# ON ADDITIVE FUNCTIONS SATISFYING SOME RELATIONS

**K. Chakraborty** (Allahabad, India)
**I. Kátai and Bui Minh Phong**[1] (Budapest, Hungary)

*Dedicated to Dr. János Fehér on his 75th anniversary*

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**Abstract.** We prove that if an additive commutative semigroup $\mathbb{G}$ (with identity element 0) and $\mathbb{G}$-valued completely additive functions $f_0,\ f_1,\ f_2$ satisfy the relation $f_0(n)+f_1(2n+1)+f_2(n+2)=0$ for all $n\in\mathbb{N}$, then $f_0(n)=f_1(2n+1)=f_2(n)=0$ for all $n\in\mathbb{N}$. The same result is proved when the relation $f_0(n)+f_1(2n-1)+f_2(n+2)=0$ holds for all $n\in\mathbb{N}$.

## 1. Introduction

Let $\mathbb{G}$ be an additive commutative semigroup with identity element 0. Let $\mathcal{A}^*_{\mathbb{G}}$ denote the set of those functions $f:\mathbb{N}\to\mathbb{G}$, for which $f(nm)=f(n)+f(m)$ holds for all $n,\ m\in\mathbb{N}$. The domain of $f\in\mathcal{A}^*_{\mathbb{G}}$ can be extended to $\mathbb{Q}_+$ (the multiplicative group of positive rationals) by

$$f\left(\frac{n}{m}\right)=f(n)-f(m).$$

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If we define $f(-\alpha) := f(\alpha)$ for $\alpha \in \mathbb{Q}_+$, then the equation $f(\alpha\beta) = f(\alpha) + f(\beta)$ remains valid for arbitrary nonzero rational numbers $\alpha, \beta$. Let $\mathcal{P}$ be the set of primes.

In case $\mathbb{G} = \mathbb{R}$, then we simply write $\mathcal{A}^*$ instead of $\mathcal{A}^*_{\mathbb{R}}$.

In an old paper written by Kátai I. [6] the following conjecture has been formulated:

**Conjecture 1.** *If* $f_0,\ f_1,\ \dots, f_k \in \mathcal{A}^*$ *and,*

$$L_n = f_0(n) + f_1(n+1) + \ldots + f_k(n+k) \equiv 0 \pmod 1 \tag{1.1}$$

*for all* $n \in \mathbb{N}$*, then*

$$f_0(n) \equiv f_1(n) \equiv \ldots \equiv f_k(n) \equiv 0 \pmod 1 \tag{1.2}$$

*are satisfied for all* $n \in \mathbb{N}$

This conjecture has been proved for $k = 2, 3$ (see [4] and [5]) and in [3] the case $k = 4$ assuming the fulfilment of relation (1.1) for every $n \in \mathbb{Z}$. Here we define $f_j(0) = 0 \quad (j = 0, \cdots, k)$. P.D.T.A. Elliott investigated the case when $f_j = f_0$ and $f_j = -f_0$ for $j = 1, \cdots, k$ is arbitrary (see [1] and [2]), and even the case when $f_j \in \{f_0,\ -f_0,\ f_1,\ -f_1\}$, $(j = 2, \cdots, k)$.

For other results we refer to works [7], [8], [9] and [10]

The following, more general problem seems to be interesting, also. Let $A_0(n), A_1(n)\ , \dots, A_k(n) \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $f_0,\ f_1,\ \dots, f_k \in \mathcal{A}^*$ for which

$$f_0(A_0(n)) + f_1(A_1(n)) + \ldots + f_k(A_k(n)) \equiv 0 \pmod 1$$

holds. Under what conditions can we assert that

$$f_0(n) \equiv f_1(n) \equiv \ldots \equiv f_k(n) \equiv 0 \pmod 1$$

are satisfied for all $n \in \mathbb{N}$.

In this short paper we investigate the simple non-trivial case

$$(A_0(n), A_1(n), A_2(n)) = (n, 2n+1, n+2)$$

and

$$(A_0(n), A_1(n), A_2(n)) = (n, 2n-1, n+2).$$

## 2. Formulation of the theorems

We shall prove the following two theorems.

