)\}.$$

Theorem 3. *Let $g \in \mathcal{A}^*_{\mathbb{G}}$ be defined on $\mathcal{K}_d$. Assume that there exists an infinite sequence of integers $1 < N_1 < N_2 < \cdots$, an infinite sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ and a sequence $a_1, a_2, \cdots$ of elements of $\mathbb{G}$ such that*

$$g(n) = a_{\ell_\nu} \quad \textit{if} \quad n \in J_c(N_\nu) \quad \textit{and} \quad n \equiv \ell_\nu \pmod d \tag{2.3}$$

$(\nu = 1, 2, \cdots)$. If $c > 2d$, then $g \in \mathcal{X}_{\mathbb{G}}$.

We provide the proofs of Theorems 1, 2 and omit the proof of Theorem 3 as it is similar.

3. Proof of Theorem 1

Assume that c, d, the sequences N_ν, ℓ_ν are as in the statement of Theorem 1 with $c > 2d$. As the sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ is infinite, there exists a reduced residue $\ell \pmod d$ such that $\ell_\nu \equiv \ell \pmod d$ holds for infinitely many ν. Consequently, the condition (2.1) can be replaced by the following:

$$f(n) = a_\ell \quad \text{if} \quad n \in J_c(N_\nu) \quad \text{and} \quad n \equiv \ell \pmod d, \quad (\nu = 1, 2, \cdots), \tag{3.1}$$

where $\ell \in \mathcal{K}_d$ is fixed integer and a_ℓ is a nonzero complex number.

First we deduce the following lemma.

Lemma 1. *Assuming (3.1) and if*

$$du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0,$$

then

$$f(u) = f(u+d). \tag{3.2}$$

Proof. Indeed, if $du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0$, then

$$c\sqrt{N_\nu}u \ge du(u+d) + N_\nu d$$

and so

$$\frac{N_\nu + c\sqrt{N_\nu}}{u+d} - \frac{N_\nu}{u} \ge d.$$

Thus there exists an r for which $ur \equiv \ell \pmod d$ and

$$r \in \Big[\frac{N_\nu}{u}, \quad \frac{N_\nu + c\sqrt{N_\nu}}{u}\Big], \quad r \in \Big[\frac{N_\nu}{u+d}, \quad \frac{N_\nu + c\sqrt{N_\nu}}{u+d}\Big],$$

i.e

$$ru \equiv r(u+d) \equiv \ell \pmod d, \quad \textit{and} \ \ ru, \ r(u+d) \in J_c(N_\nu).$$

Hence, from (3.1) we have

$$f(ru) = a_\ell \quad \text{and} \quad f(r(u+d)) = a_\ell,$$

which with $a_\ell \ne 0$ implies that $f(u) = f(u+d)$. Thus the assertion follows.

Now we shall verify that

$$du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0. \tag{3.3}$$

Clearly the condition $c > 2d$ implies that

$$(c\sqrt{N_\nu} - d^2)^2 - 4d^2 N_\nu \ge \Big[(c^2 - 4d^2)\sqrt{N_\nu} - 2cd^2\Big] + d^4 > 0$$

for all $\nu > \nu_0$, where

$$N_{\nu_0} \ge \Big(\frac{2cd^2}{c^2 - 4d^2}\Big)^2.$$

Let

$$\xi_{1,2} = \frac{(c\sqrt{N_\nu} - d^2) \mp \sqrt{(c\sqrt{N_\nu} - d^2)^2 - 4d^2 N_\nu}}{2d}$$

and let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d^2}}{2d}, \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4d^2}}{2d}.$$

It is clear that

$$\xi_1 = (1 + o_\nu(1))\lambda_1\sqrt{N_\nu} \quad \text{and} \quad \xi_2 = (1 + o_\nu(1))\lambda_2\sqrt{N_\nu}$$

and that (3.3) holds for all $u \in [\xi_1,\ \xi_2]$.

Let $\epsilon > 0$ be an arbitrary constant such that $\lambda_1 + \epsilon < \lambda_2 - \epsilon$. Let

$$S = [\lambda_1 + \epsilon, \lambda_2 - \epsilon].$$

Next we denote by ν_1 the least index for which $\nu_1 > \nu_0$ and

$$S\sqrt{N_\nu} = [(\lambda_1 + \epsilon)\sqrt{N_\nu}, (\lambda_2 - \epsilon)\sqrt{N_\nu}] \subseteq [\xi_1,\ \xi_2]$$

is satisfied for all $\nu > \nu_1$.

Let $t_j = \sqrt{N_{j+\nu_1}}$. Then (3.3) holds for each $u \in St_j$, which implies from (3.2) that $f(u) = f(u + d)$. Thus, we have proved that

$$f(n_1) = f(n_2) \ne 0 \quad \text{if} \quad n_1 \equiv n_2 \pmod{d},\ n_1, n_2 \in St_j, n_1 \in \mathcal{K}_d. \tag{3.4}$$

Now we complete the proof of Theorem 1.

