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ON THE PRIME DIVISORS OF THE EULER PHI AND THE SUM OF DIVISORS FUNCTIONS

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Abstract. The following assertion is proved. Given an arbitrary constant $\lambda > 2$, let $x_1 = \log x$, $x_{k+1} = \log x_k$ $(k = 1, 2, \dots)$, $\varphi(n)$ be Euler's totient, and $\sigma(n)$ the sum of divisors function. Let $I_x = \left[\frac{\lambda x_2}{x_3}, x_2\right]$, $Q_1, Q_2 \in I_x$ be primes,

$$E_{Q_1,Q_2}(x) := \#\{n \leq x \mid Q_1 \nmid \varphi(n),\ Q_2 \nmid \varphi(n+1)\}.$$

Then, uniformly for $Q_1, Q_2 \in I_x$,

$$\frac{1}{x} E_{Q_1,Q_2}(x) = (1 + o_x(1)) \frac{B}{2} \kappa_1 \kappa_2,$$

where $\kappa_j = \exp\left(-\frac{x_2}{Q_j - 1}\right)$ $(j = 1, 2)$, B is a given constant. Some other assertions are formulated without proof.

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1. Introduction

Let $\varphi(n)$ be Euler's totient function, $\sigma(n)$ be the sum of divisors function. We shall define the iterates of φ and σ as follows:

$$\varphi_{k+1}(n) = \varphi(\varphi_k(n)), \quad \sigma_{k+1}(n) = \sigma(\sigma_k(n)),$$
$$\varphi_1(n) = \varphi(n), \quad \sigma_1(n) = \sigma(n).$$

Let $\mathcal{P}$ be the set of primes. It is known that φ and σ are multiplicative functions, and if p^α is a prime power, then $\varphi(p^\alpha) = p^{\alpha-1}(p-1), \quad \sigma(p^\alpha) = = 1 + p + \ldots + p^\alpha$. In particular, $\varphi(p) = p-1, \quad \sigma(p) = p+1$. The letters p, Q with and without suffixes always denote prime numbers. As usual let $p(n)$ be the smallest and $P(n)$ be the largest prime divisor of n.

Let $x_1 = \log x, \quad x_2 = \log x_1, \ldots$.

The letters $c, c_1, c_2, \ldots$ denote suitable, $d, d_1, d_2, \ldots$ be arbitrary positive constants, not necessarily the same at every occurrence.

Let $(\xi_p =)\ \xi_p(x) = e^{-\frac{x_2}{p-1}} \quad (p \in \mathcal{P})$,

$$\tau(Q \mid x) = \xi_Q(x) \prod_{\substack{p<Q \\ p\in\mathcal{P}}} (1-\xi_p(x)), \quad Q \in \mathcal{P}.$$

Let

$$\mathcal{P}_\pm(Q) := \{p \mid p \in \mathcal{P}, \quad p \equiv \pm 1 \pmod{Q}\}.$$

Since $Q \nmid \varphi(n), \quad n \le x$ holds if and only if $(n, \mathcal{P}_+(Q)) = 1$ and $Q^2 \nmid n$, thus, by the sieve of Eratosthenes-Brun we obtain that

$$\begin{aligned} \frac{E_Q(x)}{x} &:= \frac{1}{x}\#\{n \le x \mid Q \nmid \varphi(n)\} = \\ &= (1+o_x(1))\left(1-\frac{1}{Q^2}\right) \prod_{\substack{p\le x \\ p\in\mathcal{P}_+(Q)}} (1-1/p) \end{aligned} \tag{1.1}$$

if $Q \le x_2^d$.

Since

$$\begin{aligned} \pi(x,k,l) &= \#\{p \le x, \quad p \equiv l \pmod{k}\} = \\ &= \frac{1}{\varphi(k)}(1+\mathcal{O}(e^{-c\sqrt{x_1}})) \end{aligned}$$

holds uniformly as $(k,l)=1, \quad k \le x_1^{d_1}$, we obtain that the right hand side of (1.1) is $\xi_q(x)\left(1+\mathcal{O}\left(\frac{\log Q}{Q}\right)\right)(1+o_x(1))$. We shall list several assertions which can be deduced by immediate application of the Brun sieve (see Theorem 2.5 in [7]).