**Theorem 2.1.** *Let $\mathbb{G}$ be an additive commutative semigroup with identity element* 0*. If $f_0,\ f_1,\ f_2 \in \mathcal{A}^*_{\mathbb{G}}$ and*

$$\mathcal{A}(n) := f_0(n) + f_1(2n+1) + f_2(n+2) = 0$$

*holds for all $n \in \mathbb{N}$, then*

$$f_0(n) = f_1(2n+1) = f_2(n) = 0$$

*hold for all $n \in \mathbb{N}$.*

**Theorem 2.2.** *Let $\mathbb{G}$ be an additive commutative semigroup with identity element* 0*. If $f_0,\ f_1,\ f_2 \in \mathcal{A}^*_{\mathbb{G}}$ and*

$$\mathcal{B}(n) := f_0(n) + f_1(2n-1) + f_2(n+2) = 0$$

*holds for all $n \in \mathbb{N}$, then*

$$f_0(n) = f_1(2n-1) = f_2(n) = 0$$

*hold for all $n \in \mathbb{N}$.*

## 3. Lemmas

Firstly we prove a few lemmas.

**Lemma 1.** *Assume that $f_0,\ f_1,\ f_2 \in \mathcal{A}^*_{\mathbb{G}}$ satisfy the condition $\mathcal{A}(n) = 0$ in Theorem 2.1 for all $n \in \mathbb{N}$. Let $f_1(2) = 0$. Then*

$$f_0(n) = f_1(n) = f_2(n) = 0$$

*holds for all $n \le 5$.*

**Proof.** Let $\mathcal{B}$ be the subgroup of $\mathbb{Q}^3_+$ generated by the element $(1, 2, 1)$ and the sequences

$$L_n = \Big(n,\ 2n+1,\ n+2\Big) \quad (n \in \mathbb{N}).$$

Since $\mathcal{A}(n) = 0$, therefore

$$f_0(a) + f_1(b) + f_2(c) = 0 \quad \text{for all} \quad (a,b,c) \in \mathcal{B}.$$

We use the following notations for a prime $p$:

$$a_p = (p,1,1),\ b_p = (1,p,1) \quad \text{and} \quad c_p = (1,1,p).$$

We show that $a_p, b_p,$ and $c_p$ are elements of $\mathcal{B}$ for all primes $p \le 19$. This assertion proves Lemma 1.

Using a simple Maple program and the relation $\mathcal{A}(n) = 0$ for $n = 4,\ 25,\ 38,\ 40$ and $n = 42$, we will get the following 5 equations.

$$E_1 := \frac{L_4}{L_2} = a_2^2 b_3 c_2 \in \mathcal{B}, \tag{3.1}$$

$$E_2 := \frac{L_3 L_{12} L_{16} L_{25}}{L_1^4 L_2^2 L_5 L_8 L_{10}} = \frac{a_3^2}{b_3^3 c_2^5} \in \mathcal{B}, \tag{3.2}$$

$$E_3 := \frac{L_1^3 L_6 L_{12} L_{38}}{L_2^2 L_3 L_{16} L_{19}} = \frac{a_3 b_3 c_2^2}{a_2^2} \in \mathcal{B}, \tag{3.3}$$

$$E_4 := \frac{L_{16} L_{40}}{L_1^3 L_5} = a_2^7 b_3^2 c_2^2 \in \mathcal{B}, \tag{3.4}$$

and

$$E_5 := \frac{L_1^3 L_2 L_{28} L_{42}}{L_7^2 L_8 L_9} = \frac{a_2 b_3^2 c_2^4}{a_3} \in \mathcal{B}. \tag{3.5}$$

This system has solutions in $a_2, a_3, b_3, c_2$, which are given in terms of $E_1, \cdots, E_5$. Thus $a_2, a_3, b_3, c_2$ are elements of $\mathcal{B}$.