Proof. Let $u_j = (\lambda_1 + \epsilon)t_j,\ v_j = (\lambda_2 - \epsilon)t_j$. It is easy to check that if

$$n \in \mathcal{K}_d, \quad n > n_0 = \frac{(\lambda_1 + \epsilon)d}{\lambda_2 - \lambda_1 - 2\epsilon},$$

then

$$\frac{v_j}{n+d} - \frac{u_j}{n} = \frac{\Big[(\lambda_2 - \lambda_1 - 2\epsilon)n - (\lambda_1 + \epsilon)d\Big]t_j}{n(n+d)} \geq d$$

for all sufficiently large integer j. This shows that

$$\Big[\frac{u_j}{n}, \quad \frac{v_j}{n}\Big] \bigcap \Big[\frac{u_j}{n+d}, \quad \frac{v_j}{n+d}\Big]$$

contains an interval of length $\geq d$, consequently there exists $m \in \mathcal{K}_d$ for which $nm \in [u_j, v_j]$ and $(n+d)m \in [u_j, v_j]$. Thus, (3.4) implies $f(nm) = f\big[(n+ +d)m\big] \neq 0$, and so $f(n) = f(n+d)$.

It implies that $f(n)$ is a periodic function $\pmod d$ on the set $n \in \mathcal{K}_d$. Since $f(n) \neq 0$, it should be a character. This completes the proof.

4. Proof of Theorem 2

The basic idea is the same as that of the proof of Theorem 1. Let us start with the following lemma.

Lemma 2. *Let $J_{N,M} = [N, N+M]$. Assume that*

$$g(\nu + d) - g(\nu) \geq 0 \quad \textit{holds for} \quad \nu \equiv \ell \pmod d, \tag{4.1}$$

$\nu \in J_{N,M}$, where $(\ell, d) = 1$.

Assume that $ur \equiv \ell \pmod d$, and that $ur \in J_{N,M}$, $(u+d)r \in J_{N,M}$. Then

$$g(u+d) - g(u) \geq 0. \tag{4.2}$$

Proof. Assume that $ur \equiv \ell \pmod d$, and that $ur \in J_{N,M}$, $(u+d)r \in \in J_{N,M}$. Then $ur + kd \equiv \ell \pmod d$ and $ur + kd \in J_{N,M}$ for $k = 0, 1, \cdots r$. Thus, we infer from (4.1) that

$$g(ur + dr) - g(ur) = \sum_{k=1}^{r} \big\{g\big(ur + kd\big) - g\big(ur + (k-1)d\big)\big\} \geq 0.$$

Proof. (Proof of Theorem 2) Repeating the argument used in the proof of Theorem 1, in this case we obtain the following assertion:

$$g(n+d)-g(n)\ge 0 \quad \text{if} \quad n\in St_j,\ n\in \mathcal{K}_d, \tag{4.3}$$

where $St_j=[u_j,v_j]=\Big[(\lambda_1+\epsilon)t_j,\quad (\lambda_2-\epsilon)t_j\Big]$.

As in the proof of Theorem 1, by using (4.3) we can prove that if j is a sufficiently large integer, then for each $n\in\mathcal{K}_d$ there exists $m\in\mathcal{K}_d$ for which $nm\in[u_j,v_j]$ and $(n+d)m\in[u_j,v_j]$. Then we have

$$nm+kd\in[u_j,v_j],\quad nm+kd\in\mathcal{K}_d,\quad (k=0,1,\cdots,m)$$

and so (4.3) implies that

$$g\big(nm+dm\big)-g\big(nm\big)=\sum_{k=1}^{r}\big\{g\big(nm+kd\big)-g\big(nm+(k-1)d\big)\big\}\ge 0,$$

Consequently, $g(n+d)-g(n)\ge 0$ for every $n\in\mathcal{K}_d$. Now we can deduce easily that $g(n)=c\log n$, if $(n,d)=1$.

Let p and q be primes, $p\ne q$, $(pq,d)=1$. Let $P=p^{k_0}\equiv 1 \pmod d$, $Q= = q^{\ell_0}\equiv 1 \pmod d$. Let u_h,v_h be such a sequence of integers for which $P^{u_h}< < Q^{v_h}<P^{u_h+1}$. It is obvious that $\frac{u_h}{v_h}\to\frac{\log Q}{\log P}$ as $h\to\infty$.

Furthermore $g(P^{u_h})\le g(Q^{v_h})\le g(P^{u_h+1})$, whence

$$\frac{u_h}{v_h}\le\frac{g(Q)}{g(P)}\le\frac{u_h+1}{v_h},$$

and so

$$\frac{k_0}{\ell_0}\frac{u_h}{v_h}\le\frac{g(q)}{g(p)}\le\frac{k_0}{\ell_0}\frac{u_h+1}{v_h}.$$

Consequently,

$$\frac{g(q)}{g(p)}=\frac{\log q}{\log p}.$$

References

[1] **Kátai, I.,** On the determination of an additive arithmetical function by its local behaviour, *Colloquium Mathematicum,* **20** (1969), 265-267.

[2] **Kátai, I.,** On the values of multiplicative functions in short intervals, *Math. Ann.,* **183** (1969), 181-184.

[3] **Kátai, I. and Iványi, A.,** On monotonic additive functions, *Acta Math. Acad. Sci. Hungar.,* **24** (1973), 203-208.

[4] **Kántor, K. and Kátai, I.,** Distinct values of number-theometic functions, *Annales Univ. Sci. Budapest. Sect. Math.,* **19** (1976), 83-86.

[5] **Erdős, P. and Kátai, I.,** On the growth of some additive functions on small intervals, *Acta Math. Acad. Sci. Hungar.,* **33** (1979), 345-359.

[6] **Kátai, I.,** Some remarks on arithmetical functions satisfying linear recursions in short intervals, *Annales Univ. Sci. Budapest. Sect. Math.,* **31** (1988), 135-139.

[7] **De Koninck, J.-M. and Kátai, I.,** On the mean value of the index of composition, *Monatshefte für Mathematik,* **145** (2005), 131-344.

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