I. Let $Q \in [x_3, x_2], \quad Q \in \mathcal{P}, \quad l \in \mathbb{Z}, \quad l \ne 0$. Then

$$\frac{1}{x}\#\{n \le x \mid \sigma(n) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.2}$$

$$\begin{aligned}\frac{1}{x}\#\{n \le x \mid \varphi(n) \not\equiv 0 \pmod{Q},\ \sigma(n) \not\equiv 0 \pmod{Q}\} &= \\ = (1+o_x(1))\xi_Q^2(x),&\end{aligned} \tag{1.3}$$

$$\frac{1}{lix}\#\{p \le x \mid \varphi(p+l) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.4}$$

$$\frac{1}{lix}\#\{p \le x \mid \sigma(p+l) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.5}$$

$$\begin{aligned}\frac{1}{lix}\#\{p \le x \mid \varphi(p+l) \not\equiv 0 \pmod{Q}, \sigma(p+l) \not\equiv 0 \pmod{Q}\} &= \\ = (1+o_x(1))\xi_Q^2(x).&\end{aligned} \tag{1.6}$$

II. Similar assertions can be proved for the set of integers $F(n)$ or $F(p)$, where $F \in \mathbb{Z}[x]$ is a polynomial the leading coefficient of which is positive.

III. Let $Q_1, \dots, Q_r; \quad Q_1^*, \dots, Q_s^* \in [x_3, x_2]$ be primes, $Q_i \ne Q_j$ if $i \ne \ne j, \quad Q_u^* \ne Q_v^*$ if $u \ne v$. Let $l \ne 0$. Then

$$\begin{aligned}\frac{1}{x}\#\{n \le x \mid (\varphi(n), Q_1 \dots Q_r) = 1, (\sigma(n), Q_1^* \dots Q_s^*) = 1\} &= \\ = (1+o_x(1))\left\{\prod_{j=1}^{r} \xi_{Q_j}(x)\right\}\left\{\prod_{l=1}^{s} \xi_{Q_l^*}(x)\right\},&\end{aligned} \tag{1.7}$$

furthermore

$$\begin{aligned}\frac{1}{lix}\#\{p \le x \mid (\varphi(p+l), Q_1 \dots Q_r) = 1,\ (\sigma(p+l), Q_1^* \dots Q_s^*) = 1\} &= \\ = (1+o_x(1))\left\{\prod_{j=1}^{r} \xi_{Q_j}(x)\right\}\left\{\prod_{l=1}^{s} \xi_{Q_l^*}(x)\right\},&\end{aligned} \tag{1.8}$$

For fixed Q, (1.1), (1.2) and (1.3) can be improved (see [2], [1]).

2. Counting those integers n for which $\varphi(n)$ and $\sigma(n)$ each avoid a given prime as their smaller prime factor

Let $u(n)$ be the smallest prime Q for which $Q\nmid\varphi(n)$, and $v(n)$ be the smallest $Q\in\mathcal{P}$, for which $Q\nmid\sigma(n)$.

Let $K_Q(x):=\#\{n\le x \mid u(n)=Q\}$, $\quad T_Q(x):=\#\{n\le x \mid v(n)=Q\}$,

$$S_{Q_1,Q_2}(x)=\#\{n\le x \mid u(n)=Q_1, v(n)=Q_2\}.$$

Theorem 1. *Assume that $Q,Q_1,Q_2\in\left[x_3,\frac{x_2}{x_3}\right]$. Then*

$$\frac{K_Q(x)}{x}=(1+o_x(1))\xi_Q(x), \tag{2.1}$$

$$\frac{T_Q(x)}{x}=(1+o_x(1))\xi_Q(x), \tag{2.2}$$

$$\frac{S_{Q_1,Q_2}(x)}{x}=(1+o_x(1))\xi_{Q_1}(x)\cdot\xi_{Q_2}(x). \tag{2.3}$$

Furthermore, if $l\neq 0$, then

$$\frac{1}{lix}\#\{p\le x \mid u(p+l)=Q\}=(1+o_x(1))\xi_Q(x), \tag{2.4}$$

$$\frac{1}{lix}\#\{p\le x \mid v(p+l)=Q\}=(1+o_x(1))\xi_Q(x), \tag{2.5}$$

$$\begin{aligned}\frac{1}{lix}\#\{p\le x \mid u(p+l)=Q_1, v(p+l)=Q_2\}=\\ =(1+o_x(1))\xi_{Q_1}(x)\cdot\xi_{Q_2}(x).\end{aligned} \tag{2.6}$$

Remark 1. Unfortunately we cannot extend Theorem 1 for the values $Q,Q_1,Q_2\ge x_2/x_3$.