The solutions of the above equations (3.1)-(3.5) are:

$$a_2 = \frac{E_1^{98} E_2^{24} E_5^{32}}{E_3^{16} E_4^{37}}, \qquad a_3 = \frac{E_1^{558} E_2^{136} E_5^{181}}{E_3^{90} E_4^{211}}$$

$$b_3 = \frac{E_3^5 E_4^{11}}{E_1^{28} E_2^8 E_5^{11}} \quad \text{and} \quad c_2 = \frac{E_3^{27} E_4^{63}}{E_1^{167} E_5^{53} E_2^{40}}.$$

Finally, we express $a_5,\ b_5,\ c_3$ and $c_5$ in the terms of $a_2, a_3, b_3, c_2$ and $L_n$. We have

$$a_5 = \frac{L_1^2 L_2^2 L_5 a_2^4 a_3}{L_{12} L_{16} b_3 c_2^2}, \quad b_5 = \frac{L_2}{a_2 c_2^2}$$

and

$$c_3 = \frac{L_1}{b_3} \qquad \text{and} \qquad c_5 = \frac{L_1^3 L_2^2 L_3 L_5 a_2^5}{L_{10} L_{12} L_{16} b_3}$$

are elements of $\mathcal{B}$. This completes the proof of Lemma 1.

**Lemma 2.** *Assume that* $f_0,\ f_1,\ f_2 \in \mathcal{A}_{\mathbb{G}}^*$ *satisfy the condition* $\mathcal{B}(n) = 0$ *in Theorem 2.2 for all* $n \in \mathbb{N}$. *Let* $f_1(2) = 0$. *Then*

$$f_0(n) = f_1(n) = f_2(n) = 0$$

*holds for all* $n \leq 7$.

**Proof.** The proof is similar to the proof of Lemma 1. Let $\mathcal{D}$ be the subgroup of $\mathbb{Q}_+^3$ generated by the element $(1, 2, 1)$ and the sequences

$$D_n := \big(n,\ 2n-1,\ n+2\big) \quad (n \in \mathbb{N}).$$

From our assumption $\mathcal{B}(n) = 0$ for all $n \in \mathbb{N}$, we have

$$f_0(a) + f_1(b) + f_2(c) = 0 \quad \text{for all} \quad (a, b, c) \in \mathcal{D}.$$

We shall use the following notations ($p$ is prime):

$$A_p := (p, 1, 1) \in \mathcal{D},\ B_p := (1, p, 1) \in \mathcal{D} \quad \text{and} \quad C_p := (1, 1, p) \in \mathcal{D}.$$

By using a simple Maple program and the relation $\mathcal{B}(n) = 0$ for $n = 8,\ 18,\ 26,\ 28$ and $n = 63$, we obtain the following 5 equations in $A_2, A_3, B_5, C_2$:

$$F_1 := \frac{D_8}{D_2 D_3} = \frac{A_2^2}{A_3 C_2} \in \mathcal{D}, \tag{3.6}$$

$$F_2 := \frac{D_1 D_{18}}{D_3 D_4} = \frac{A_3 C_2}{A_2} \in \mathcal{D}, \tag{3.7}$$

$$F_3 := \frac{D_1^3 D_3 D_{14} D_{20} D_{26}}{D_2^3 D_5 D_7 D_9 D_{13}} = \frac{A_2 C_2}{A_3 B_5} \in \mathcal{D}, \tag{3.8}$$

$$F_4 := \frac{D_2^3 D_{28}}{D_1 D_3 D_6 D_{14}} = \frac{A_2^3}{A_3^2} \in \mathcal{D}, \tag{3.9}$$

$$F_5 := \frac{D_2^5 D_3^3 D_6 D_{10}^2 D_{63}}{D_1^2 D_{14} D_{48}^2 D_{50}} = \frac{A_3^4 B_5^4 C_2^9}{A_2^2} \in \mathcal{D}. \tag{3.10}$$

This system has solutions in $A_2, A_3, B_5, C_2$, which are given in terms of $F_1, \cdots, F_5$. Thus $A_2, A_3, B_5, C_2$ are elements of $\mathcal{D}$.