Conjecture 1. *Let d be a positive constant. Then, uniformly as $x_3 < < Q, Q_1, Q_2 \leq dx_2$ we have*

$$\frac{1}{x}\#\{n \leq x \mid u(n) = Q\} = (1 + o_x(1))\tau(Q|x), \tag{2.7}$$

$$\frac{1}{x}\#\{n \leq x \mid v(n) = Q\} = (1 + o_x(1))\tau(Q|x), \tag{2.8}$$

$$\begin{aligned}\frac{1}{x}\#\{n \leq x \mid u(n) = Q_1, v(n) = Q_2\} = \\ = (1 + o_x(1))\tau(Q_1|x)\tau(Q_2|x).\end{aligned} \tag{2.9}$$

Remark. Similar assertion seems to hold for the set of shifted primes as well.

In [2] we considered $N_k(Q|x)$, the number of those $n \leq x$ for which $Q \not| \varphi_{k+1}(n)$. We determined the asymptotic of $N_k(Q|x)$ in the range $Q \in \in (x_2^{k+\varepsilon}, x_2^{k+1-\varepsilon})$. By using the same method with some generalization we could prove

Theorem 2. *Let $\varepsilon > 0$, $k \geq 2$ be fixed, $l \neq 0$, $l \in \mathbb{Z}$, and let $x_2^{k+\varepsilon} \leq \leq Q \leq x_2^{k+1-\varepsilon}$, $Q \in \mathcal{P}$. Then, setting $\eta_{k,Q}(x) := \exp\left(-\frac{x_2^{k+1}}{(k+1)!(Q-1)}\right)$, we obtain*

$$\frac{1}{x}\#\{n \leq x \mid Q \not| \sigma_{k+1}(n)\} = (1 + o_x(1))\eta_{k,Q}(x), \tag{2.10}$$

$$\frac{1}{lix}\#\{p \leq x \mid Q \not| \sigma_{k+1}(p + l)\} = (1 + o_x(1))\eta_{k,Q}(x), \tag{2.11}$$

$$\begin{aligned}\frac{1}{x}\#\{n \leq x \mid Q \not| \sigma_{k+1}(n), Q \not| \varphi_{k+1}(n)\} = \\ = (1 + o_x(1))\eta_{k,Q}^2(x),\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{1}{lix}\#\{p \leq x \mid Q \not| \sigma_{k+1}(p + l), Q \not| \varphi_{k+1}(p + l)\} = \\ = (1 + o_x(1))\eta_{k,Q}^2(x).\end{aligned} \tag{2.13}$$

Theorem 3. *Let $\varepsilon > 0$, $k \geq 2$, $r, s \geq 1$. Let $Q_1, \ldots, Q_r$ and $Q_1^*, \ldots, Q_s^*$ be distinct primes from the interval $\left[x_2^{k+\frac{1}{2}+\varepsilon}, x_2^{k+1-\varepsilon}\right]$. Then*

$$\frac{1}{x}\#\{n \leq x \mid (Q_1 \ldots Q_r, \varphi_{k+1}(n)) = 1,\ (Q_1^* \ldots Q_s^*, \sigma_{k+1}(n)) = 1\} = \tag{2.14}$$
$$= (1 + o_x(1)) \left\{\prod_{j=1}^{r} \eta_{k,Q_j}(x)\right\} \left\{\prod_{l=1}^{s} \eta_{l,Q_l^*}(x)\right\},$$

and

$$\frac{1}{lix}\#\{p \leq x \mid (Q_1 \ldots Q_r, \varphi_{k+1}(p+l)) = 1,\ (Q_1^* \ldots Q_s^*, \sigma_{k+1}(p+l)) = 1\} = \tag{2.15}$$
$$= (1 + o_x(1)) \left\{\prod_{j=1}^{r} \eta_{k,Q_j}(x)\right\} \left\{\prod_{l=1}^{s} \eta_{l,Q_l^*}(x)\right\}.$$

We shall not prove these theorems.

3. Counting those integers n for which $\varphi(n)$ and $\varphi(n+1)$ each avoid given primes in their respective prime factorizations

The problem of giving the asymptotic of those $n \leq x$ for which $Q_1 \nmid \varphi(n)$ and $Q_2 \nmid \varphi(n+1)$ simultaneously for given primes Q_1, Q_2 seems to be much harder. We are unable to determine it for example if $Q_1 = Q_2 = 3$.