The solutions of the above equations (3.6)-(3.10) are:

$$A_2 = F_1 F_2, \quad A_3 = \frac{F_2^{10} F_4^6}{F_1^3 F_3^4 F_5},$$

$$B_5 = \frac{F_3^7 F_5^2 F_1^8}{F_2^{17} F_4^{12}} \quad \text{and} \quad C_2 = \frac{F_3^4 F_5 F_1^4}{F_2^8 F_4^6}.$$

Now, we express $A_5,\ A_7, B_3,\ B_7, C_3, C_5$ and $C_7$ in the terms of $A_2, A_3, B_5, C_2$ and $L_n$. We have

$$A_5 = \frac{L_3^2 L_{10} A_2^3}{L_1 L_{48} A_3 B_5 C_2}, \quad A_7 = \frac{L_{14} A_2^2 C_2^2}{L_2^3},$$

$$B_3 = \frac{L_2}{A_2 C_2^2}, \quad B_7 = \frac{L4}{L_1 A_2^2 C_2}$$

and

$$C_3 = L1, \quad C5 = \frac{L_3}{A_3 B_5}, \quad C_7 = \frac{L_1 L_5 L_{48} A_3 B_5 C_2^5}{L_2^2 L_3^2 L_{10} A_2}$$

are elements of $\mathcal{D}$. This completes the proof of Lemma 2.

## 4. Proof of Theorem 2.1

Let $\mathbb{G}$ be an additive commutative semigroup with identity element 0. If $f_0,\ f_1,\ f_2 \in \mathcal{A}_{\mathbb{G}}^*$ and

$$\mathcal{A}(n) := f_0(n) + f_1(2n+1) + f_2(n+2) = 0$$

holds for all $n \in \mathbb{N}$. By using Lemma 1, we have $f_0(p) = f_1(q) = f_2(p) = 0$ for primes $p \leq 5$ and $q = 3, 5$.

Assume indirectly that the theorem is not true. Let $n_0$ be the smallest positive integer for which $f_j(n_0) \neq 0$. Then $n_0 = P \in \mathcal{P},\ P > 5$ and either $f_0(P) \neq 0$ or $f_2(P) \neq 0$.

**Case I.** $f_0(P) = \xi\ (\neq 0)$.

If $P \equiv 1 \pmod 3$, then $3|P+2, 3|2P+1$, thus $f_1(P+2) = 0, f_2(2P+1) = 0$, consequently $\mathcal{A}(P) = 0$ implies that $f_0(P) = 0$.

It remains to consider the case $P \equiv -1 \pmod 3$. Let $4P+1 = 3Q$. Then it follows from the fact $P > 5$ that $\frac{Q-1}{2} < \frac{Q+3}{2} < P$, consequently $\mathcal{A}(\frac{Q-1}{2}) = 0$ implies that

$$0 = f_0\Big(\frac{Q-1}{2}\Big) + f_1(Q) + f_2\Big(\frac{Q+3}{2}\Big) = f_1(Q),$$

thus we infer from $\mathcal{A}(2P) = 0$ that

$$0 = f_0(2P) + f_1(3Q) + f_2\Big(4\frac{P+1}{2}\Big)$$

and so $f_0(P) = 0$.

**Case II.** $f_2(P) = \nu\ (\neq 0)$.

| | From $\mathcal{A}(n) = 0$ | we obtain that | and that |
|---|---|---|---|
| (1) | $\mathcal{A}(P-2) = 0$ | $2P-3 \in \mathcal{P}$ | $f_1(2P-3) = -\nu$ |
| (2) | $\mathcal{A}(2P-2) = 0$ | $4P-3 \in \mathcal{P}$ | $f_1(4P-3) = -\nu$ |
| (3) | $\mathcal{A}(6P-2) = 0$ | $4P-1 \in \mathcal{P}$ | $f_1(4P-1) = -\nu$ |
| (4) | $\mathcal{A}(3P-5) = 0$ | $\frac{3P-5}{2} \in \mathcal{P}$ | $f_0(\frac{3P-5}{2}) = \nu$ |
| (5) | | $P \equiv 2 \pmod 3$ | |
| (6) | $\mathcal{A}(4P-2) = 0$ | $8P-3 \in \mathcal{P}$ | $f_1(8P-3) = -\nu$ |
| (7) | | $P \equiv 3 \pmod 5$ | |

The assertions (1) and (2) are clear.