Theorem 4. *Let $\lambda > 2$ be an arbitrary constant. Let $\mathcal{I}_x = \left[\frac{\lambda x_2}{x_3}, x_2\right]$,*

$$B = \prod_{p \geq 3} \left(1 - \frac{2}{p(p-1)}\right).$$

Let $Q_1, Q_2 \in \mathcal{I}_x$ be arbitrary primes. Let

$$E_{Q_1,Q_2}(x) := \#\{n \leq x \mid Q_1 \nmid \varphi(n),\ Q_2 \nmid \varphi(n+1)\}. \tag{3.1}$$

Then, uniformly as $Q_1, Q_2 \in \mathcal{I}_x$,

$$\frac{1}{x} E_{Q_1,Q_2}(x) = (1 + o_x(1)) \frac{B}{2} \kappa_1 \kappa_2, \tag{3.2}$$

where $\kappa_1 = \exp\left(-\frac{x_2}{Q_1-1}\right)$, $\kappa_2 = \exp\left(-\frac{x_2}{Q_2-1}\right)$.

Remark. A similar assertion can be proved for $\sigma(n)$ instead of $\varphi(n)$.

Proof. It is clear that

$$\frac{\kappa_1\kappa_2}{Q_1} \to 0, \quad \frac{\kappa_1\kappa_2}{Q_2} \to 0, \qquad \text{as} \quad x \to \infty. \tag{3.3}$$

Let $\mathcal{P}_j = \{p \mid p \equiv 1 \pmod{Q_j}\}$ $(j = 1,2)$, $\mathcal{N}(\mathcal{P}_j) = \{n \mid p|n \Rightarrow p \in \in \mathcal{P}_j\}$, $\mathcal{N}_{Q_j}(\mathcal{P}_j) = \{n \mid n \in \mathcal{N}(\mathcal{P}_j) \text{ and } Q_j^2 \nmid n\}$, $(j = 1,2)$.

Let $\mathcal{E}_{Q_1,Q_2} = \{n \mid n \le x,\ n \in \mathcal{N}_{Q_1}(\mathcal{P}_1),\ n+1 \in \mathcal{N}_{Q_2}(\mathcal{P}_2)\}$. Let $Y = = x^{1/\gamma_x}$, $\gamma_x = 40x_2$.

For some $n \in \mathcal{E}_{Q_1,Q_2}$ we write $n = \xi u$, $n+1 = \eta v$, where $\xi \in \mathcal{N}_{Q_1}(\mathcal{P}_1)$, $\eta \in \in \mathcal{N}_{Q_2}(\mathcal{P}_2)$, $P(\xi) \le Y$, $P(\eta) \le Y$, $p(u) > Y$, $p(v) > Y$. Let $\mathcal{T}(\xi,\eta)$ be the set of those $n \in \mathcal{E}_{Q_1,Q_2}$ for which ξ and η are fixed. Let $T(\xi,\eta) = \#\mathcal{T}(\xi,\eta)$. If $T(\xi,\eta) \ne 0$, then $(\xi,\eta) = 1$ and $2 \mid \xi\eta$.

It is well-known that $\psi(x,y) \ll xe^{-u/2}$, $u = \frac{\log x}{\log y}$, where

$$\psi(x,y) = \#\{n \le x \mid P(n) \le y\}.$$

It is clear that

$$E_{Q_1,Q_2}(x) \le \sum_{\xi,\eta} T(\xi,\eta). \tag{3.4}$$

We shall overestimate the contribution of those terms standing on the right hand side of (3.4), for which $\xi > x^{1/10}$, or $\eta > x^{1/10}$ holds. This is less than

$$\begin{aligned} 2x \sum_{\substack{m > x^{1/10} \\ P(m) < Y}} 1/m &\le 2x \sum_{j=0}^{\infty} \frac{1}{2^j x^{1/10}} \psi(2^{j+1}x^{1/10}, Y) \le \\ &\le 2x \sum_{j\ge 0} \exp\left(-\frac{\frac{1}{10}x_1 + j\log 2}{2\log Y}\right) = 2xe^{-\frac{\gamma_x}{20}} \cdot \frac{1}{1 - e^{-\frac{\log 2}{\log Y}}} \le \\ &\le 4xe^{-\frac{\gamma_x}{20}} \cdot \frac{x_1}{\log 2} \cdot \frac{1}{\gamma_x} \ll \frac{x}{x_1^{3/2}}. \end{aligned}$$

Thus

$$E_{Q_1,Q_2}(x) \le \sum_{\max(\xi,\eta) \le x^{1/10}} T(\xi,\eta) + \mathcal{O}\left(\frac{x}{x_1^{3/2}}\right). \tag{3.5}$$