In order to show (3), let $Q := \frac{3P-1}{2}$. Then we have

$$Q \equiv 1 \pmod 3, 3|Q+2,\ \frac{Q+2}{3} = \frac{P+1}{2} < P \quad \text{and} \quad 2Q+1 = 3P,$$

which with $\mathcal{A}(Q) = 0$ shows that

$$f_0(Q) + f_1(2Q+1) + f_2(Q+2) = f_0(Q) + f_1(P) = 0.$$

It is clear from $\mathcal{A}(\frac{P-1}{2}) = 0$ that $f_1(P) = 0$, consequently $f_0(Q) = f_0(3P-1) = f_0(6P-2) = 0$, thus $\mathcal{A}(6P-2) = 0$ implies

$$0 = f_0(6P-2) + f_1(12P-3) + f_2(6P) = f_1(4P-1) + f_2(P),$$

which proves (3).

From $\mathcal{A}(3P-5)=0$ we have

$$0=f_0(3P-5)+f_1(6P-9)+f_2(3P-3)=f_0\Big(\frac{3P-5}{2}\Big)+f_1(2P-3),$$

from (1) we obtain (4).

Since $P\in\mathcal{P}$, the assertion (5) follows from (3).

From (5), we have $3|2P-1$ and $\frac{2P-1}{3}<P$, consequently $f_0(4P-2)= =f_0(2P-1)=0$. Thus we obtain from $\mathcal{A}(4P-2)=0$ that

$$0=f_0(4P-2)+f_1(8P-3)+f_2(4P)=f_1(8P-3)+f_2(P),$$

which proves (6). Since $P\in\mathcal{P}$, the assertion (7) follows from (1), (2) and (6).

Let $T:=\frac{3P-5}{2}$. Then we infer from (4) and (7) that

$$T\in\mathcal{P},\quad f_0(T)=\nu \tag{4.1}$$

and

$$T\equiv -1 \pmod 3,\ T\equiv 2 \pmod 5. \tag{4.2}$$

From (4.2) we have $5|2T+1$, $\frac{2T+1}{5}=\frac{3P-4}{5}<P$, consequently we obtain from $\mathcal{A}(T)=0$ that

$$\mathcal{A}(T)=f_0(T)+f_1(2T+1)+f_2(T+2)=f_0(T)+f_2(T+2)=0.$$

This with (4.1) implies

$$f_2(T+2)=-\nu. \tag{4.3}$$

From (4.2), we have $5|3T+4$, $\frac{3T+4}{5}=\frac{9P-7}{10}<P$ and $f_0(3T+4)=f_0(\frac{3T+4}{5})=0$. Thus we obtain from $\mathcal{A}(3T+4)=0$ that

$$0=f_0(3T+4)+f_1(6T+9)+f_2(3T+6)=f_1(2T+3)+f_2(T+2),$$

which with (4.3) implies

$$f_1(2T+3)=\nu. \tag{4.4}$$

Finally, $\mathcal{A}(T+1)=0$ implies that

$$f_0(T+1)+f_1(2T+3)+f_2(T+3)=0.$$

Since $\frac{T+1}{2}=3\frac{P-1}{4}<P$, $\frac{T+3}{2}=\frac{3P+1}{4}<P$, we deduce that $f_1(2T+3)=0$. This contradicts to (4.4).

The proof of Theorem 1 is complete.

## 5. Proof of Theorem 2.2

Let $\mathbb{G}$ be an additive commutative semigroup with identity element 0. If $f_0,\ f_1,\ f_2 \in \mathcal{A}^*_{\mathbb{G}}$ and

$$\mathcal{B}(n) := f_0(n) + f_1(2n+1) + f_2(n+2) = 0$$

holds for all $n \in \mathbb{N}$. By using Lemma 2, we have $f_0(p) = f_1(q) = f_2(p) = 0$ for primes $p \le 5$ and $q = 3, 5$.

Assume indirectly that the theorem is not true. Let $n_0$ be the smallest positive integer for which $f_j(n_0) \ne 0$. Then $n_0 = P \in \mathcal{P}, P > 7$ and either $f_0(P) \ne 0$ or $f_2(P) \ne 0$.

**Case I.** $f_2(P) = \nu\ (\ne 0)$.

We infer from $\mathcal{B}(P-2) = 0$ that $f_1(2P-5) = -\nu$, $2P-5 \in \mathcal{P}$ and so $P \equiv -1 \pmod 3$. We have

$$\mathcal{B}(2P-2) = f_0(2P-2) + f_1(4P-5) + f_2(2P) = 0.$$

Since $f_0(2P-2) = 0$, therefore $f_1(4P-5) = -\nu, Q = \frac{4P-5}{3} \in \mathcal{P}$ and

$$\mathcal{B}\Big(\frac{Q+1}{2}\Big) = f_0\Big(\frac{Q+1}{2}\Big) + f_1(Q) + f_2\Big(\frac{Q+5}{2}\Big) = 0.$$

Since $P > 5$, we have $\frac{Q+1}{2} = \frac{2P-1}{3} < P$ and $\frac{Q+5}{2} = \frac{2P+5}{3} < P$, consequently

$$f_1(Q) = f_1\Big(\frac{4P-5}{3}\Big) = f_1(4P-5) = 0.$$

This cannot occur.

**Case II.** $f_0(P) = \xi\ (\ne 0)$.

Since $4|2P+2$ and $\frac{2P+2}{4} = \frac{P+1}{2} < P$, we infer from $\mathcal{B}(2P) = 0$ that

$$0 = \mathcal{B}(2P) = f_0(2P) + f_1(4P-1) + f_2\Big(2P+2\Big) = \xi + f_1(4P-1).$$

Thus $f_1(4P-1) = -\xi$ and either $4P-1 \in \mathcal{P}$ or $\frac{4P-1}{3} \in \mathcal{P}$.

If $Q := \frac{4P-1}{3} \in \mathcal{P}$, then $P > 7$ shows that $\frac{Q+1}{2} = \frac{2P+1}{3} < P$ and $\frac{Q+5}{2} = \frac{2P+7}{3} < P$, consequently

$$0 = \mathcal{B}\Big(\frac{Q+1}{2}\Big) = f_0\Big(\frac{Q+1}{2}\Big) + f_1(Q) + f_2\Big(\frac{Q+5}{2}\Big) = f_1(Q).$$

This cannot occur. Thus we have proved that $4P-1\in\mathcal{P}$, and so

$$P\equiv -1 \pmod 3). \tag{5.1}$$

From (5.1) we have $3|2P-1$, consequently $\mathcal{B}(P)=0$ implies that $f_2(P+2)= =-\xi,\ P+2\in\mathcal{P}$.

Since

$$0=\mathcal{B}(2P)=f_0(2P)+f_1(4P-1)+f_2\Big(2P+2\Big)$$

and $f_2\Big(2P+2\Big)=0$, we have

$$0=\mathcal{B}(2P)=f_0(2P)+f_1(4P-1)+f_2\Big(2P+2\Big)=\xi+f_1(4P-1),$$

consequently $f_1(4P-1)=-\xi,\ 4P-1\in\mathcal{P}$.

On the other hand, we have

$$\mathcal{B}(2P+2)=f_0(2P+2)+f_1(4P+3)+f_2\Big(2P+4\Big)=0,$$

which implies that $f_1(4P+3)=\xi,\ 4P+3\in\mathcal{P}$. Since $P,P+2,4P-1,4P+3\in\mathcal{P}$, therefore $P\equiv 1 \pmod 5$ or $P\equiv 2 \pmod 5$.

**Case II.a.** $P\equiv 2 \pmod 5$.

Since $15|8P-1, 5|2P+1$, therefore

$$\mathcal{B}(4P)=f_0(4P)+f_1\Big(\frac{8P-1}{15}\Big)+f_2\Big(\frac{2P+1}{5}\Big)=0,$$

therefore $f_0(P)=0$.