We shall estimate $T(\xi,\eta)$ for $\max(\xi,\eta)<x^{1/10}$. We have to count those $n\le x$, for which $n=\xi u,\quad n+1=\eta v,\quad n\le x$ and $p(u)>Y,\quad p(v)>Y$. Let u_0,v_0 be the smallest pair of those positive integers u,v, for which $\eta v-\xi u=1$. Let $F_1(t)=u_0+\eta t,\quad F_2(t)=v_0+\xi t$. Then

$$T(\xi,\eta)=\#\left\{t\le\frac{x}{\xi\eta}\ \Big|\ p(F_1(t))>Y,\quad p(F_2(t))>Y\right\}.$$

By using Theorem 2.6 in Halberstam - Richert [7], we deduce that

$$T(\xi,\eta)=(1+o_x(1))x\cdot\frac{1}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)\cdot\theta_{\xi,\eta}\tag{3.6}$$

uniformly for all possible ξ,η, where

$$\theta_{\xi,\eta}=\frac{1}{\xi\eta}\prod_{\substack{p\mid\xi\eta\\ p\ne 2}}\frac{1-1/p}{1-2/p}.\tag{3.7}$$

The implied constants in the error term of (8.4) of Theorem 2.6 in [7] may depend on the coefficients of F_1,F_2, i.e. on ξ and on η, but reading the proof carefully one can see that estimate (3.6) will hold for all ξ,η provided that we add an error term, thus implying that (3.6) holds with that particular restriction. Hence, from (3.5), we obtain that

$$E_{Q_1,Q_2}(x)\le(1+o_x(1))\frac{x}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)\Sigma_1,\tag{3.8}$$

where

$$\Sigma_1=\sum\frac{1}{\xi\eta}\prod_{\substack{p\mid\xi\eta\\ p>2}}\frac{1-1/p}{1-2/p}.\tag{3.9}$$

Let Σ_2 be the sum of those terms on the right hand side of (3.9), for which additionally $Q_1\nmid\xi,\quad Q_2\nmid\eta$ holds. It is clear that

$$0<\Sigma_1-\Sigma_2\ll\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right)\Sigma_1\ll\frac{x_3}{x_2}\Sigma_1.\tag{3.10}$$

We shall write

$$\Sigma_2=A(Y)B(Y)C(Y),\tag{3.11}$$

where

$$A(Y)=\prod_{\substack{3\le p<Y\\(p-1,Q_1Q_2)=1}}\left(1+\frac{2}{p}+\frac{2}{p^2}+\dots\right)= \tag{3.12}$$
$$=\prod_{\substack{3\le p<Y\\(p-1,Q_1Q_2)=1}}\left(1+\frac{2}{p-1}\right),$$

$$B(Y)=\prod_{\substack{p\equiv 1 \pmod{Q_1}\\(p-1,Q_2)=1\\p<Y}}(1+1/p);\quad C(Y)=\prod_{\substack{p\equiv 1 \pmod{Q_2}\\(p-1,Q_1)=1\\p<Y}}(1+1/p). \tag{3.13}$$

Thus we proved that

$$E_{Q_1,Q_2}(x)\le(1+o_x(1))\frac{x}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y)B(Y)C(Y)+\mathcal{O}\left(\frac{x}{x_1^3}\right). \tag{3.14}$$

Let $\mathcal{T}^*(\xi,\eta)$ be the set of those $n=\xi u\ \le x$, for which $n+1=\eta v,\ \ p(u)>$ $>Y,\ p(v)>Y$, and $u\in\mathcal{N}(\mathcal{P}_1),\ \ v\in\mathcal{N}(\mathcal{P}_2)$. Let

$$\Delta(\xi,\eta)=\#(\mathcal{T}(\xi,\eta)\setminus\mathcal{T}^*(\xi,\eta)). \tag{3.15}$$

If $n\in\mathcal{T}(\xi,\eta)\setminus\mathcal{T}^*(\xi,\eta)$, then there exists $p_1|n$ such that $Y<p_1,\ \ p_1\equiv 1 \pmod{Q_1}$, or $p_2|n+1$, such that $Y<p_2,\ \ p_2\equiv 1 \pmod{Q_2}$.

We shall prove that

$$\sum_{\xi,\eta<x^{1/10}}\Delta(\xi,\eta)=o_x(1)x\kappa_1\kappa_2. \tag{3.16}$$

Let $\xi,\eta<x^{1/10}$ be fixed. By using Theorem 2.6 in [7] we can overestimate those solutions of $n=\xi u,\ \ n+1=\eta v$ counted in $\mathcal{T}(\xi,\eta)$ for which there exists either a $p_1\in\mathcal{P}_1$ such that $p_1|u$, and $p_1<x^{0,75}$, or a $p_2\in\mathcal{P}_2$, such that $p_2|v$ and $p_2<x^{0,75}$. We consider the first case. The second case is similar. If $p_1|n$, then let $u=p_1m$. For fixed ξ,p_1,η we should estimate those m,v for which $\eta v-(p_1\xi)m=1,\ \ p(v)>Y,\ \ \ p(m)>Y$. Arguing as above, by using Theorem 2.6 in [7] we obtain the number of the integers is less than $\frac{c}{p_1}T(\xi,\eta)$. Since

$$\sum_{\substack{Y<p_1<x\\p_1\equiv 1(Q_1)}}1/p_1\le\frac{c_1}{Q_1}\log\frac{\log x}{\log Y}\ll\frac{x_3}{Q_1},$$

the contribution of these types of integers to (3.16) is less than

$$\frac{x_3}{Q_1}\sum_{\xi,\eta<x^{1/10}} T(\xi,\eta). \tag{3.17}$$

Let us observe that the number of those $n\le x$ for which there exists $p_1\in\mathcal{P}_1,\ p_1|n,\quad p_1>\sqrt{x}$, or $p_2\in\mathcal{P}_2$, such that $p_2|n+1,\quad p_2>\sqrt{x}$ is $o_x(1)x\kappa_1\kappa_2$. The number of these integers is less than

$$\sum_{\substack{p_1\equiv 1(Q_1)\\ \sqrt{x}<p_1<x}}\frac{x}{p_1}+\sum_{\substack{p_2\equiv 1(Q_2)\\ \sqrt{x}<p_2<x}}\frac{x}{p_2}\ll x\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right),$$

and the right hand side is $o_x(1)x\kappa_1\kappa_2$.

(3.17) is proved, whence we obtain that

$$\begin{aligned}E_{Q_1,Q_2}(x)=&(1+o_x(1))\frac{x}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y)B(Y)C(Y)+\\ &+o_x(1)x\kappa_1\kappa_2.\end{aligned}\tag{3.18}$$

We have

$$\begin{aligned}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y)=&\prod_{3\le p<Y}\left(1-\frac{2}{p(p-1)}\right)\cdot\prod_{\substack{3\le p<Y\\ p\equiv 1(Q_1)}}\frac{1}{1+\frac{2}{p-1}}\times\\ &\times\prod_{\substack{3\le p<Y\\ p\equiv 1(Q_2)}}\frac{1}{1+\frac{2}{p-1}}\cdot\prod_{\substack{3\le p<Y\\ p\equiv 1(Q_1,Q_2)}}\left(1+\frac{2}{p-1}\right).\end{aligned}\tag{3.19}$$

Furthermore,

$$\prod_{\substack{3\le p<Y\\ p\equiv 1\pmod D}}\frac{1}{1+\frac{2}{p-1}}=e^{-\frac{2\log\log Y}{\varphi(D)}+\mathcal{O}\left(\frac{1}{D}\right)} \tag{3.20}$$

and

$$\prod_{\substack{3\le p<Y\\ p\equiv 1\pmod D}}(1+1/p)=e^{\frac{\log\log Y}{\varphi(D)}+\mathcal{O}\left(\frac{1}{D}\right)} \tag{3.21}$$

uniformly as $D \leq x_2^2$. Let

$$B = \prod_{p \geq 3} \left(1 - \frac{2}{p(p-1)}\right). \tag{3.22}$$

From (3.19), (3.20), (3.21) we have:

(i) the right hand side of (3.19) equals to

$$\begin{aligned} &(1 + o_x(1))B \cdot \kappa_1^2 \kappa_2^2 e^{\frac{2x_2}{(Q_1-1)(Q_2-1)}} = \\ &\qquad = (1 + o_x(1))B \cdot \kappa_1^2 \kappa_2^2, \end{aligned} \tag{3.23}$$

(ii)

$$B(Y) = (1 + o_x(1))\frac{1}{\kappa_1}, \tag{3.24}$$

(iii)

$$C(Y) = (1 + o_x(1))\frac{1}{\kappa_2}. \tag{3.25}$$

Hence the theorem follows immediately.

4. Final remark

The distribution of the prime power divisors of the iterates of $\varphi(n)$, $\sigma(n)$ are investigated in [8].

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