**Case II.b.** $P\equiv 1 \pmod 5$.

In this case $6P-1=5Q$, $\frac{Q+1}{2}=\frac{3P+2}{5}<P$ and $\frac{Q+5}{2}=\frac{3P+12}{5}<P$. Thus

$$\mathcal{B}\Big(\frac{Q+1}{2}\Big)=f_0\Big(\frac{Q+1}{2}\Big)+f_1(Q)+f_2\Big(\frac{Q+5}{2}\Big)=f_1(Q).$$

Hence

$$f_1(Q)=f_1(6P-1)=0. \tag{5.2}$$

Since $5|3P+2,\ f_2(3P+2)=0$, therefore $\mathcal{B}(3P)=0$ with (5.2) implies that $f_0(P)=0$.

The proof of Theorem 2 is complete.

## 6. Final remarks

**Theorem 6.1.** *Let* $\alpha_0 = \beta_0 = (1, 2, 1)$ *and* $\alpha_n = (n, 2n + 1, n + 2)$, $\beta_n = (n, 2n-1, n+2)$. *Let* $\mathcal{B}$ *be the subgroup of* $\mathbb{Q}_+^3$ *generated by* $\alpha_n$ $(n = 0, 1, 2, \cdots)$ *and* $\mathcal{D}$ *be the subgroup of* $\mathbb{Q}_+^3$ *generated by* $\beta_n$ $(n = 0, 1, 2, \cdots)$. *Then*

$$\mathcal{B} = \mathbb{Q}_+^3 \quad \text{and} \quad \mathcal{D} = \mathbb{Q}_+^3.$$

*It means that for every* $(r_1, r_2, r_3) \in \mathbb{Q}_+^3$ *there exist* $n_1, n_2, \cdots, n_k \in \mathbb{N}_0$, $\epsilon_1, \epsilon_2, \cdots, \epsilon_k \in \{-1, 1\}$ *and* $m_1, m_2, \cdots, m_l \in \mathbb{N}_0$, $\delta_1, \delta_2, \cdots, \delta_l \in \{-1, 1\}$ *such that*

$$(r_1, r_2, r_3) = \prod_{i=1}^{k} \alpha_{n_i}^{\epsilon_i},$$

*and that*

$$(r_1, r_2, r_3) = \prod_{i=1}^{l} \alpha_{m_i}^{\delta_i}.$$

## References


[1] **Elliott, P.D.T.A.**, *Arithmetic functions and integer products,* Springer-Verlag, New York, 1985.

[2] **Elliott, P.D.T.A.**, On representing integers as products of integers of a preseribed type, *J. Austral. Math. Soc. (Series A),* **35** (1983), 143-161.

[3] **Chakraborty, K., Kátai, I. and Phong, B. M.**, On real valued additive functions modulo 1, *Annales Univ. Sci. Budapest. Sect. Comp.,* **36** (2012).

[4] **Kátai, I.**, On additive functions satisfying a congruence, *Acta Sci. Math.,* **47** (1984), 85-92.

[5] **Kátai, I.**, A remark on additive functions satisfying a relation, *Annales Univ. Sci. Budapest. Sect. Math.,* **27** (1985), 229-233.

[6] **Kátai, I.**, Arithmetical functions satisfying some relations, *Acta Sci. Math.,* **55** (1991), 249-268.

[7] **Kátai, I. and van Rossum-Wijsmuller, M.**, Additive functions satisfying congruences, *Acta Sci. Math.,* **56** (1992), 63-72.

[8] **Kovács, A. and Phong, B.M.**, On completely additive functions satisfying a congruence, *Annales Univ. Sci. Budapest. Sect. Comp.,* **19** (2000), 237-249.

[9] **van Rossum-Wijsmuller, M.**, Additive functions on the Gaussian integers, *Publ. Math. Debrecen,* **38** (1991), 255-262.

[10] **Styer, R.**, A problem of Kátai on sums of additive functions, *Acta Sci. Math.,* **55** (1991), 269-286.

**K. Chakraborty**
Department of Mathematics
Harish-Chandra Research Institute
Chhatnag Road, Jhusi,
Allahabad 211 019, India
kalyan@hri.res.in

**I. Kátai and Bui Minh Phong**
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/C
1117 Budapest, Hungary
katai@compalg.inf.elte.hu
bui@compalg.inf.elte.hu